Priscila Dombrovski Zen **Métodos Aproximados em Engenharia**

3º Trabalho da disciplina

Resolver usando o método dos resíduos ponderados a seguinte equação diferencial:

$$\frac{d^2u}{dx^2} - u = 0$$

Onde as condições de contorno são: $u(0) = 0$ e $u(1) = 1$

Solução Analítica

$$\frac{d^2u}{dx^2} - u = 0$$

$$t^2 - 1 = 0$$

$$t = \pm 1$$

$$u(x) = C_1e^x + C_2e^{-x}$$

$$u(0) = C_1e^0 + C_2e^{-0} = 0 \ \rightarrow \ C_1 = -C_2$$

$$u(1) = C_1e^1 + C_2e^{-1} = 1 \rightarrow -C_2e^1 + C_2e^{-1} = 1$$

$$C_2(e^1 - e^{-1}) = -1 \ \rightarrow \ C_2 = -\frac{1}{(e^1 - e^{-1})}$$

$$C_1 = \frac{1}{(e^1 - e^{-1})}$$

$$u(x) = \frac{1}{(e^1 - e^{-1})}e^x - \frac{1}{(e^1 - e^{-1})}e^{-x}$$

$$u(x) = \frac{e^x - e^{-x}}{(e^1 - e^{-1})} = \frac{2\sinh x}{2\sinh 1} = \frac{\sinh x}{\sinh 1}$$

Solução Aproximada

Utilizando o método de Galerkin e uma solução aproximada do tipo:

$$\bar{u} = \beta + \sum_{m=1}^{M} \alpha_m sen\ m\pi x$$

Escolhendo $\boldsymbol{\beta = x}$ e $\boldsymbol{m = 1}$, um parâmetro, temos:

$$\bar{u} = x + \alpha_1 sen\ (\pi x)$$

$$\frac{d\bar{u}}{dx} = 1 + \pi\alpha_1 cos\ (\pi x)$$

$$\frac{d\bar{u}}{dx} = -\pi^2\alpha_1 sen\ (\pi x)$$

$$\varepsilon_\Omega = -\pi^2\alpha_1 sen(\pi x) - \left(x + \alpha_1 sen\ (\pi x)\right) = -x - (\pi^2 + 1)sen(\pi x)$$

Onde $\Phi_1 = sen\ \pi x$

A sentença dos resíduos ponderados pode ser escrita como:

$$\int_0^1 [\omega_1]\ \varepsilon_\Omega dx = \int_0^1 [\Phi_1]\ \varepsilon_\Omega dx = 0$$

$$\int_0^1 [sen\ (\pi x)][-(\pi^2 + 1)sen\ (\pi x)].\{\alpha_1\}\, dx - \int_0^1 \big[[sen\ (\pi x)]\big]\, xdx = 0$$

$$-\int_0^1 [sen\ (\pi x)][(\pi^2 + 1)sen\ (\pi x)].\{\alpha_1\}\, dx - \int_0^1 \big[[sen\ (\pi x)]\big]\, xdx = 0$$

$$-\int_0^1 [sen\ (\pi x)][(\pi^2 + 1)sen\ (\pi x)].\{\alpha_1\}\, dx = \int_0^1 \big[[sen\ (\pi x)]\big]\, xdx$$

$$-\int_0^1 [(\pi^2 + 1)(sen\ (\pi x))^2].\{\alpha_1\}\, dx = \int_0^1 \big[[sen\ (\pi x)]\big]\, xdx$$

Integrando as funções separadamente:

$$-\int_0^1 [(sen\ (\pi x))^2]\, dx = -\int_0^1 \left[\frac{1}{2} - \frac{\cos(2\pi x)}{4\pi}\right] dx = -\frac{x}{2} + \frac{\mathrm{sen}(2\pi x)}{4\pi}\bigg|_{x=0}^{x=1} = -\frac{1}{2}$$

E

$$\int_0^1 \big[[sen\ (\pi x)]\big]\, xdx = \frac{1}{\pi}$$

Logo, temos:

$$\frac{-(\pi^2+1)}{2}\alpha_1 = \frac{1}{\pi}$$

$$\alpha_1 = \frac{-2}{\pi(\pi^2+1)}$$

Assim, no solução aproximada para um parâmetro:

$$\bar{u} = x - \frac{2}{\pi(\pi^2+1)} sen\ (\pi x)$$

Escolhendo $\boldsymbol{\beta = x}$ e $\boldsymbol{m = 2}$, dois parâmetros, temos:

$$\bar{u} = x + \alpha_1 sen\ (\pi x) + \alpha_2 sen\ (2\pi x)$$

$$\frac{d\bar{u}}{dx} = 1 + \pi\alpha_1 cos\ (\pi x) + 2\pi\ \alpha_2 cos\ (2\pi x)$$

$$\frac{d\bar{u}}{dx} = -\pi^2\alpha_1 sen\ (\pi x) - 4\pi^2\ \alpha_2 sen\ (2\pi x)$$

$$\varepsilon_\Omega = -\pi^2\alpha_1 sen\ (\pi x) - 4\pi^2\ \alpha_2 sen\ (2\pi x) - (x + \alpha_1 sen\ (\pi x) + \alpha_2 sen\ (2\pi x))$$

$$\varepsilon_\Omega = -(x + (\pi^2+1)\alpha_1 sen\ (\pi x) + (4\pi^2+1)\alpha_2 sen\ (2\pi x))$$

Onde $\Phi_1 = sen\ \pi x$ e $\Phi_2 = sen\ 2\pi x$

A sentença dos resíduos ponderados pode ser escrita como:

$$\int_0^1 \begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} \varepsilon_\Omega dx = \int_0^1 \begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix} \varepsilon_\Omega dx = 0$$

$$-\int_0^1 \begin{bmatrix} sen\ (\pi x)\\ sen\ (2\pi x)\end{bmatrix} [(\pi^2+1) sen\ (\pi x) \quad (4\pi^2+1) sen\ (2\pi x)].\begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix} dx - \int_0^1 \left[\begin{bmatrix} sen\ (\pi x)\\ sen\ (2\pi x)\end{bmatrix}\right] x dx = 0$$

$$-\int_0^1 \begin{bmatrix} (\pi^2+1) sen^2\ (\pi x) & (4\pi^2+1) sen\ (\pi x). sen\ (2\pi x)\\ (\pi^2+1) sen\ (2\pi x) sen(\pi x) & (4\pi^2+1) sen^2\ (2\pi x)\end{bmatrix} .\begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix} dx - \int_0^1 \left[\begin{bmatrix} x. sen\ (\pi x)\\ x. sen\ (2\pi x)\end{bmatrix}\right] dx = 0$$

Integrando as funções separadamente:

$$-\int_0^1 [sen^2\ (\pi x)]\, dx = -\int_0^1 \left[\frac{1}{2} - \frac{\cos(2\pi x)}{4\pi}\right] dx = \left. -\frac{x}{2} + \frac{\text{sen}(2\pi x)}{4\pi}\right|_{x=0}^{x=1} = -\frac{1}{2}$$

$$-\int_0^1 [sen^2\ (2\pi x)]\, dx = -\int_0^1 \left[\frac{1}{2} - \frac{\cos(4\pi x)}{2}\right] dx = \left. -\frac{x}{2} + \frac{\text{sen}(4\pi x)}{8\pi}\right|_{x=0}^{x=1} = -\frac{1}{2}$$

$$-\int_0^1 sen\ (\pi x).sen\ (2\pi x)\, dx = \left. \frac{sen(\pi - 2\pi)x}{2(\pi - 2\pi)} - \frac{sen(\pi + 2\pi)x}{2(\pi + 2\pi)}\right|_{x=0}^{x=1} = \left. \frac{sen(-\pi)x}{-2\pi} - \frac{sen(3\pi)x}{6\pi}\right|_{x=0}^{x=1} = 0$$

$$\int_0^1 [[sen\ (\pi x)]]\, x dx = \frac{1}{\pi}$$

$$\int_0^1 [[sen\ (2\pi x)]]\, x dx = \frac{-1}{2\pi}$$

Logo, temos:

$$\begin{bmatrix} \frac{-(\pi^2 + 1)}{2} & 0 \\ 0 & \frac{(4\pi^2 + 1)}{2} \end{bmatrix} . \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\pi} \\ \frac{-1}{2\pi} \end{bmatrix}$$

$$\frac{-(\pi^2 + 1)}{2} \alpha_1 = \frac{1}{\pi} \rightarrow \alpha_1 = \frac{-2}{\pi(\pi^2 + 1)}$$

$$\frac{(4\pi^2 + 1)}{2} \alpha_2 = -\frac{1}{2\pi} \rightarrow \alpha_2 = \frac{1}{\pi(4\pi^2 + 1)}$$

Assim, a solução aproximada para dois parâmetros:

$$\bar{u} = x - \frac{2}{\pi(\pi^2 + 1)} sen\ (\pi x) + \frac{1}{\pi(4\pi^2 + 1)} sen\ (2\pi x)$$

Escolhendo $\boldsymbol{\beta} = x$ e $\boldsymbol{m} = 3$, três parâmetros, temos:

$$\bar{u} = x + \alpha_1 sen\,(\pi x) + \alpha_2 sen\,(2\pi x) + \alpha_3 sen\,(3\pi x)$$

$$\frac{d\bar{u}}{dx} = 1 + \pi\alpha_1 cos\,(\pi x) + 2\pi\,\alpha_2 cos(2\pi x) + 3\pi\alpha_3 cos\,(3\pi x)$$

$$\frac{d\bar{u}}{dx} = -\pi^2\alpha_1 sen\,(\pi x) - 4\pi^2\,\alpha_2 sen\,(2\pi x) - 9\pi^2\,\alpha_3 sen\,(3\pi x)$$

$$\varepsilon_\Omega = -\pi^2\alpha_1 sen\,(\pi x) - 4\pi^2\,\alpha_2 sen\,(2\pi x) - 9\pi^2\,\alpha_3 sen\,(3\pi x) - (x + \alpha_1 sen\,(\pi x) + \alpha_2 sen\,(2\pi x))\; + \alpha_3 sen\,(3\pi x)$$

$$\varepsilon_\Omega = -(x + (\pi^2+1)\alpha_1 sen\,(\pi x) + (4\pi^2+1)\alpha_2 sen\,(2\pi x) + (9\pi^2+1)\alpha_3 sen\,(3\pi x))$$

Onde $\Phi_1 = sen\,\pi x$, $\Phi_2 = sen\,2\pi x$ e $\Phi_3 = sen\,3\pi x$

A sentença dos resíduos ponderados pode ser escrita como:

$$\int_0^1 \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \varepsilon_\Omega dx = \int_0^1 \begin{bmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\end{bmatrix} \varepsilon_\Omega dx = 0$$

$$-\int_0^1 \begin{bmatrix} sen\,(\pi x)\\ sen\,(2\pi x)\\ sen\,(3\pi x)\end{bmatrix} [(\pi^2+1)sen\,(\pi x) \quad (4\pi^2+1)sen\,(2\pi x) \quad (9\pi^2+1)sen\,(3\pi x)].\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix} dx - \int_0^1 \left[\begin{bmatrix} sen\,(\pi x)\\ sen\,(2\pi x)\\ sen\,(3\pi x)\end{bmatrix}\right] xdx = 0$$

$$-\int_0^1 \begin{bmatrix} (\pi^2+1)sen^2\,(\pi x) & (4\pi^2+1)sen\,(2\pi x)sen(\pi x) & (9\pi^2+1)sen\,(\pi x)sen(3\pi x)\\ (\pi^2+1)sen\,(2\pi x)sen(\pi x) & (4\pi^2+1)sen^2\,(2\pi x) & (9\pi^2+1)sen\,(2\pi x)sen(3\pi x)\\ (\pi^2+1)sen\,(3\pi x)sen(\pi x) & (4\pi^2+1)sen\,(2\pi x)sen(3\pi x) & (9\pi^2+1)sen^2\,(3\pi x)\end{bmatrix}.\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix} dx - \int_0^1 \begin{bmatrix} x.sen\,(\pi x)\\ x.sen\,(2\pi x)\\ x.sen\,(3\pi x)\end{bmatrix} dx = 0$$

Integrando as funções separadamente:

$$-\int_0^1 [sen^2\,(\pi x)]\,dx = -\int_0^1 \left[\frac{1}{2} - \frac{\cos(2\pi x)}{4\pi}\right] dx = -\frac{x}{2} + \frac{\text{sen}(2\pi x)}{4\pi}\bigg|_{x=0}^{x=1} = -\frac{1}{2}$$

$$-\int_0^1 [sen^2\,(2\pi x)]\,dx = -\int_0^1 \left[\frac{1}{2} - \frac{\cos(4\pi x)}{8\pi}\right] dx = -\frac{x}{2} + \frac{\text{sen}(4\pi x)}{8\pi}\bigg|_{x=0}^{x=1} = -\frac{1}{2}$$

$$-\int_0^1 [sen^2\,(3\pi x)]\,dx = -\int_0^1 \left[\frac{1}{2} - \frac{\cos(6\pi x)}{12\pi}\right] dx = -\frac{x}{2} + \frac{\text{sen}(6\pi x)}{12\pi}\bigg|_{x=0}^{x=1} = -\frac{1}{2}$$

$$-\int_0^1 sen\,(\pi x).sen\,(2\pi x)\,dx = \frac{sen(\pi-2\pi)x}{2(\pi-2\pi)} - \frac{sen(\pi+2\pi)x}{2(\pi+2\pi)}\bigg|_{x=0}^{x=1} = \frac{sen(-\pi)x}{-2\pi} - \frac{sen(3\pi)x}{6\pi}\bigg|_{x=0}^{x=1} = 0$$

$$-\int_0^1 sen\,(\pi x).sen\,(3\pi x)\,dx = \frac{sen(\pi-3\pi)x}{2(\pi-3\pi)} - \frac{sen(\pi+3\pi)x}{2(\pi+3\pi)}\bigg|_{x=0}^{x=1} = \frac{sen(-2\pi)x}{-4\pi} - \frac{sen(4\pi)x}{8\pi}\bigg|_{x=0}^{x=1} = 0$$

$$-\int_0^1 sen\ (2\pi x).sen\ (3\pi x)\ dx = \frac{sen(2\pi-3\pi)x}{2(2\pi-3\pi)} - \frac{sen(2\pi+3\pi)x}{2(2\pi+3\pi)}\bigg|_{x=0}^{x=1} = \frac{sen(-\pi)x}{-2\pi} - \frac{sen(5\pi)x}{10\pi}\bigg|_{x=0}^{x=1} = 0$$

$$\int_0^1 [[sen\ (\pi x)]]\, xdx = \frac{1}{\pi}$$

$$\int_0^1 [[sen\ (2\pi x)]]\, xdx = \frac{-1}{2\pi}$$

$$\int_0^1 [[sen\ (3\pi x)]]\, xdx = \frac{1}{3\pi}$$

Logo, temos:

$$\begin{bmatrix} \frac{-(\pi^2+1)}{2} & 0 & 0 \\ 0 & \frac{(4\pi^2+1)}{2} & 0 \\ 0 & 0 & \frac{(9\pi^2+1)}{2} \end{bmatrix} . \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\pi} \\ \frac{-1}{2\pi} \\ \frac{1}{3\pi} \end{bmatrix}$$

$$\frac{-(\pi^2+1)}{2}\alpha_1 = \frac{1}{\pi} \rightarrow \alpha_1 = \frac{-2}{\pi(\pi^2+1)}$$

$$\frac{(4\pi^2+1)}{2}\alpha_2 = -\frac{1}{2\pi} \rightarrow \alpha_2 = \frac{1}{\pi(4\pi^2+1)}$$

$$\frac{(9\pi^2+1)}{2}\alpha_3 = \frac{1}{3\pi} \rightarrow \alpha_3 = \frac{-2}{3\pi(9\pi^2+1)}$$

Assim, a solução aproximada para três parâmetros:

$$\bar{u} = x - \frac{2}{\pi(\pi^2+1)} sen\ (\pi x) + \frac{1}{\pi(4\pi^2+1)} sen\ (2\pi x) - \frac{2}{3\pi(9\pi^2+1)}\ sen\ (3\pi x)$$

Representação Gráfica

Comparativo entre Solução Analítica e Soluções Aproximadas

	Analítica	Galerkin m=1	Galerkin m=2	Galerkin m=3
0,1	0,085234	0,081901243	0,086204478	0,085989997
0,2	0,17132	0,165574118	0,172536899	0,172284762
0,3	0,259122	0,252616839	0,25957962	0,259497695
0,4	0,349517	0,344297754	0,348600989	0,348756818
0,5	0,443409	0,441431192	0,441431192	0,441696305
0,6	0,54174	0,544297754	0,539994518	0,540150348
0,7	0,645493	0,652616839	0,645654058	0,645572134
0,8	0,755705	0,765574118	0,758611338	0,7583592
0,9	0,873482	0,881901243	0,877598008	0,877383527
1	1	1	1	1