Priscila Dombrovski Zen **Métodos Aproximados em Engenharia**

Flexão de Vigas –

Pela teoria de Euler-Bernoulli, a equação que governa o problema de flexão de vigas é:

$$EI\,\frac{d^4u}{dx^4} = q(x)$$

Onde *E* é o módulo da Elasticidade do material, *I* é o Momento de Inércia da seção transversal e *q(x)* o carregamento.

Para as vigas mais comuns, as condições de contorno, admitindo uma viga de vão L, são:

- Viga simplesmente apoiada:

$$u(0) = u(L) = 0$$

$$u''(0) = u''(L) = 0$$

- Viga engastada:

$$u(0) = u(L) = 0$$

$$u'(0) = u'(L) = 0$$

- Viga em balanço:

$$u(0) = u'(0) = 0$$

$$u''(L) = u'''(L) = 0$$

- Viga engastada em *x=0* e simplesmente apoiada em *x=L*:

$$u(0) = u'(0) = 0$$

$$u(L) = u''(L) = 0$$

Problema:

Discretizar a equação de *Euler-Bernoulli* para as situações listadas anteriormente.

Iniciaremos atualizando as derivadas conhecidas para diferenças centrais:

$$\frac{du}{dx} = u'(x) = \frac{u_{i+1} - u_{i-1}}{2h}$$

$$\frac{d^2u}{dx^2} = u''(x) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

$$\frac{d^3u}{dx^3} = u'''(x) = \frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2h^3}$$

$$\frac{d^4u}{dx^4} = u''''(x) = \frac{u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}}{h^4}$$

Tomando *EI* como uma constante *K*, e *L= 8*

Partindo da equação original

$$EI \frac{d^4u}{dx^4} = q(x)$$

Substituindo a 4ª derivada e o EI na equação temos:

$$K \frac{u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}}{h^4} = q(x)$$

$$\frac{K}{h^4}(u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}) = q(x)$$

$$(u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}) = q(x)\frac{h^4}{K}$$

$$u_i = \frac{1}{6}\left[-u_{i+2} + 4u_{i+1} + 4u_{i-1} - u_{i-2}) + q(x)\frac{h^4}{K}\right]$$

a) Viga Simplesmente Apoiada
   Condições:

$$u(0) = u(8) = 0$$

$$u''(0) = u''(8) = 0$$

Assim:

$$u''(0) = u''(8) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0$$

Para $u''(0) = \frac{u_1 - 2u_0 + u_{-1}}{h^2} = 0$, assim: $u_1 = -u_{-1}$

Para $u''(8) = \frac{u_9 - 2u_8 + u_7}{h^2} = 0$, assim: $u_9 = -u_7$

Discretizando, temos:

Para $i = 1$: $u_1 = \frac{1}{6}\left[(-u_3 + 4u_2 + 4u_0 - u_{-1}) + q(x_1)\frac{h^4}{K}\right]$

$$u_1 = \frac{1}{6}\left[(-u_3 + 4u_2 + u_1) + q(x_1)\frac{h^4}{K}\right]$$

$$u_1 = \frac{1}{5}\left[(-u_3 + 4u_2 + q(x_1)\frac{h^4}{K}\right]$$

Para $i = 2$: $u_2 = \frac{1}{6}\left[(-u_4 + 4u_3 + 4u_1 - u_0) + q(x_2)\frac{h^4}{K}\right]$

$$u_2 = \frac{1}{6}\left[(-u_4 + 4u_3 + 4u_1) + q(x_2)\frac{h^4}{K}\right]$$

Para $i = 3$: $u_3 = \frac{1}{6}\left[(-u_5 + 4u_4 + 4u_2 - u_1) + q(x_3)\frac{h^4}{K}\right]$

Para $i = 4$: $u_4 = \frac{1}{6}\left[(-u_6 + 4u_5 + 4u_3 - u_2) + q(x_4)\frac{h^4}{K}\right]$

Para $i = 5$: $u_5 = \frac{1}{6}\left[(-u_7 + 4u_6 + 4u_4 - u_3) + q(x_5)\frac{h^4}{K}\right]$

Para $i = 6$: $u_6 = \frac{1}{6}\left[(-u_8 + 4u_7 + 4u_5 - u_4) + q(x_6)\frac{h^4}{K}\right]$

$$u_6 = \frac{1}{6}\left[(4u_7 + 4u_5 - u_4) + q(x_6)\frac{h^4}{K}\right]$$

Para $i = 7$: $u_7 = \frac{1}{6}\left[(-u_9 + 4u_8 + 4u_6 - u_5) + q(x_7)\frac{h^4}{K}\right]$

$$u_7 = \frac{1}{6}\left[(u_7 + 4u_6 - u_5) + q(x_7)\frac{h^4}{K}\right]$$

$$u_7 = \frac{1}{5}\left[(+4u_6 - u_5) + q(x_7)\frac{h^4}{K}\right]$$

Em sistema de equações teremos:

$$\begin{cases} q(x_1) = \frac{K}{h^4}(u_3 - 4u_2 + 5u_1) \\ q(x_2) = \frac{K}{h^4}(u_4 - 4u_3 + 6u_2 - 4u_1) \\ q(x_3) = \frac{K}{h^4}(u_5 - 4u_4 + 6u_3 - 4u_2 + u_1) \\ q(x_4) = \frac{K}{h^4}(u_6 - 4u_5 + 6u_4 - 4u_3 + u_2) \\ q(x_5) = \frac{K}{h^4}(u_7 - 4u_6 + 6u_5 - 4u_4 + u_3) \\ q(x_6) = \frac{K}{h^4}(-4u_7 + 6u_6 - 4u_5 + u_4) \\ q(x_7) = \frac{K}{h^4}(5u_7 - 4u_6 + u_5) \end{cases}$$

$$\frac{K}{h^4}\begin{pmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & 1 & -4 & 6 & -4 & 1 & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{pmatrix} . \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} = \begin{pmatrix} q(x_1) \\ q(x_2) \\ q(x_3) \\ q(x_4) \\ q(x_5) \\ q(x_6) \\ q(x_7) \end{pmatrix}$$

b) Viga engastada:

$$u(0) = u(L) = 0$$

$$u'(0) = u'(L) = 0$$

Assim:

$$u'(0) = u'(8) = \frac{u_{i+1} - u_{i-1}}{2h} = 0$$

Para $u'(0) = \frac{u_1 - u_{-1}}{2h} = 0$, assim: $u_1 = u_{-1}$

Para $u'(8) = \frac{u_9 - u_7}{2h} = 0$, assim: $u_9 = u_7$

Discretizando, temos:

Para $i = 1$: $u_1 = \frac{1}{6}\left[(-u_3 + 4u_2 + 4u_0 - u_{-1}) + q(x_1)\frac{h^4}{K}\right]$

$$u_1 = \frac{1}{6}\left[(-u_3 + 4u_2 - u_1) + q(x_1)\frac{h^4}{K}\right]$$

$$u_1 = \frac{1}{7}\left[(-u_3 + 4u_2 + q(x_1)\frac{h^4}{K}\right]$$

Para $i = 2$: $u_2 = \frac{1}{6}\left[(-u_4 + 4u_3 + 4u_1 - u_0) + q(x_2)\frac{h^4}{K}\right]$

$$u_2 = \frac{1}{6}\left[(-u_4 + 4u_3 + 4u_1) + q(x_2)\frac{h^4}{K}\right]$$

Para $i = 3$: $u_3 = \frac{1}{6}\left[(-u_5 + 4u_4 + 4u_2 - u_1) + q(x_3)\frac{h^4}{K}\right]$

Para $i = 4$: $u_4 = \frac{1}{6}\left[(-u_6 + 4u_5 + 4u_3 - u_2) + q(x_4)\frac{h^4}{K}\right]$

Para $i = 5$: $u_5 = \frac{1}{6}\left[(-u_7 + 4u_6 + 4u_4 - u_3) + q(x_5)\frac{h^4}{K}\right]$

Para $i = 6$: $u_6 = \frac{1}{6}\left[(-u_8 + 4u_7 + 4u_5 - u_4) + q(x_6)\frac{h^4}{K}\right]$

$$u_6 = \frac{1}{6}\left[(4u_7 + 4u_5 - u_4) + q(x_6)\frac{h^4}{K}\right]$$

Para $i = 7$: $u_7 = \frac{1}{6}\left[(-u_9 + 4u_8 + 4u_6 - u_5) + q(x_7)\frac{h^4}{K}\right]$

$$u_7 = \frac{1}{6}\left[(-u_7 + 4u_6 - u_5) + q(x_7)\frac{h^4}{K}\right]$$

$$u_7 = \frac{1}{7}\left[(+4u_6 - u_5) + q(x_7)\frac{h^4}{K}\right]$$

Em sistema de equações teremos:

$$\begin{cases} q(x_1) = \frac{K}{h^4}(u_3 - 4u_2 + 7u_1) \\ q(x_2) = \frac{K}{h^4}(u_4 - 4u_3 + 6u_2 - 4u_1) \\ q(x_3) = \frac{K}{h^4}(u_5 - 4u_4 + 6u_3 - 4u_2 + u_1) \\ q(x_4) = \frac{K}{h^4}(u_6 - 4u_5 + 6u_4 - 4u_3 + u_2) \\ q(x_5) = \frac{K}{h^4}(u_7 - 4u_6 + 6u_5 - 4u_4 + u_3) \\ q(x_6) = \frac{K}{h^4}(-4u_7 + 6u_6 - 4u_5 + u_4) \\ q(x_7) = \frac{K}{h^4}(7u_7 - 4u_6 + u_5) \end{cases}$$

$$\frac{K}{h^4}\begin{pmatrix} 7 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & 1 & -4 & 6 & -4 & 1 & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 7 \end{pmatrix} . \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} = \begin{pmatrix} q(x_1) \\ q(x_2) \\ q(x_3) \\ q(x_4) \\ q(x_5) \\ q(x_6) \\ q(x_7) \end{pmatrix}$$

c) Viga em balanço:

$$u(0) = u'(0) = 0$$

$$u''(L) = u'''(L) = 0$$

Assim:

$$u'(0) = \frac{u_{+1} - u_{-1}}{2h} = 0,\ u_1 = u_{-1}$$

$$u''(8) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0,\ \ u_9 = -u_7$$

$$u'''(8) = \frac{u_{10} - 2u_9 + 2u_7 - u_6}{2h^3} = 0, u_{10} = -2u_7 - 2u_7 + u_6, u_{10} = -4u_7 + u_6$$

Discretizando, temos:

Para $i = 1$: $u_1 = \frac{1}{6}\left[(-u_3 + 4u_2 + 4u_0 - u_{-1}) + q(x_1)\frac{h^4}{K}\right]$

$$u_1 = \frac{1}{6}\left[(-u_3 + 4u_2 - u_1) + q(x_1)\frac{h^4}{K}\right]$$

$$u_1 = \frac{1}{7}\left[(-u_3 + 4u_2 + q(x_1)\frac{h^4}{K}\right]$$

Para $i = 2$: $u_2 = \frac{1}{6}\left[(-u_4 + 4u_3 + 4u_1 - u_0) + q(x_2)\frac{h^4}{K}\right]$

$$u_2 = \frac{1}{6}\left[(-u_4 + 4u_3 + 4u_1) + q(x_2)\frac{h^4}{K}\right]$$

Para $i = 3$: $u_3 = \frac{1}{6}\left[(-u_5 + 4u_4 + 4u_2 - u_1) + q(x_3)\frac{h^4}{K}\right]$

Para $i = 4$: $u_4 = \frac{1}{6}\left[(-u_6 + 4u_5 + 4u_3 - u_2) + q(x_4)\frac{h^4}{K}\right]$

Para $i = 5$: $u_5 = \frac{1}{6}\left[(-u_7 + 4u_6 + 4u_4 - u_3) + q(x_5)\frac{h^4}{K}\right]$

Para $i = 6$: $u_6 = \frac{1}{6}\left[(-u_8 + 4u_7 + 4u_5 - u_4) + q(x_6)\frac{h^4}{K}\right]$

Para $i = 7$: $u_7 = \frac{1}{6}\left[(-u_9 + 4u_8 + 4u_6 - u_5) + q(x_7)\frac{h^4}{K}\right]$

$$u_7 = \frac{1}{6}\left[(+u_7 + 4u_8 + 4u_6 - u_5\ ) + q(x_7)\frac{h^4}{K}\right]$$

$$u_7 = \frac{1}{5}\left[(2u_8 + 4u_6 - u_5) + q(x_7)\frac{h^4}{K}\right]$$

Para $i = 8$: $u_8 = \frac{1}{6}\left[(-u_{10} + 4u_9 + 4u_7 - u_6) + q(x_8)\frac{h^4}{K}\right]$

$$u_8 = \frac{1}{6}\left[(-(2u_9 - 2u_7 + u_6) + 4(2u_8 - u_7) + 4u_7 - u_6) + q(x_8)\frac{h^4}{K}\right]$$

$$u_8 = \frac{1}{2}\left[+4u_7 - 2u_6 + q(x_8)\frac{h^4}{K}\right]$$

Em sistema de equações teremos:

$$\begin{cases} q(x_1) = \frac{K}{h^4}(u_3 - 4u_2 + 7u_1) \\ q(x_2) = \frac{K}{h^4}(u_4 - 4u_3 + 6u_2 - 4u_1) \\ q(x_3) = \frac{K}{h^4}(u_5 - 4u_4 + 6u_3 - 4u_2 + u_1) \\ q(x_4) = \frac{K}{h^4}(u_6 - 4u_5 + 6u_4 - 4u_3 + u_2) \\ q(x_5) = \frac{K}{h^4}(u_7 - 4u_6 + 6u_5 - 4u_4 + u_3) \\ q(x_6) = \frac{K}{h^4}(u_8 - 4u_7 + 6u_6 - 4u_5 + u_4) \\ q(x_7) = \frac{K}{h^4}(-2u_8 + 5u_7 - 4u_6 + u_5) \\ q(x_8) = \frac{K}{h^4}(2u_8 - 4u_7 + 2u_6) \end{cases}$$

$$\frac{K}{h^4}\begin{pmatrix} 7 & -4 & 1 & & & & & \\ -4 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & 1 & -4 & 6 & -4 & 1 & & \\ & & 1 & -4 & 6 & -4 & 1 & \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & 5 & 2 \\ & & & & & 2 & -4 & 2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} = \begin{pmatrix} q(x_1) \\ q(x_2) \\ q(x_3) \\ q(x_4) \\ q(x_5) \\ q(x_6) \\ q(x_7) \\ q(x_8) \end{pmatrix}$$

d) Viga engastada em *x=0* e simplesmente apoiada em *x=L*:

$$u(0) = u'(0) = 0$$

$$u(L) = u''(L) = 0$$

Assim:

$$u'(0) = \frac{u_{+1} - u_{-1}}{2h} = 0,\; u_1 = u_{-1}$$

$$u''(8) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0,\;\; u_9 = -u_7$$

Discretizando, temos:

Para $i = 1$: $u_1 = \frac{1}{6}\left[(-u_3 + 4u_2 + 4u_0 - u_{-1}) + q(x_1)\frac{h^4}{K}\right]$

$$u_1 = \frac{1}{6}\left[(-u_3 + 4u_2 - u_1) + q(x_1)\frac{h^4}{K}\right]$$

$$u_1 = \frac{1}{7}\left[(-u_3 + 4u_2 + q(x_1)\frac{h^4}{K}\right]$$

Para $i = 2$: $u_2 = \frac{1}{6}\left[(-u_4 + 4u_3 + 4u_1 - u_0) + q(x_2)\frac{h^4}{K}\right]$

$$u_2 = \frac{1}{6}\left[(-u_4 + 4u_3 + 4u_1) + q(x_2)\frac{h^4}{K}\right]$$

Para $i = 3$: $u_3 = \frac{1}{6}\left[(-u_5 + 4u_4 + 4u_2 - u_1) + q(x_3)\frac{h^4}{K}\right]$

Para $i = 4$: $u_4 = \frac{1}{6}\left[(-u_6 + 4u_5 + 4u_3 - u_2) + q(x_4)\frac{h^4}{K}\right]$

Para $i = 5$: $u_5 = \frac{1}{6}\left[(-u_7 + 4u_6 + 4u_4 - u_3) + q(x_5)\frac{h^4}{K}\right]$

Para $i = 6$: $u_6 = \frac{1}{6}\left[(-u_8 + 4u_7 + 4u_5 - u_4) + q(x_6)\frac{h^4}{K}\right]$

$$u_6 = \frac{1}{6}\left[(4u_7 + 4u_5 - u_4) + q(x_6)\frac{h^4}{K}\right]$$

Para $i = 7$: $u_7 = \frac{1}{6}\left[(-u_9 + 4u_8 + 4u_6 - u_5) + q(x_7)\frac{h^4}{K}\right]$

$$u_7 = \frac{1}{6}\left[(u_7 + 4u_6 - u_5) + q(x_7)\frac{h^4}{K}\right]$$

$$u_7 = \frac{1}{5}\left[(+4u_6 - u_5) + q(x_7)\frac{h^4}{K}\right]$$

Em sistema de equações teremos:

$$\begin{cases} q(x_1) = \dfrac{K}{h^4}(u_3 - 4u_2 + 7u_1) \\ q(x_2) = \dfrac{K}{h^4}(u_4 - 4u_3 + 6u_2 - 4u_1) \\ q(x_3) = \dfrac{K}{h^4}(u_5 - 4u_4 + 6u_3 - 4u_2 + u_1) \\ q(x_4) = \dfrac{K}{h^4}(u_6 - 4u_5 + 6u_4 - 4u_3 + u_2) \\ q(x_5) = \dfrac{K}{h^4}(u_7 - 4u_6 + 6u_5 - 4u_4 + u_3) \\ q(x_6) = \dfrac{K}{h^4}(-4u_7 + 6u_6 - 4u_5 + u_4) \\ q(x_7) = \dfrac{K}{h^4}(5u_7 - 4u_6 + u_5) \end{cases}$$

$$\frac{K}{h^4}\begin{pmatrix} 7 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & 1 & -4 & 6 & -4 & 1 & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{pmatrix} . \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} = \begin{pmatrix} q(x_1) \\ q(x_2) \\ q(x_3) \\ q(x_4) \\ q(x_5) \\ q(x_6) \\ q(x_7) \end{pmatrix}$$