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Modified Picard with multigrid method for two-phase flow problems in rigid porous media

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Abstract

Two-phase flow problems in porous media can be found in several areas, such as Geomechanics, Hydrogeology, Engineering and Biomedicine, for example. Typically, these processes are mathematically modeled by a highly nonlinear system of coupled partial differential equations. The nonlinearity of the system makes the design and implementation of robust numerical solvers a challenging task. In this work we consider the flow of two immiscible and incompressible fluids within a non-deformable porous medium. A mixed pressure-saturation formulation is adopted, allowing the transition from the unsaturated to saturated zones and maintaining numerical mass conservation. A cell-centered finite volume method and an implicit Euler scheme are considered for the spatial and time discretization of the problem. In this work, we propose a solution method for two-phase flow problems which is based on the combination of the modified Picard linearization method and a very simple cell-centered multigrid algorithm that performs efficiently even for heterogeneous random media. This is shown in the numerical experiments, where two test problems are presented to demonstrate the robustness of the proposed solver.

KEYWORDS

coupled problem, finite volume method, heterogeneous random media, nonlinear problem

1 | INTRODUCTION

Numerical simulation of multiphase flow problems in porous media is highly important in different areas as Geomechanics, Hydrogeology, Petroleum Engineering, and Biomechanics, for example. Nuclear waste management, CO2 storage, groundwater remediation, and tumor growth are just a few examples of its applications. These problems can be mathematically modeled by a coupled system of time-dependent nonlinear partial differential equations (PDEs), including the Darcy's law, the equation of mass conservation for each phase, the constraint of the saturations, and the dependency of capillary pressure on the wetting saturation. Typically there are a number of different possibilities to select a set of primary variables for the system, resulting in different mathematical formulations for the same model. The choice of the proper formulation can strongly influence both the performance of the numerical scheme and the solver. Anyway, the representation in mathematical models becomes complex, as generate systems of highly nonlinear coupled partial differential equations. The major computational bottleneck lies in the solution of the resulting large systems of equations. Thus, numerical simulation is often difficult, resulting a great challenge to find efficient and robust solvers to enable large-scale simulations.

Various degrees of implicitness in the discretization and coupling in the nonlinear solver can be considered in the numerical solution of two-phase flow models. The fully implicit method (FIM)[1-5] and the implicit pressure explicit saturation (IMPES)[6] method are the main strategies to solve numerically this nonlinear coupled system of partial differential equations. This latter is one of the most common solution methods, despite its stability problems and need of restrictions on the size of the time step. The computation becomes remarkably simple because only one linear system for the pressure must be solved per time step. Improved versions of IMPES scheme have been presented in References 7-13. In contrast, a larger and highly nonlinear system of equations must be solved per time step if a fully implicit approach is used. These methods seem to be more reliable with respect to robustness, due to the fully implicit and fully coupled treatment of the governing equations. They avoid any restrictions on the time step, but need an efficient algorithm for solving the resulting nonlinear system. The main iterative linearization methods used for this type of nonlinear problems are the Newton method, modified Picard, L-scheme, or combinations of them.

A common approach is the use of Newton method[14,15] which has quadratic convergence, but paying the price of a costly computation of derivatives at each iteration. It is worthy to mention that the convergence of the Newton method for degenerate cases was proved rigurously in Reference 16. Picard linearization techniques are also widely used. The linearization based on the mixed form of the equation, which allows the transition from the unsaturated to saturated zones and maintains numerical mass conservation, is referred to as a "modified Picard" linearization. It was introduced by Celia in 1990, first for Richard's equation[17] and later for the two-phase flow problem in porous media.[18] A concern of Newton's and Picard's methods is the need to use a regularization term for degenerate problems which may alter the quality of the solution.[16] Finally, the L-scheme[3,5,19] is another linearization method which is extremely common due to its simplicity, since it replaces the Newton iteration by a simple fixed point iteration. Its robustness, however, comes at the price of slower convergence (generally linear convergence). The modified L-scheme, proposed in Reference 20, shows stability similar to the L-scheme while having much faster convergence rates (scaling with respect to the time step size). A comparison among Newton, modified Picard and L-scheme approaches for linearization of the surfactant transport in porous media was presented in Reference 21. With any of these linearization techniques, linear iterative schemes are required. In this paper, we will focus on how to efficiently solve these resulting linear problems.

The development of efficient iterative solvers for fully-implicit simulation of complex multiphase flow problems has been widely studied over the last decades. One of the most popular approaches in the reservoir simulation community is based on the combination of Krylov subspace methods with a constrained pressure residual (CPR) multistage preconditioning technique.[22,23] Its standard implementation[24-30] couples an algebraic multigrid (AMG) preconditioner for the usually elliptic pressure block and an incomplete factorization, that is effective for the hyperbolic part. Moreover, different variants of CPR preconditioning have been proposed.[31-33] Other alternatives, however, can be found in the literature. In Reference 34, the authors consider an approximate Jacobian construction as an alternative to the conventional Newton method, with exact Jacobian construction, as a nonlinear solver. In Reference 35, a highly parallel solver in which the resultant nonlinear system arising at each time step is solved in a monolithic way by using a Newton-Krylov-Schwarz type algorithm was proposed. Recently, different methods based on a Schur Complement Reduction have been presented.[36,37] In Reference 38, the authors consider an original block preconditioner which exploits the block structure of the Jacobian matrix while coping with the nonsymmetric nature of the individual blocks. Same authors propose in Reference 39 a novel preconditioning technique (EDFA) based on the approximation of the decoupling factors of the system matrix by using appropriate restriction operators for the sake of the Schur complement computation.

In this work, our focus is on the iterative solution of the linear systems arising in a fully implicit cell-centered finite volume discretization of a two-phase flow system, with the phases being incompressible and immiscible, and the skeleton matrix non-deformable. In particular, we consider a combination of the modified Picard iteration and a cell-centered multigrid method for the solution of the resulting linear system. Here, we would like to remark that the proposed linear solver is not restrictive to the use of the modified Picard method. The same linear solver can also be used if L-scheme or modified L-scheme are the chosen linearization techniques. In fact, in the numerical experiments section, results of the combination of L-scheme with the proposed cell-centered multigrid method are presented.

It is well known that multigrid methods[40-42] are among the fastest numerical techniques for solving the large systems of equations arising from the discretization of PDEs, and they have shown optimal complexity in solving many problems in different areas of application.[43] In the context of two-phase flow problems in porous media, multigrid methods have been applied, on the one hand, for the solution of the pressure equation within a decoupled approach (IMPES) in References 44 and 45. On the other hand, multigrid has been also applied to the fully implicit fully coupled approach in References 46-48. Although the solution of nonlinear equations by multigrid is possible with two different approaches: a linearization-multigrid approach (a global linearization, usually Newton's method, is performed and the resulting linear

equations are solved with multigrid) or nonlinear multigrid (the linearization is only done within the smoothing iteration), in Reference 48 Molenaar compared both approaches finding that the first approach is computationally more efficient. Here, we follow this latter approach and we propose the use of a basic cell-centered multigrid algorithm for solving the linear system of equations obtained after linearizing the mixed formulation of the problem by means of the modified Picard method. It is well-known that the performance of a multigrid method strongly depends on the choice of its components, and therefore they have to be carefully chosen. The proposed algorithm considers a simple Gauss-Seidel iteration as the smoother, a piecewise constant prolongation operator and its adjoint as the restriction, and a direct discretization technique to define the discrete operators on the coarse grids. We will see that such a basic multigrid algorithm converges well even in the context of random heterogeneous fields. A similar multigrid algorithm was proposed in Reference 49 for solving Richard's equation.

The rest of the paper is organized as follows. Section 2 is devoted to introduce the mathematical model together with its numerical approximation, including the linearization of the mixed formulation of the two-phase flow model and the spatial and time discretization techniques for the resulting system of PDEs. The cell-centered multigrid method proposed for the solution of the obtained linear system is described in Section 3. In Section 4, we present two numerical tests to show the robustness of the proposed solver. Finally, conclusions are drawn in Section 5.

2 | MATHEMATICAL MODEL AND NUMERICAL APPROXIMATION

The flow of two immiscible fluids in a rigid porous medium is described by two coupled partial differential equations

$$\frac{\partial(\phi\rho_\alpha S_\alpha)}{\partial t} + \nabla\cdot(\rho_\alpha \mathbf{v}_\alpha) = F_\alpha, \quad \alpha = w, n, \tag{1}$$

where $\alpha = w, n$ represents the wetting and non-wetting fluid phases respectively, ϕ is the porosity, S_α is the saturation, ρ_α is the density, $\mathbf{v}_\alpha$ is the Darcy velocity and F_α is the source term of phase α. The symbol $\nabla\cdot$ represents the divergence operator. The Darcy velocity is given by the generalized Darcy's Law for the multiphase case

$$\mathbf{v}_\alpha = -\frac{k_{r\alpha}}{\mu_\alpha}\mathbf{K}\big(\nabla p_\alpha - \rho_\alpha \mathbf{g}\big), \quad \alpha = w, n, \tag{2}$$

where $k_{r\alpha} = k_{r\alpha}(S_\alpha)$ is the relative permeability, μ_α is the viscosity, $\mathbf{K}$ is the intrinsic permeability tensor, p_α is the pressure, $\mathbf{g}$ is the gravitational acceleration vector.

To close the system, we have the auxiliary relation with the capillary pressure $p_c(S_w) = p_n - p_w$ and the constraint $S_w + S_n = 1$.

Substitution of Equation (2) into Equation (1) yields the following system of partial differential equations:

$$\frac{\partial(\phi\rho_\alpha S_\alpha)}{\partial t} - \nabla\cdot\left(\rho_\alpha\frac{k_{r\alpha}}{\mu_\alpha}\mathbf{K}\nabla p_\alpha - \rho_\alpha^2\mathbf{g}\right) = F_\alpha, \quad \alpha = w, n. \tag{3}$$

To complete the formulation of the problem, functions that relate capillary pressure to saturation and relative permeability to saturation are required. Here, we consider the van Genuchten parametrization.[50,51] For the saturation, van Genuchten proposed the following analytic formula

$$S_w(p_c) = (1 + \left(\left|\frac{p_c}{p_e}\right|^n\right)^{-m}, \tag{4}$$

where p_e is a scaling parameter, and n and $m = 1 - n^{-1}$ are parameters related to the texture of the considered porous medium. For the relative permeability k_{rw}, the following analytic expression was proposed

$$k_{rw}(p_c) = S_w(p_c)^{1/2}(1 - (1 - S_w(p_c)^{1/m})^m)^2. \tag{5}$$

Different mathematical formulations for this model can be chosen. Moreover, there exist mathematical formulations using new (artificial) unknowns that have favorable mathematical properties. The selection of the proper formulation can

strongly influence the behavior of the numerical simulations and in particular of the efficiency of the solver. Therefore such choice is of primary importance. Following Reference 17, we consider the mixed formulation given in (3). Numerical methods based on the hybrid form like finite volumes are popular as they result in mass conservation schemes. The linearization based on the hybrid form of the system (3) is known in the literature as modified Picard linearization.[17] Next, we briefly describe the implicit Euler modified Picard linearization for solving (3). For simplicity we consider density constant and gravitation is neglected. Let τ be the time discretization parameter of an uniform temporal grid. The time level will be denoted by the superscript j and the iteration number within a time step is denoted by the superscript k. By simplicity we introduce the moisture content $\theta_\alpha(p_c) = \phi S_\alpha(p_c)$, and we will use the simplified notation $\theta_\alpha^{j,k} = \theta_\alpha(p_c^{j,k})$ and $K_\alpha^{j,k} = \mathbf{K}\frac{k_{r\alpha}\left(p_\alpha^{j,k}\right)}{\mu_\alpha}$. The implicit Euler scheme of system (3) is then written by the system

$$\frac{\theta_\alpha^{j+1,k+1} - \theta_\alpha^{j}}{\tau} - \nabla \cdot K_\alpha^{j+1,k} \nabla p_\alpha^{j+1,k+1} = F_\alpha. \tag{6}$$

The key idea of the modified Picard iteration is the use of a first order Taylor expansion for $\theta_\alpha^{j+1,k+1}$ with respect to p_c, i.e.,

$$\theta_\alpha^{j+1,k+1} \approx \theta_\alpha^{j+1,k} + \frac{\partial \theta_\alpha^{j+1,k}}{\partial p_c}\left(p_c^{j+1,k+1} - p_c^{j+1,k}\right), \tag{7}$$

where $\frac{\partial \theta_\alpha}{\partial p_c} = C_\alpha$ is analytically computed from (4). Substituting (7) in (6), we get

$$C_w^{j+1,k}\frac{\delta p_w^{j+1,k} - \delta p_n^{j+1,k}}{\tau} - \nabla \cdot \left[K_w^{j+1,k}\nabla\left(\delta p_w^{j+1,k}\right)\right] = \nabla \cdot \left[K_w^{j+1,k}\nabla\left(p_w^{j+1,k}\right)\right] + F_w^{j+1} - \frac{\theta_w^{j+1,k} - \theta_w^{j}}{\tau}, \tag{8}$$

$$C_w^{j+1,k}\frac{\delta p_n^{j+1,k} - \delta p_w^{j+1,k}}{\tau} - \nabla \cdot \left[K_n^{j+1,k}\nabla\left(\delta p_n^{j+1,k}\right)\right] = \nabla \cdot \left[K_n^{j+1,k}\nabla\left(p_n^{j+1,k}\right)\right] + F_n^{j+1} - \frac{\theta_n^{j+1,k} - \theta_n^{j}}{\tau}, \tag{9}$$

where $\delta p_\alpha^{j+1,k} = p_\alpha^{j+1,k+1} - p_\alpha^{j+1,k}$.

As a result, at time t^{j+1}, a coupled system of PDEs with variables $\delta p_n^{j+1,k}$ and $\delta p_w^{j+1,k}$ has to be solved. With this purpose, we consider a cell-centered finite volume method (FVM) for spatial discretization, which is known for its mass conservation property. For simplicity, we consider a uniform partitioning of the domain $\Omega = (0, L_x) \times (0, L_y)$ into equally sized cells, that is

$$\Omega_h = \left\{(x_i, y_j) \mid x_i = (i - 1/2)h_x,\ i = 1, \ldots, N_x,\ y_j = (j - 1/2)h_y,\ j = 1, \ldots, N_y\right\},$$

with N_x and N_y being the number of volumes in space in the $x-$ and $y-$directions, respectively, $h_x = \frac{L_x}{N_x}$ and $h_y = \frac{L_y}{N_y}$. Thus, for each interior cell with center (x_i, y_j), we obtain two equations associated to variables $\delta p_\alpha^{j+1,k}$. For simplicity in the presentation, we show below the equation corresponding to the wetting phase ($\alpha = w$), that is,

$$c_{i,j}^{h,w}\delta\widetilde{p}_{i,j}^{w} + w_{i,j}^{h,w}\delta\widetilde{p}_{i-1,j}^{w} + e_{i,j}^{h,w}\delta\widetilde{p}_{i+1,j}^{w} + s_{i,j}^{h,w}\delta\widetilde{p}_{i,j-1}^{w} + n_{i,j}^{h,w}\delta\widetilde{p}_{i,j+1}^{w} + c_{i,j}^{h,n}\delta\widetilde{p}_{i,j}^{n} = \widetilde{f}_{i,j}^{h,w}, \tag{10}$$

with $\delta\widetilde{p}^{\alpha}$ denoting $\delta p_\alpha^{j+1,k}$, and

$$w_{i,j}^{h,w} = -\frac{2\tau}{h_x h_y}\frac{\widetilde{K}_{i,j}^{w}\widetilde{K}_{i-1,j}^{w}}{\widetilde{K}_{i,j}^{w} + \widetilde{K}_{i-1,j}^{w}}, \quad e_{i,j}^{h,w} = -\frac{2\tau}{h_x h_y}\frac{\widetilde{K}_{i,j}^{w}\widetilde{K}_{i+1,j}^{w}}{\widetilde{K}_{i,j}^{w} + \widetilde{K}_{i+1,j}^{w}},$$

$$s_{i,j}^{h,w} = -\frac{2\tau}{h_x h_y}\frac{\widetilde{K}_{i,j}^{w}\widetilde{K}_{i,j-1}^{w}}{\widetilde{K}_{i,j}^{w} + \widetilde{K}_{i,j-1}^{w}}, \quad n_{i,j}^{h,w} = -\frac{2\tau}{h_x h_y}\frac{\widetilde{K}_{i,j}^{w}\widetilde{K}_{i,j+1}^{w}}{\widetilde{K}_{i,j}^{w} + \widetilde{K}_{i,j+1}^{w}},$$

$$c_{i,j}^{h,w} = -\left(w_{i,j}^{h,w} + e_{i,j}^{h,w} + n_{i,j}^{h,w} + s_{i,j}^{h,w}\right) + \widetilde{C}_{i,j}^{w}, \quad c_{i,j}^{h,n} = -\widetilde{C}_{i,j}^{w},$$

where $\widetilde{C}^w = C_w^{j+1,k}$ and the interblock conductivities have been computed based on the harmonic averaging of the hydraulic conductivities from the adjacent cells,[52,53] denoting $\widetilde{K}_{i,j}^w$ the conductivity $K_w^{j+1,k}$ associated with cell (i,j). Notice that the above formulae are valid if uniform meshes are used. Of course, when non-uniform grids are considered the way to compute such conductivities has to be properly adapted.

Finally, the source term in (10) is an approximation of the right-hand side in (8) in the corresponding cell (i,j). The equation corresponding to the non-wetting phase is analogous, and the previous five-point scheme is modified appropriately for cells close to the boundary.

Thus, in each time step, we must solve a linear system of the form

$$\begin{bmatrix} A_w & B \\ B & A_n \end{bmatrix} \begin{bmatrix} \delta p_w \\ \delta p_n \end{bmatrix} = \begin{bmatrix} f_w \\ f_n \end{bmatrix}, \tag{11}$$

where, A_w and A_n are five-banded symmetric and positive definite matrices of order $N_x \times N_y$, and B is a diagonal matrix of the same size.

3 | LINEAR SOLVER: CELL-CENTERED MULTIGRID METHOD

To solve the linear system arising from the above discretization of the modified Picard iteration, we consider a geometric multigrid method. This algorithm is based on the two-grid cycle, which performs the following steps:

1. Apply ν_1 iterations of a classical iterative method, called smoother, on the target fine grid Ω_h (pre-smoothing step).
2. Compute the residual of the current fine grid approximation, and restrict it to the coarse grid Ω_{2h} by using a restriction operator R_h^{2h}.
3. Solve the residual equation on the coarse grid.
4. Interpolate the obtained correction to the fine grid Ω_h by using a prolongation operator P_{2h}^h and add the interpolated correction to the current fine grid approximation.
5. Apply ν_2 iterations of a classical iterative method (post-smoothing step).

The standard multigrid algorithm is obtained by applying the same algorithm in a recursive way by using a hierarchy of coarser meshes. The performance of the method depends on the choice of the components involved in the previously defined steps. In particular, we need to specify how to construct the hierarchy of grids, the coarse grid operators, the type of cycle, the intergrid transfer operators, and the smoother. Thus, in the following we explain our choices in this work.

The multigrid hierarchy is based on a standard coarsening, i.e., the ratio between the cell-width on the fine and coarse grid is two. Therefore, each coarse cell is the union of four fine cells, and, since the unknowns are located in the cell centers, this results in a nonnested hierarchy of grids. As the smoothing method, a collective lexicographic Gauss-Seidel relaxation scheme is used, i.e., at each cell the two associated equations are solved simultaneously. This means that a small linear 2×2 system of equations has to be solved at each cell. A simple piecewise constant interpolator P_{2h}^h is chosen as prolongation operator. In classical stencil notation, it is given by

$$P_{2h}^h = \begin{bmatrix} 1 & & 1 \\ & \star & \\ 1 & & 1 \end{bmatrix}_{2h}^{h}, \tag{12}$$

where $\star$ denotes the position of a coarse cell center. The classical stencil notation shows the contribution of the coarse grid node to the neighboring fine grid nodes. The restriction operator R_h^{2h} is chosen as the adjoint of the prolongation, given in stencil form by

$$R_h^{2h} = \frac{1}{4} \begin{bmatrix} 1 & & 1 \\ & \star & \\ 1 & & 1 \end{bmatrix}_{h}^{2h}. \tag{13}$$

The coarse grid operators are obtained by direct discretizations defining the diffusion coefficients at the edges of the coarse cells appropriately. As explained above, the flux over an edge, dependent on the solution in the two adjacent cells, is calculated based on the harmonic average. However, the diffusion coefficient at coarse edges, is calculated as the arithmetic average of the corresponding fine grid coefficients. As it was pointed in References 53 and 54, this direct discretization procedure is equivalent to the Galerkin approach, i.e., $L_{2h} = \frac{1}{2} R_h^{2h} L_h P_{2h}^h$, but computationally more efficient. The factor $1/2$ in the previous expression is due to the lack of consistency of the operator $R_h^{2h} L_h P_{2h}^h$ with the differential operator.[55] We have chosen a $W(2,2)-$cycle since it was found in References 54 and 49 to be a very robust and efficient multigrid cycling strategy. The stopping criterion for the multigrid solver is set to reduce the initial residual by a factor TOL_{MG}, i.e. $\|R^m\|_\infty / \|R^0\|_\infty \leqslant TOL_{MG}$, where R^0 and R^m are the initial residual and the residual in the iteration m respectively.

Remark 1. Notice that a similar multigrid approach can be extended to different types of grids in order to deal with more complex configurations of the problem. As done in Reference 56 by some of the authors for other problem, logically rectangular grids can be considered to take advantage of the recent trends in computer architectures: many-core and accelerated architectures that achieve their best performance when structured data can be used. In addition, the geometric multigrid method can be easily extended to semi-structured triangular grids in order to deal with more complex domains. The idea would be to construct an unstructured coarse triangulation in such a way that the grid fits the irregular structure of the domain, and then to apply a regular refinement to the input triangles in order to obtain a totally unstructured grid composed of structured patches where the geometric multigrid method can be easily applied (see References 57-59).

4 | NUMERICAL EXPERIMENTS

In this section, we consider two different numerical experiments to test the behavior of the proposed solution method. First, we consider a problem with an analytical solution to test our solver, and later, a more realistic problem in order to show its robustness and good behavior.

In the following, we will consider a stopping criterion for the linerization method with a tolerance of $TOL_{Lin} = 10^{-9}$. Within the multigrid iterative procedure, the stopping criterion will be to reduce the initial residual in a factor of $TOL_{MG} = 10^{-5}$. Moreover, the number of levels employed in the multigrid strategy is the highest possible. We consider standard coarsening ratio, i.e. coarsening ratio equal 2, so the grids need to be created hierarchically as power of 2, in which the coarsest mesh is composed of 2×2 volumes.

The proposed solver has been implemented in our in-house Matlab code, and the numerical computations have been carried out on a computer with Intel Core i7 2.6 GHz processor, 8 GB RAM, running Windows 10, and using double precision arithmetic.

4.1 | Numerical test with analytical solution

First, we consider a test problem with analytical solution proposed by Kvashchuk in Reference 60. In that work, Kvashchuk considered the averaged pressure formulation, with $\overline{p} - S_w$ being the primary variables, where $\overline{p} = \frac{p_w + p_n}{2}$. In that case, they choose the right-hand side functions such that the following analytical solution is considered:

$$\overline{p}(\mathbf{x}, t) = xt(1-x)y(1-y),$$

$$S_w(\mathbf{x}, t) = \frac{1}{2} + xt(1-x)y(1-y),$$

in the spatial-temporal domain $D = (0,1) \times (0,1) \times [0,T]$, with $T = 1$. Since we consider the modified Picard linearization of the mixed formulation of the problem, the system to solve is given in terms of the two pressures p_w–p_n, and then we have to make some adaptations to use p_w and p_n instead of $\overline{p}$. For this, using the capillary pressure equation ($p_c = p_n - p_w$) and $\overline{p}$, we obtain that

$$p_w(\mathbf{x}, t) = \overline{p} - \frac{p_c(S_w)}{2} \tag{14}$$

and

$$p_n(\mathbf{x}, t) = \overline{p} + \frac{p_c(S_w)}{2}, \tag{15}$$

with initial and Dirichlet boundary conditions given by the analytical solutions (Equations 14 and 15).

As in Reference 60, we choose van Genuchten parametrization for the dependence of the relative permeabilities (16) and the capillary pressure (17), that is,

$$\begin{aligned} k_{r,w}(S_w) &= \sqrt{S_w}\left(1 - \left(1 - S_w^{1/m}\right)^m\right)^2, \\ k_{r,n}(S_w) &= \sqrt{1 - S_w}\left(1 - S_w^{1/m}\right)^{2m}, \end{aligned} \tag{16}$$

$$p_c(S_w) = p_e\left(S_w^{-1/m} - 1\right)^{1/n}, \tag{17}$$

where $m = 1 - 1/n$, n and p_e are the van Genuchten parameters, which are chosen as $n = 2$ and $p_e = 2$ MPa. The rest of the values for the parameters are $K = 1(m^2)$, $\phi = 1$ and $\mu_\alpha = 1$ (Pa s).

First, we compute the errors between the analytical and the numerical solution for different values of the discretization steps in time (τ) and in space ($h = h_x = h_y$). In Table 1, we show the L_2-norm errors for the saturation and both pressure unknowns $\left(E^i_{S_w}, E^i_{p_w}\right.$ and $\left.E^i_{p_n}\right.$, respectively$\left.\right)$, together with the corresponding error reduction $\left(E^{i-1}_\gamma / E^i_\gamma\right.$, being γ the corresponding variable$\left.\right)$. It can be seen that second order of convergence is obtained in contrast to the expected theoretical first order. Similar results were observed in Reference 60. This behavior can be due to the simple problem considered.

Regarding the convergence of the proposed solver, in Table 2 we show, for different values of the discretization parameters ($h = \tau$), the number of multigrid cycles in the last Picard iteration of the last time step (it_{MG}), the arithmetic mean of the number of multigrid cycles ($\overline{it}_{MG}$), the number of linearization iterations in the last time step (it_{Lin}) and the arithmetic mean of the number of linearization iterations ($\overline{it}_{Lin}$). We can observe that the number of modified Picard linearization iterations that we need to perform is small in all the cases, with modified Picard being better, as it converges with less linearizations, giving rise to a robust algorithm independently of the values of the discretization parameters. Moreover, for sufficiently small values of h and τ, the number of multigrid iterations is always a small number. Thus, we notice that in all the cases, the proposed solver for this specific problem proves to be robust with respect to the discretization parameters. As mentioned in the introduction, the proposed linear solver can also be applied to other linearization techniques

TABLE 1 L_2-norm errors of pressures and saturation functions for different time steps and mesh sizes, $T = 1$.

i	$h = \tau$	$E^i_{p_w}$	$E^i_{p_n}$	$E^i_{S_w}$	$E^{i-1}_{p_w}/E^i_{p_w}$	$E^{i-1}_{p_n}/E^i_{p_n}$	$E^{i-1}_{S_w}/E^i_{S_w}$
1	0.1	1.261e−03	2.755e−03	1.883e−04			
2	0.05	3.276e−04	6.995e−04	4.726e−05	3.8483	3.9389	3.9847
3	2.5e−02	8.274e−05	1.756e−04	1.183e−05	3.9599	3.9843	3.9943
4	1.25e−02	2.073e−05	4.393e−05	2.960e−06	3.9903	3.9960	3.9980
5	6.25e−03	5.186e−06	1.099e−05	7.400e−07	3.9979	3.9990	3.9992

TABLE 2 Number of iterations for multigrid and for the modified Picard and L-scheme linearization method for different values of the discretization parameters.

	Modified Picard–MG				L-scheme–MG			
$h = \tau$	it_{MG}	$\overline{it}_{MG}$	it_{Lin}	$\overline{it}_{Lin}$	it_{MG}	$\overline{it}_{MG}$	it_{Lin}	$\overline{it}_{Lin}$
0.1	34	50.22	6	5.7	46	52.03	7	8.3
0.05	20	20.27	6	5.25	20	20.19	8	8.8
2.5e−02	8	9.63	5	4.78	10	9.58	8	8.8
1.25e−02	5	5.82	5	4.51	5	5.6	8	8.74
6.25e−03	4	4.00	5	4.03	4	4	8	8.54

as the L-scheme. Thus, In Table 2, we also show similar results for the combination of the L-scheme with the proposed cell-centered multigrid method. As it can be observed, the proposed solver is robust with respect to the discretization parameters if sufficiently small values of h and τ are considered.

4.2 | Immiscible two-phase flow in a heterogeneous media

We consider a problem without analytical solution in the spatial-temporal domain $(0,1)\times(0,8)\times[0,T]$, where $T=1$. The source terms in Equation (3) are considered equal to zero, that is, $F_w = F_n = 0$. We assume the following initial conditions: $p_w(\mathbf{x},0) = -p_c/2$, $p_n(\mathbf{x},0) = p_c/2$ and $S_w(\mathbf{x},0) = 0.5$, and boundary conditions:

$$p_n(\mathbf{x},t) = -p_w(\mathbf{x},t) = -\frac{p_c}{2}, \text{ for } \quad x=0, x=1, y=0, \tag{18}$$

$$\frac{\partial p_w(\mathbf{x},t)}{\partial x} = -1, \text{ for } \quad y=8, \tag{19}$$

$$\frac{\partial p_n(\mathbf{x},t)}{\partial x} = 0, \text{ for } \quad y=8. \tag{20}$$

We use again Van Genuchten parametrization for the dependence of the capillary pressure in Equation (17), with parameters $n = p_e = 2$. In this experiments, we want to test the robustness of the proposed solver in heterogeneous random media. Thus, relative permeabilities are modeled as a random field.[61] As an example, in Figure 1, we present a possible sample of the permeability field k_{rw}. We consider permeabilities of the order of 10^{-5} to 10^{-8}, which consist of characteristic values for very fine sands, silts and clay-silt laminate (desiccated and fissured clays).[62] The values for the rest of the parameters are $K = 10^{-4}$ for the absolute permeability, and for the porosity and viscosity $\phi = \mu_\alpha = 1$.

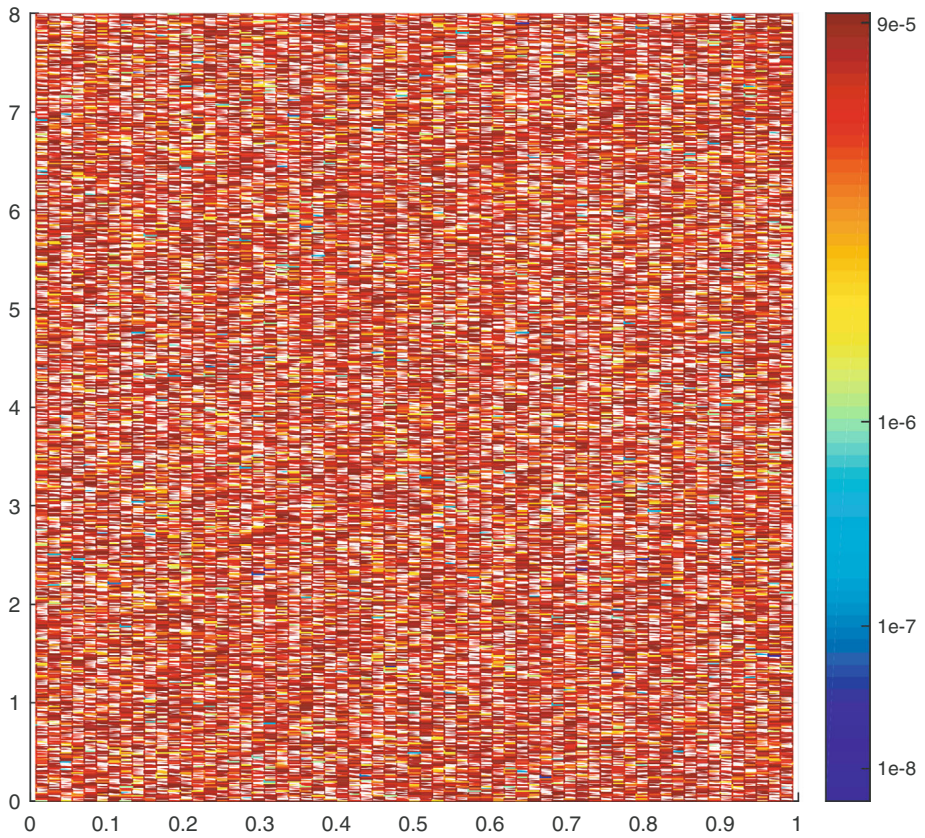

FIGURE 1 Permeability field k_{rw}, for $N_x \times N_y = 64 \times 512$.

We solve this test problem by considering the proposed modified Picard linearization method combined with multigrid. In Table 3, we show, for different target grids, the number of multigrid cycles in the last linearization iteration of the last time step (it_{MG}), the arithmetic mean of multigrid cycles ($\overline{it}_{MG}$), the number of modified Picard linearization iterations in the last time step (it_{Lin}) and the arithmetic mean of these latter ($\overline{it}_{Lin}$). Also, in the last column of the table, we display the arithmetic mean of the multigrid asymptotic convergence factor ρ_M, computed as follows,[63]

$$\rho_{\mathbf{M}} = \frac{1}{it_{Lin}} \sum_{j=1}^{it_{Lin}} \rho_m^j, \text{ where } \rho_m^j = \sqrt[it_{MG}]{\frac{\|R^{it_{MG}}\|_\infty}{\|R^0\|_\infty}}. \tag{21}$$

As we can observe, the proposed solver needs only a few linearization iterations to achieve the desired stopping criterion, as well as a small number of multigrid iteration in each linearization step. Notice that the presented results are robust with respect to the discretization parameters, too.

In order to demonstrate the robustness of the solver with respect to the discretization parameters, next, different refinement strategies for τ and h are considered. In Tables 4 and 5 we present the corresponding results for a fixed value of τ and different refinements of the spatial grid, that is for different values of h.

In Table 6, on the contrary, we fix the spatial grid such that $h = 0.016$ and different numbers of time steps are considered. In all the cases, we can observe that the iterations remain robust with respect to the mesh refinement and the time step, showing the robustness of the iterative method with respect to the discretization parameters.

TABLE 3 Number of iterations for multigrid and for the modified Picard linearization method, together with multigrid asymptotic convergence factors, for different values of the discretization parameters.

$N_x \times N_y$	N_t	$h = \tau$	it_{MG}	$\overline{it}_{MG}$	it_{Lin}	$\overline{it}_{Lin}$	$\rho_{\mathbf{M}}$
4×32	4	0.25	4	3.25	4	4	0.0271
8×64	8	0.125	4	3	4	4	0.0155
16×128	16	0.063	4	3	4	4	0.0257
32×256	32	0.031	4	3	4	4	0.0224
64×512	64	0.016	4	3.25	4	4	0.0212

TABLE 4 Number of iterations for multigrid and for the modified Picard linearization method, together with multigrid asymptotic convergence factors, for $\tau = 0.016$ and different values of the discretization parameters.

$N_x \times N_y$	N_t	h	it_{MG}	$\overline{it}_{MG}$	it_{Lin}	$\overline{it}_{Lin}$	$\rho_{\mathbf{M}}$
4×32	64	0.25	4	3.25	4	4	0.0311
8×64	64	0.125	4	2.67	3	3	0.0157
16×128	64	0.063	4	3	4	4	0.0247
32×256	64	0.031	4	3	4	4	0.0206
64×512	64	0.016	4	3.25	4	4	0.0212

TABLE 5 Number of iterations for multigrid and for the modified Picard linearization method, together with multigrid asymptotic convergence factors, for $\tau = 0.125$ and different values of the discretization parameters.

$N_x \times N_y$	N_t	h	it_{MG}	$\overline{it}_{MG}$	it_{Lin}	$\overline{it}_{Lin}$	$\rho_{\mathbf{M}}$
4×32	8	0.25	4	3.25	4	4	0.0283
8×64	8	0.125	4	3	4	4	0.0155
16×128	8	0.063	4	3	4	4	0.0262
32×256	8	0.031	4	3.25	4	4	0.0173
64×512	8	0.016	4	3.50	4	4.13	0.0245

TABLE 6 Number of iterations for multigrid and for the modified Picard linearization method, together with multigrid asymptotic convergence factors, for $h = 0.016$ and different values of the discretization parameters.

$N_x \times N_y$	N_t	τ	it_{MG}	$\overline{it}_{MG}$	it_{Lin}	$\overline{it}_{Lin}$	ρ_M
64×512	4	0.25	4	3.80	5	5	0.0308
64×512	8	0.125	4	3.50	4	4.13	0.0245
64×512	16	0.063	4	3.25	4	4	0.0220
64×512	32	0.031	4	3.25	4	4	0.0205
64×512	64	0.016	4	3.25	4	4	0.0212

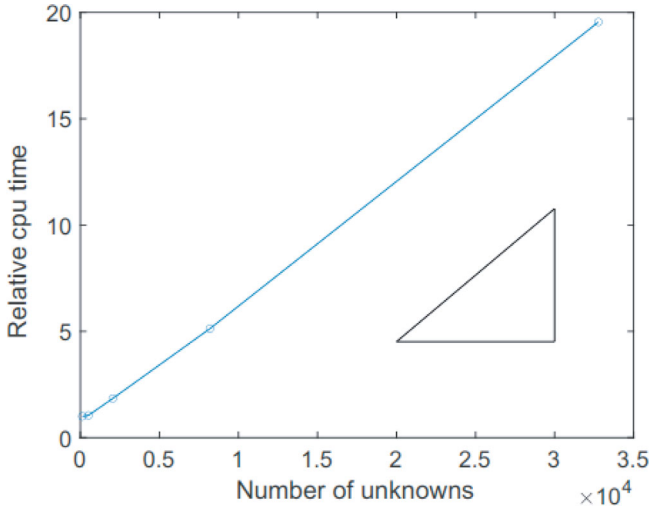

FIGURE 2 Relative cpu time in terms of the number of unknowns.

Finally, to show the optimality of the proposed linear solver, in Figure 2 we display the relative cpu time in terms of the number of unknowns. We can see from the picture that the computational cost of the linear solver is close to $\mathcal{O}(N)$, being N the number of unknowns. All these results show the robustness and efficiency of the proposed solution method.

5 | CONCLUSION

In this work, we have proposed a solution method for the mixed formulation of the two-phase flow model in rigid porous media. The solver combines the modified Picard linearization method of this formulation with a simple cell-centered multigrid algorithm for the resulting linear problem, which provides a good convergence even in the context of random fields. The proposed linear solver can be also combined with other linearization methods as the L-scheme, providing an efficient solver as well. Two numerical tests are presented showing the robustness and efficiency of the proposed solver for problems even with heterogeneous random media. A small number of iterations of both the linearization and the linear solution methods are enough to achieve convergence in all the cases. The proposed solver provides a robust solution method with respect to both spatial and temporal discretization parameters, leading to a robust solver for two-phase flow models. We can conclude that the combination of the modified Picard linearization method and the cell-centered multigrid algorithm provides a robust and efficient method for this type of problems. Although the results provided in this work are shown for relatively simple problems with the aim of carefully testing mainly the behavior of the linear solver, the proposed techniques can be extended to more complex and challenging situations. In order to deal with more complex configurations of the problem, the proposed multigrid approach can be extended to different types of grids, as logically rectangular grids or even semi-structured triangular grids, so that the method can deal with more complex domains. This extension is a topic for future research, as well as the extension of the method to deal with more realistic numerical experiments, including degenerate cases, and multiphase flow in deformable porous media.

AUTHOR CONTRIBUTIONS

Michely L. de Oliveira works with multigrid, linearization methods, space and time discretization. Marcio A. V. Pinto works with multigrid and porous media. Carmen Rodrigo has experience in multigrid and multiphase flow. Francisco J. Gaspar has experience in multigrid, porous media and coupled problem.

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CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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