MÉTODOS NUMÉRICOS PARA PROBLEMAS EM MEIOS POROSOS

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PARTE I

We consider the following two-dimensional equation on the square domain $\Omega = (0,1)^2$: (Darcy's Problem)

$$\begin{cases} -\nabla \cdot \left(\frac{K(x)}{\eta} \nabla p(x) \right) = f(x), & x \in \Omega \\ p(x) = g(x), & x \in \partial\Omega \end{cases}$$

~~Aquí p(x) es la presión, K(x)~~

Here, $p(x)$ is the pressure, $K(x)$ is the permeability and η is the viscosity. The permeability is a measure of the ability of a porous medium to transmit fluids. It only depends on properties of the soil and can be discontinuous. In general, the permeability is a second order tensor given by the 2x2 matrix:

$$K = \begin{pmatrix} K_{xx} & K_{xy} \\ K_{xy} & K_{yy} \end{pmatrix}$$

- If the medium is isotropic: $K = \kappa I$.

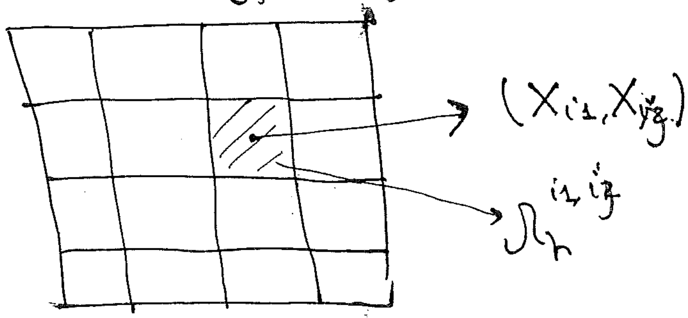
① $$\begin{cases} -\nabla \cdot (K(x) \nabla p) = f(x), & x \in \Omega \quad (1) \\ p(x) = g(x), & x \in \partial\Omega, \end{cases}$$

where $K(x)$ is a function which may be discontinuous.

Cell-centered Finite Volume discretization:

We consider a uniform grid Ω_h with the same step grid $h = \frac{1}{N}$, $N \in \mathbb{N}$ in both directions,

$$\Omega_h = \{ (x_{i_1}, x_{i_2}) \,;\; x_{i_\alpha} = (i_\alpha - \tfrac{1}{2})h, \; i_\alpha = 1, \dots, N, \; \cancel{\alpha = 1, 2} \}$$
$$y_j = (j - \tfrac{1}{2})h \qquad j = 1 \dots N$$



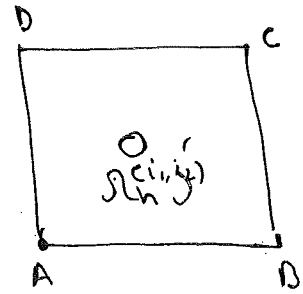
Therefore, we have square blocks of size $h \times h$ denoted by $\Omega_h^{(i_1,i_2)}$.

By integrating (1) on $\Omega_h^{(i_1,i_2)}$ we obtain:

$$\int_{\Omega_h^{(i_1,i_2)}} -\nabla\cdot(K\nabla p)\,dx\,dy = \int_{\Omega_h^{(i_1,i_2)}} f\,dx\,dy.$$

Using the Gauss-divergence theorem, the left-hand side of the above integral is reformulated as a boundary integral for boundary $\partial\Omega_h^{(i_1,i_2)}$ and $\vec{n}_K$ is the unit normal vector to that face

$$\int_{\Omega_h^{(i_1,i_2)}} -\nabla\cdot(K\nabla p)\,dx\,dy = \int_{\partial\Omega_h^{(i_1,i_2)}} K\nabla p\cdot\vec{n}\,ds$$



For cell $\Omega_h^{(i_1,i_2)} = ABCD$, the boundary integrals are given by:

$$-\int_{\partial\Omega_h^{(i_1,i_2)}} K\nabla p\cdot\vec{n}\,ds = \int_A^B K\frac{\partial p}{\partial y}\,dx - \int_B^C K\frac{\partial p}{\partial x}\,dy - \int_C^D K\frac{\partial p}{\partial y}\,dx + \int_D^A K\frac{\partial u}{\partial x}\,dy$$

To calculate these integrals, we approximate them as:

$$\int_B^C K\frac{\partial p}{\partial x}\,dy \approx \left(K_{i+1/2,j}\,\frac{p_{i+1,j}-p_{i,j}}{h}\right)h$$

If the coefficient field is smooth we can approximate $K_{i+1/2,j}$ by the mean of $K_{i,j}$ and $K_{i+1,j}$. In case of discontinuous coefficients, a harmonic average of $K_{i,j}$ and $K_{i+1,j}$ works well.

$$K_{i+1/2,j} = \frac{2 K_{ij} K_{i+1,j}}{K_{ij} + K_{i+1,j}}$$

The right-hand integral is approximated using the midpoint rule

$$\int_{\Omega_h^{(i,j)}} f \, dx\,dy \approx h^2 f_{ij}$$

Finally, we get the discrete equation for the interior volume $\Omega_h^{(i,j)}$ as:

$$c_{ij}^h p_{ij} + w_{ij}^h p_{i-1,j} + e_{ij}^h p_{i+1,j} + s_{ij}^h u_{i,j-1} + n_{ij}^h u_{i,j+1} = f_{ij}^h$$

where:

$$w_{ij}^h = -\frac{2}{h^2}\left(\frac{K_{ij} K_{i-1,j}}{K_{ij} + K_{i-1,j}}\right); \quad e_{ij}^h = -\frac{2}{h^2}\left(\frac{K_{ij} K_{i+1,j}}{K_{ij} + K_{i+1,j}}\right);$$

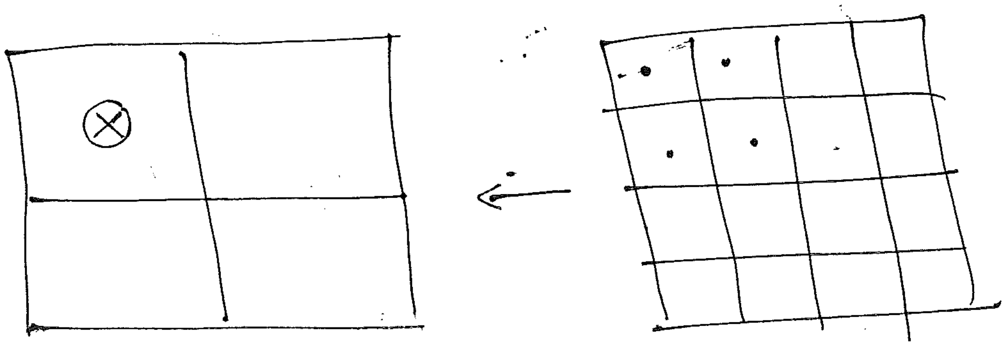
$$s_{ij}^h = -\frac{2}{h^2}\left(\frac{K_{ij} K_{ij-1}}{K_{ij} + K_{ij-1}}\right); \quad n_{ij}^h = -\frac{2}{h^2}\left(\frac{K_{ij} K_{ij+1}}{K_{ij} + K_{ij+1}}\right)$$

$$c_{ij}^h = -\left(w_{ij}^h + e_{ij}^h + s_{ij}^h + n_{ij}^h\right).$$

<u>Note</u>: the scheme is changed appropriately for the cells close to the boundary.

<u>Multigrid</u>:

- Standard coarsening obtained by doubling the mesh size in both directions.



Each coarse cell is the union of four fine cells, and since the unknowns are located in the cell-centres, this results in a non-nested hierarchy of grids.

- The simplest prolongation operator: piecewise constant interpolation operator.

$$P_{2h}^{h} = \left]\begin{matrix} 1 & & 1 \\ & * & \\ 1 & & 1 \end{matrix}\right[_{2h}^{h}$$

where $*$ denotes the position of a coarse grid unknown. The classical stencil notation shows the contribution of the coarse grid node to the neighbouring fine grid nodes.

- the restriction operator R_h^{2h} is chosen as the adjoint of the prolongation. given in stencil form by:

$$R_h^{2h} = \frac{1}{4}\begin{bmatrix} 1 & & 1 \\ & * & \\ 1 & & 1 \end{bmatrix}$$

- Operators on the coarse grids: $L_{2h} = \frac{1}{2} R_h^{2h} L_h P_{2h}^{h}$.
- Smoother: Lexicographic Gauss-Seidel.

Mixed-Formulation.

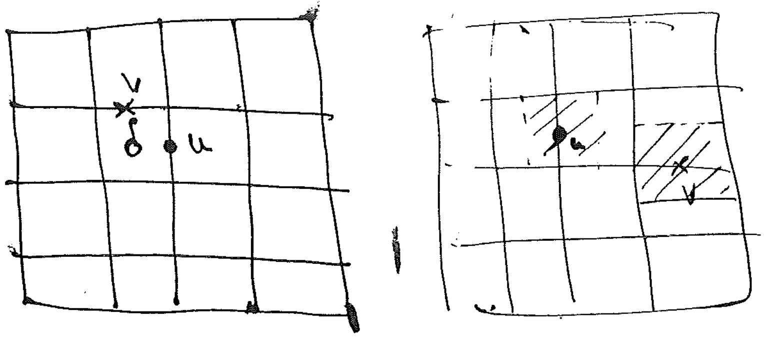
The mixed formulation of the Darcy problem is natural for computations in the porous medium region since it allows to directly approximate the velocity. this formulation reads:

$$(2)\quad \begin{cases} K^{-1}u + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot u = f & \text{in } \Omega. \end{cases}$$

with the following b. conditions

$$\begin{cases} p = g & \text{on } \partial\Omega_D \\ \vec{u}\cdot\vec{n} = g & \text{on } \partial\Omega_N. \end{cases}$$

Discretization: Finite volume on staggered grid:



Different control volumes are defined depending on which variable is considered

Regarding the horizontal velocity unknown, let us consider control volume $\Omega_{i+1/2,j}$ for variable $u_{i+1/2,j}$:

$$K^{-1} u_{i+1/2,j} + \frac{p_{i+1,j} - p_{i,j}}{h} = 0$$

Similarly, by discretizing the equation for v over a control volume $\Omega_{i,j+1/2}$, one obtains the discrete equation for the vertical velocity unknown, $V_{i,j+1/2}$:

$$K^{-1} V_{i,j+1/2} + \frac{p_{i,j+1} - p_{i,j}}{h} = 0.$$

Finally, the discrete equation corresponding to the pressure unknown $p_{i,j}$ is obtained by discretizing the second equation in (2) over control volume $\Omega_{i,j}$

resulting in:

$$\frac{U_{i+1/2,j} - U_{i-1/2,j}}{h} + \frac{V_{i,j+1/2} - V_{i,j-1/2}}{h} = f_{i,j}$$

The discretization leads to a so called <u>saddle point linear system</u>:
of the form:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} g \\ f \end{pmatrix}$$

B^T and B represent the discrete gradient and the minus discrete divergence operators, and A is the discrete operator for $K^{-1}I$.

Multigrid:

- The sequence of coarse grids is obtained by doubling the mesh size in each spatial direction.
- Restriction operators:

$$R^u_{h,2h} = \frac{1}{8}\begin{pmatrix} 1 & 2 & 1 \\ & \ast & \\ 1 & 2 & 1 \end{pmatrix}; \quad R^v_{h,2h} = \frac{1}{8}\begin{pmatrix} 1 & & 1 \\ 2 & \ast & 2 \\ 1 & & 1 \end{pmatrix}; \quad R^p_{h,2h} = \frac{1}{4}\begin{pmatrix} 1 & & 1 \\ & \ast & \\ 1 & & 1 \end{pmatrix}.$$

- Prolongation operators: the adjoints.
- Suavizadores $\begin{cases} \text{Acoplados} \\ \text{Desacoplados} \end{cases}$
- Acoplados: Overlapping block smoothers have been widely applied to saddle point problems, and in particular in the field of CFD. (Known as Vanka smoothers). It consists of simultaneously updating all unknowns appearing in the discrete divergence operator in the second equation of the system.

This approach implies that 4 unknowns corresponding to velocities and one pressure unknown are relaxed simultaneously and therefore a 5×5 system has to be solved for each cell for the pressure.
The local system to solve for each box has the following form:

$$\begin{pmatrix} K_{i+1/2} & 0 & 0 & 0 & -1/h \\ 0 & K_{i-1/2} & 0 & 0 & 1/h \\ 0 & 0 & K_{i,j+1/2} & 0 & -1/h \\ 0 & 0 & 0 & K_{i,j-1/2} & 1/h \\ 1/h & -1/h & 1/h & -1/h & \end{pmatrix} \begin{pmatrix} \delta U_{i+1/2,j} \\ \delta U_{i-1/2,j} \\ \delta V_{i,j+1/2} \\ \delta V_{i,j-1/2} \\ \delta p_{ij} \end{pmatrix} = \begin{pmatrix} r^u_{i+1/2,j} \\ \vdots \\ \\ \\ \\ \end{pmatrix}$$

- Uzawa smoother: It is based on the splitting:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} = \begin{pmatrix} M_A & 0 \\ B & -\omega^{-1} I \end{pmatrix} - \begin{pmatrix} M_A - A & -B^T \\ 0 & -\omega^{-1} I \end{pmatrix}$$

where M_A is a typical smoother for A and ω is some positive parameter.
From a given approximation of the solution to the system (u^k, p^k), the relaxed approximation (u^{k+1}, p^{k+1}) is computed according to the decoupled Uzawa smoother in the following way:

$$\begin{pmatrix} M_A & 0 \\ B & -\omega^{-1} I \end{pmatrix} \begin{pmatrix} u^{k+1} \\ p^{k+1} \end{pmatrix} = \begin{pmatrix} M_A - A & B^T \\ 0 & -\omega^{-1} I \end{pmatrix} \begin{pmatrix} u^k \\ p^k \end{pmatrix} + \begin{pmatrix} g \\ f \end{pmatrix}$$

① Relax the velocities by applying M_A:

$$u^{k+1} = u^k + M_A^{-1}(g - A u^k - B^t p^k)$$

$$\omega = \frac{h^2}{5K}$$

② Update the pressure: $p^{k+1} = p^k + \omega(B u^{k+1} - f)$. LFA

PARTE II

Poroelasticity:

Equilibrium equation: $\operatorname{div} \sigma' - \alpha \nabla p = \vec{f}$

Hooke's law: $\sigma' = \lambda \operatorname{tr}(\varepsilon) I + 2\mu \varepsilon$

Compatibility equation: $\varepsilon(u) = \frac{1}{2}(\nabla \vec{u} + \nabla \vec{u}^t)$

Darcy's law: $\vec{w} = -\frac{1}{\mu_f} K (\nabla p - \rho_f \vec{g})$

Continuity equation: $\frac{\partial}{\partial t}\left(\frac{1}{M} p + \alpha \nabla \cdot u\right) + \nabla \cdot \vec{w} = f$ en Ω.

• stress tensor: $\sigma' = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}$

Equations of equilibrio:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} - \alpha \frac{\partial p}{\partial x} = f_1$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \alpha \frac{\partial p}{\partial y} = f_2$$

Consider a body where we identify a point P and denote its location within the body by $\vec{x} = (x_i, y)$. When an external load is applied to the body, the point P will shift to a new location $\vec{x}' = (x_i', y')$. The vector that connects the initial location $\vec{x}$ and the final location $\vec{x}'$ is called the displacement vector $\vec{u} = (u, v)$

$$\vec{u} = \vec{x} - \vec{x}' ; \quad u = x_i - x_i', \quad v = y - y'$$

In general the displacement may vary from point to point.

In addition to displacement, the deformation is also quantified by the second order strain tensor:

$$\varepsilon = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{pmatrix}$$

where $\varepsilon_{xx}, \varepsilon_{yy}$ are known as the normal strains and ε_{xy} as the shear strains.

- The normal strains are related to the spatial derivatives of the displacement vector:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad ; \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}.$$

- the shear strain:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

- The strain tensor can be written:

$$\varepsilon(u) = \frac{1}{2}\left(\nabla\vec{u} + (\nabla\vec{u})^T\right)$$

where ~~grad~~ $\nabla\vec{u}$ is the displacement gradient, which is also a second-order tensor.

$$\varepsilon = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} \end{pmatrix}$$

$$\nabla\vec{u} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \qquad (\nabla u)^T = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Hooke's law: There is a connection between the stress and strain tensors.

$$\sigma = C\,\varepsilon.$$

If the elastic properties of the material are the same in all directions, such material is called isotropic.

The elastic behaviour of isotropic materials can be described by only two parameters, λ and μ known as Lame's parameters, and therefore the stress-strain relation is simplified as follows:

$$\sigma = 2\mu \varepsilon + \lambda \varepsilon_v I$$

where I is the identity tensor and ε_v is the volumetric strain:

$$\varepsilon_v = tr(\varepsilon) = \varepsilon_{xx} + \varepsilon_{yy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla \cdot \vec{u}.$$

Taking into account this, we have

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} = 2\mu \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{pmatrix} + \lambda \left(\begin{matrix} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} & 0 \\ 0 & \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \end{matrix} \right)$$

The equilibrio equations therefore can be written as:

- $$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} - \alpha \frac{\partial p}{\partial x} = f_1 \Rightarrow$$

$$\Rightarrow 2\mu \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \lambda \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{2\mu}{2}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \alpha \frac{\partial p}{\partial x} = f_1$$

$$\Rightarrow \boxed{(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial^2 y} + (\lambda+\mu)\left(\frac{\partial^2 v}{\partial x \partial y}\right) - \alpha \frac{\partial p}{\partial x} = f_1}$$

- $$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \alpha \frac{\partial p}{\partial y} = f_2 \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial x}\left(2\mu \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(2\mu \frac{\partial v}{\partial y} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right) - \alpha \frac{\partial p}{\partial y} = f_2$$

$$\boxed{\mu \frac{\partial^2 v}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + (\lambda+\mu)\frac{\partial^2 u}{\partial x \partial y} - \alpha \frac{\partial p}{\partial y} = f_2}$$

Depending on the variables that you are interested in, we have the two-field or three field-formulation of the poroelasticity:

- two-field (displacement-pressure velocity) formulation.

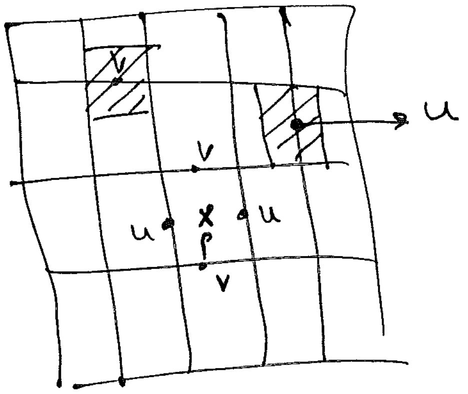
$$\begin{cases} -\mu \Delta u - (\lambda+\mu)\nabla\nabla\cdot u - \alpha\nabla p = f, \\ \frac{\partial}{\partial t}\left(\frac{1}{M}p + \alpha\nabla\cdot\vec{u}\right) - \nabla\cdot\left(\frac{1}{\mu_f}K(\nabla p - \rho_f\vec{g})\right) = 0 \end{cases}$$

- Three-field (displacement-pressure-velocity) formulation.

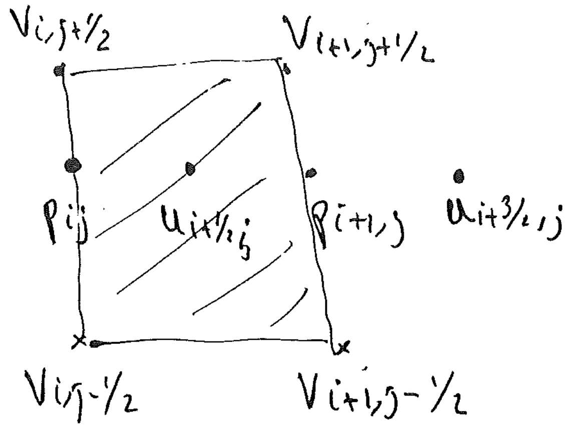
$$\begin{cases} -\mu\Delta u - (\lambda+\mu)\nabla\nabla\cdot u - \alpha\nabla p = f. \\ \vec{w} = -\frac{1}{\mu_f}K(\nabla p - \rho_f\vec{g}) \\ \frac{\partial}{\partial t}\left(\frac{1}{M}p + \alpha\nabla\cdot\vec{u}\right) + \nabla\cdot\vec{w} = 0. \end{cases}$$

- Discretization of the two-field formulation by finite volume on staggered grids.

We consider the finite volume method on a staggered grid. The computational domain is partitioned into square blocks of size $h\times h$. Different control volumes are defined depending on the variable considered.



We take the horizontal displacement $u_{i+1/2,j}$ as an example, since the discretization for the vertical component can be deduced in a similar way. We consider the control volume $V_{i+1/2,j}$



We have.

$$-\left(\frac{\sigma_{xx}/_e - \sigma_{xx}/_w}{h} + \frac{\sigma_{xy}/_n - \sigma_{xy}/_s}{h}\right) + \alpha\left(\frac{p_{i+1,j} - p_{ij}}{h}\right) = (f_1)_{i+1/2,j}$$

$$(\sigma_{xx})/_e \approx \frac{\lambda+2\mu}{h}\left(u_{i+3/2,j} - u_{i+1/2,j}\right) + \frac{\lambda}{h}\left(v_{i+1,j+1/2} - v_{i+1,j-1/2}\right)$$

$$(\sigma_{xx})_w \approx \frac{\lambda+2\mu}{h}\left(u_{i+1/2,j} - u_{i-1/2,j}\right) + \frac{\lambda}{h}\left(v_{i,j+1/2} - v_{i,j-1/2}\right)$$

$$(\sigma_{xy})_n = \frac{\mu}{h}\left(u_{i+1/2,j+1} - u_{i+1/2,j} + v_{i+1,j+1/2} - v_{i,j+1/2}\right)$$

$$(\sigma_{xy})_s = \frac{\mu}{h}\left(u_{i+\frac{1}{2},j} - u_{i+\frac{1}{2},j-1} + v_{i+1,j-\frac{1}{2}} - v_{i,j-1/2}\right)$$

For the pressure, the backward Euler scheme is considered for the time-dependent term:

$$\frac{1}{h\tau}\left(u^m_{i+1/2,j} - u^m_{i-1/2,j} + v^m_{i,j+1/2} - v^m_{i,j-1/2}\right) - \frac{1}{h\tau}\left(u^{m-1}_{i+1/2,j} - u^{m-1}_{i-1/2,j} + v^{m-1}_{i,j+1/2} - v^{m-1}_{i,j-1/2}\right) - \frac{k}{h^2}\left(p^m_{i+1,j} + p_{i-1,j} + p_{i,j+1} + p_{i,j-1} - 4p_{ij}\right) = f$$

Multigrid:

- Fixed-stress split method: (Slides).
- Vanka (the same than Darcy).
- Uzawa $\quad w = \dfrac{h^2(\lambda+2\mu)}{5k\tau(\lambda+2\mu)+h^2}$

Multipoint-flux approximation approach.

We consider the two-dimensional pressure equation obtained by combining the continuity equation ($\nabla \cdot u = f$) and Darcy's law ($u = -K\nabla p$) for steady state incompressible single phase flows in porous media:

$$-\nabla \cdot (K \nabla p) = f \text{ en } \Omega$$

The permeability tensor is full, symmetric and uniformly positive definite

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} \quad . \quad 0 < C_1 \sum_{\alpha=1}^{2} \xi_\alpha^2 \leq \sum_{\alpha,\beta=1}^{2} K_{\alpha\beta} \xi_\alpha \xi_\beta \leq C_2 \sum_{\alpha=1}^{2} \xi_\alpha^2$$

The entries of the permeability tensor K may have jump discontinuities along

Finite volume discretization.

The domain Ω is partitioned into blocks Ω_{ij} so that the discontinuities of the permeability tensor K are aligned with cell boundaries. The centers of the cells Ω_{ij} are denoted by (x_i, y_j) and the cell vertexes are the points $(x_i \pm \frac{h_1}{2}, y_j \pm \frac{1}{2}h_2)$. The mesh that will be used to approximate the pressure will include all cell centers (x_i, y_j). This mesh will be call "primary mesh" $\omega_h = \{(x_i, y_j) : \Omega_{ij}\}$. Similarly we shall use also the mesh of all cell vertexes, called often "dual mesh". The velocities will be calculated at the points $(x_i \pm \frac{h_1}{2}, y_j)$ and $(x_i, y_j \pm \frac{h_2}{2})$.

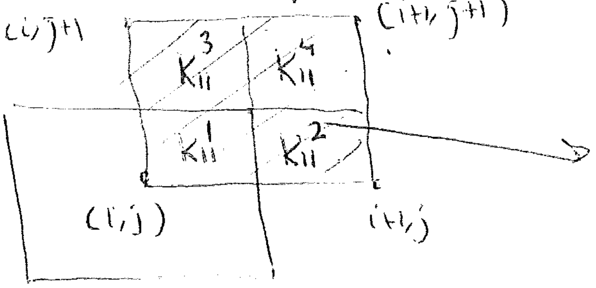
The continuity equation ($\nabla \cdot \vec{u} = f$) is integrated over control volume Ω_{ij} and making use of the divergence theorem, we obtain

$$\int_{\Omega_{ij}} \nabla \cdot \vec{u}\, dx = \int_{\Omega_{ij}} f\, dx \Rightarrow \int_{\partial\Omega_{ij}} \vec{u} \cdot \vec{n}\, ds = \int_{\Omega_{ij}} f\, dx$$

Replacing la velocidad $\vec{u}$ ~~en la fórmula anterior~~ por cierta aproximación in the previous formula involving p by using the Darcy's relation $\vec{u} = -K \cdot \nabla p$ we get a conservative method.

According to the multipoint flux approximation this is done in the following way.

We introduce a dual grid by drawing lines from each ~~cell~~ cell center to the midpoints of the cell face. The cells of the dual grid are termed "interaction volumes". The interaction volume divides the cell interfaces in two parts:



interaction volume.

Consider an internal vertex that is surrounded by four subcells with $p_{i,j}$, $p_{i+1,j+1}$, $p_{i+1,j}$, $p_{i,j+1}$ at the corners. (see figure).

We require continuity of normal components of the flux.

$$K_{11}^1 \frac{p_{i+1/2,j} - p_{ij}}{h_1/2} + K_{12}^1 \frac{p_{i,j+1/2} - p_{i,j}}{h_2/2} = K_{11}^2 \frac{p_{i+1,j} - p_{i+1/2,j}}{h_1/2} + K_{12}^2 \frac{p_{i+1,j+1/2} - p_{i+1,j}}{h_2/2} = f_1$$

$\cancel{K_{12} \, p_{i+1/2,j} - p_{i,j}}$

$$K_{11}^3 \frac{p_{i+1/2,j+1} - p_{i,j+1}}{h_1/2} + K_{12}^3 \frac{p_{i,j+1} - p_{i,j+1/2}}{h_2/2} = K_{11}^4 \frac{p_{i+1,j+1} - p_{i+1/2,j+1}}{h_1/2} + K_{12}^4 \frac{p_{i+1,j+1} - p_{i+1,j+1/2}}{h_2/2} =$$

$$K_{12}^1 \frac{p_{i+1/2,j} - p_{i,j}}{h_1/2} + K_{22}^1 \frac{p_{i,j+1/2} - p_{i,j}}{h_2/2} = K_{12}^3 \frac{p_{i+1/2,j+1} - p_{i,j+1}}{h_1/2} + K_{22}^3 \frac{p_{i,j+1} - p_{i,j+1/2}}{h_2/2} = f_3$$

$$K_{12}^2 \frac{p_{i+1,j} - p_{i+1/2,j}}{h_1/2} + K_{22}^2 \frac{p_{i+1,j+1/2} - p_{i+1,j}}{h_2/2} = K_{12}^4 \frac{p_{i+1,j+1} - p_{i+1/2,j+1}}{h_1/2} + K_{22}^4 \frac{p_{i+1,j+1} - p_{i+1,j+1/2}}{h_2/2} =$$

Simplification: $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}$ homogeneous and $h_1 = h_2$

Flux. $\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = Cv + Fu$, where:

$$V = \begin{pmatrix} p_{i+1/2,j} \\ p_{i+1/2,j+1} \\ p_{i,j+1/2} \\ p_{i+1,j+1/2} \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} p_{ij} \\ p_{i+1,j} \\ p_{i,j+1} \\ p_{i+1,j+1} \end{pmatrix}$$

$$C = \begin{pmatrix} -K_{11} & 0 & -K_{12} & 0 \\ 0 & K_{11} & 0 & K_{12} \\ 0 & -K_{12} & K_{22} & 0 \\ K_{12} & 0 & 0 & -K_{22} \end{pmatrix} \qquad F = \begin{pmatrix} K_{11}+K_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & -(K_{11}+K_{12}) \\ 0 & 0 & -(K_{22}-K_{12}) & 0 \\ 0 & K_{22}-K_{12} & 0 & 0 \end{pmatrix}$$

From page 9 we have:

$$Av = Bu \quad \text{where:}$$

$$A = \begin{pmatrix} 2K_{11} & 0 & K_{12} & -K_{12} \\ 0 & 2K_{11} & -K_{12} & K_{12} \\ K_{12} & -K_{12} & 2K_{22} & 0 \\ -K_{12} & K_{12} & 0 & 2K_{22} \end{pmatrix} ; \quad B = \begin{pmatrix} K_{11}+K_{12} & K_{11}-K_{12} & 0 & 0 \\ 0 & 0 & K_{11}-K_{12} & K_{11}+K_{12} \\ K_{22}+K_{12} & 0 & K_{22}-K_{12} & 0 \\ 0 & K_{22}-K_{12} & 0 & K_{22}+K_{12} \end{pmatrix}$$

$Av = Bu \Rightarrow v = A^{-1}Bu$ and therefore:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \left(CA^{-1}B + F \right) u = Tu$$

The expression for the flux through an entire edge in x-direction is constructed from the flux across half edge 1 in the upper interaction volume and from the flux across half edge 2 in the lower interaction volume. The flux in the x-direction reads.

$$f_i = (t_{2,1} + t_{2,3}) p_{i,j} + (t_{1,2} + t_{2,4}) p_{i+1,j} + t_{1,4} p_{i+1,j+1} + t_{1,3} p_{i,j+1}$$
$$+ t_{2,1} p_{i,j-1} + t_{2,2} p_{i+1,j-1}$$

the expression for the flux through an entire edge in y-direction is constructed from the flux across half edge 3 in the right interaction volume and from the flux across half edge 4 in the left interaction volume. The flux in y-direction:

$$f_j = (t_{3,1} + t_{4,2}) p_{i,j} \quad \text{etc.}$$

Finally, we construct the difference equation of a cell by way.

$$\vec{f}_1 + \vec{f}_2 + \vec{f}_3 - \vec{f}_4 = h^2 q_{ij}$$

and where $\vec{f}_i$ is calculated as above.