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A NEW MULTIGRID APPROACH USING SIMPLEC PRESSURE-VELOCITY COUPLING FOR INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Jean Michael Borges de Oliveira
Postgraduate Program in Numerical Methods in Engineering - UFPR - Curitiba, PR, Brazil
jmboliveira86@gmail.com
Luciano Kiyoshi Araki
Department of Mechanical Engineering - UFPR - Curitiba, PR, Brazil
lucaraki@ufpr.br
Marcio Augusto Villela Pinto
Department of Mechanical Engineering - UFPR - Curitiba, PR, Brazil
marcio_villela@ufpr.br
Simone de Fátima Tomazzoni Gonçalves
Department of Mechanical Engineering - UFPR - Curitiba, PR, Brazil
simone.tg@ufpr.br

Abstract. *This work reports on the analysis of a multigrid method applied to the solution of the incompressible Navier-Stokes equations, modelling the flow in the lid-driven square cavity. The mathematical model is discretized by the Finite Volume Method, with second order approximation schemes, uniform non-staggered grids, pressure-velocity coupling by the SIMPLEC algorithm and Gauss-Seidel as main solver. In order to accelerate (speed-up) the convergence, a Full Approximation Scheme (FAS) multigrid method, with V-cycles, is applied on the SIMPLEC iterations, by a new developed scheme. It consists of keeping on the SIMPLEC algorithm but the pressure-correction and momentum equations are solved by independent V-cycles. The performances of the new scheme and the standard multigrid SIMPLEC approaches are compared. The efficiency and the accuracy of the proposed scheme are also assessed by residual decayments and comparisons with results from literature, respectively. Tests are performed for different grid refinements and for different Reynolds numbers. Finally, the performance of the new developed scheme is compared with a standard scheme from literature.*

Keywords: *Multigrid, Navier-Stokes, V-cycle, Finite Volume, SIMPLEC.*

1. INTRODUCTION

Due to its complexity and importance in Computational Fluid Dynamics (CFD), in the last three decades, the Navier-Stokes equations, which model fluid flows, have been the focus of work of several researchers, who mainly seek accurate numerical solutions through computational codes. However, using traditional iterative methods (also called solvers) in conjunction with conventional pressure-velocity coupling methods (such as those from the SIMPLE family) in order to solve problems on high resolution meshes and for a varied range of Reynolds numbers often leads to slow convergence and high computational times.

Multigrid methods have been widely used in CFD to improve performance of numerical models used for the solution of flow problems involving partial differential equations (PDEs). Thus, many authors have obtained substantial reduction in convergence rates and, by consequence, in computational times by use of this class of methods. Among these, the works of Ghia *et al.* (1982), Ferziger *et al.* (1989), Zeng and Wesseling (1994) and Yan *et al.* (2007) stand out. Multigrid methods have optimal efficiency when applied to elliptic PDEs, however, for non-ellipic problems, in which the Navier-Stokes equations are included, obtention of theoretical optimal efficiency still represents a great challenge (Santiago *et al.*, 2015).

In multigrid methods, a set of coarser grids is used in order to eliminate the components (or Fourier modes) of the errors associated with the numerical solutions (Trottenberg *et al.*, 2001). The way in which grids with different spacings are crossed is called a cycle. Amongst several types of existing cycles (Briggs *et al.*, 2000), the V-cycle is the one that provides the best representation of the multigrid idea and, therefore, is perhaps the most known and used. For non-linear (and non-elliptic) problems, the best multigrid performance is obtained through the full approximation scheme (FAS) and the full multigrid (FMG) algorithm, FAS-FMG (Trottenberg *et al.*, 2001). When only V-cycles are used, worse performances are obtained: the convergence generally takes $O(10^2)$ (sometimes $O(10^3)$) cycles; the speedups are generally $O(10^1)$ and the computational times do not increase linearly with the size of the problem. Examples of these cases can be found in Yuan (2002); Darwish *et al.* (2004) and Kumar *et al.* (2009). Still, as the V-cycle is, perhaps, the most used basis for the FMG algorihtm, its study is worthy.

In this study, a new form of applying V-cycles (and therefore multigrid) to the SIMPLE family of methods is presented. In this scheme of application the SIMPLE-like algorithms structure is maintained, but the momentum equations and the pressure-correction equations are solved by means of individual V-cycles. This form contrasts with the traditional ones, where the same V-cycle is used for all the equations and, that way, coarse-grid representations of the initial fine-grid equations are solved on intermediate meshes (Ferziger *et al.*, 1989). Thus, in this new proposed way, the theoretical form of multigrid can be applied on the equations, that is, the right hand sides of equations on the coarser grids receive the restricted information of either the residual alone (correction scheme - CS) or the residual and solution (FAS) from the finer grid Briggs *et al.* (2000).

The lid-driven square cavity flow problem (Rubin and Khosla, 1981; Ghia *et al.*, 1982; Vanka, 1986) is used for the comparison and verification tests. This problem has great importance in CFD and has been used by many researchers for testings and validation of their new methods and techniques due to its simplicity and the richness of fluid flows phenomena it contains, including the formation of sub-vortexes against the main vortex in the corners of the cavity (Erturk, 2009).

The text of the present study is divided as follows: in section 2, the methods and techniques employed in the construction of the numerical model are presented; in section 3 the results obtained can be analised and the performance of the new developed model can be assessed; finally, in section 4, a conclusion of this work is done.

2. METHODOLOGY

Considering the two-dimensional permanent flow of an incompressible fluid without heat transfer in the unity square cavity, the non-dimensional conservations of momentum and mass, representing the Navier-Stokes equations, can be written as follows (Greenberg, 1998):

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) = \frac{1}{\text{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial p}{\partial x} \tag{1}$$

$$\frac{\partial}{\partial x}(v^2) + \frac{\partial}{\partial y}(uv) = \frac{1}{\text{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial p}{\partial y} \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

where x, y are the cartesian coordinates, u, v are the horizontal and vertical components of the velocity, respectively, p is the pressure and Re is the Reynolds number. The classic problem in the cavity considers that the values of all variables are null on all the boundaries of the cavity, except by the unitary value of the component u at the top lid. Thus, the boundary conditions are given by :

$$u(0, y) = u(1, y) = u(x, 0) = 0 \tag{4a}$$

$$u(x, 1) = 1, \tag{4b}$$

$$v(0, y) = v(1, y) = v(x, 0) = v(x, 1) = 0 \tag{4c}$$

$$p(0, y) = p(1, y) = p(x, 0) = p(x, 1) = 0 \tag{4d}$$

2.1 NUMERICAL MODEL

As was mentioned earlier, in this work, the finite volume method (FVM) (Maliska, 2004) is used for the discretization of the mathematical model. In this method, the discrete version of the initial domain (grid) is divided into control volumes such that each grid point is surrounded by only one volume and there is no overlapping between volumes. Figure 1 illustrates typical control volumes surrounding the central point P and its neighbors. The grid adopted here has the same features of the figure, this is, a uniform grid with colocated arrangement (non-staggered) is used, and the nodal points are located in the exact center of each volume. The colocated arrangement has advantages over the staggered one when a multigrid method is being used, which includes the fact that only one set of equations are needed to be refined and coarsened (Ferziger *et al.*, 1989).

After the discretization of the domain, the governing equations are integrated over each control volume and, by means of the Gaussian divergence theorem (Greenberg, 1998), the integrals of the space derivatives are evaluated through corresponding fluxes across the volumes faces. These fluxes, in turn, are approximated by discrete differences involving neighbor points. In this work, the diffusive fluxes are approximated by the central difference scheme of second order, and the convective fluxes are approximated by the upwind scheme, of first order, with deferred correction in order to obtain

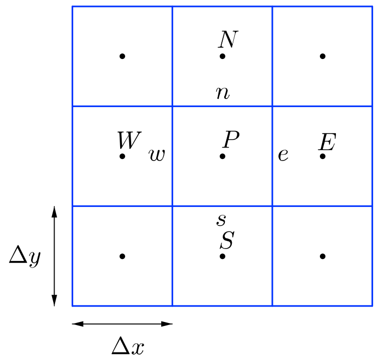

Figure 1: Typical control volumes surrounding the grid points.

second order approximations. For more information about these schemes, Ferziger and Perić (2002) can be consulted. After the integration over the control volumes, the conservation of mass (or continuity equation), Eq. (3), is transformed in a pressure-correction equation by the use of the SIMPLEC method (Doormal and Raithby, 1984) for pressure-velocity coupling. Thus, the variables of the equations are now the velocity components u, v and the pressure correction p'.

The discretized equations for u and v can be written as:

$$A_P^V u_P = \Sigma A_{nb}^V u_{nb} - \frac{p_E - p_W}{2}\Delta y + S^u \tag{5}$$

$$A_P^V v_P = \Sigma A_{nb}^V v_{nb} - \frac{p_N - p_S}{2}\Delta x + S^v \tag{6}$$

where the index nb runs over the neighbor points W, S, E and N. The coefficients A^V are the same for both velocity components u and v and can be checked in Maliska (2004). The pressure differences in the two coordinate directions come from the pressure spacial derivative approximations on the point P (Fig. 1). The source terms S^u and S^v contain the terms from the deferred corretion. As can be seen from Fig. 1, Δx and Δy are the lenghts of the volumes in the corresponding coordinate directions.

Similarly to the momentum equations, the discretized pressure correction equation can be written as:

$$A_P^p p'_P = \Sigma A_{nb}^p p'_{nb} + S^p \tag{7}$$

where the coefficients A^p can be found in Ferziger and Perić (2002) and Maliska (2004). The source term S^p contains the mass imbalance $\dot{m}$ of the velocity components obtained from the solution of the momentum equations, therefore:

$$\begin{aligned} S^p = \dot{m} &= -\left(\rho u_e \Delta y - \rho u_w \Delta y + \rho v_n - \rho u_s \Delta x\right) \\ &= -\left(\dot{m}_e - \dot{m}_w + \dot{m}_n - \dot{m}_s\right) \end{aligned} \tag{8}$$

where $\dot{m}_f$, $f = w, e, s, n$, are the mass fluxes at the control volume faces, ρ is the constant density of the fluid. According to the SIMPLE family of methods, after solving the momentum equations, Eqs. (5) and (6), the new velocity field, that will be indicated by u^* and v^*, will not, in general, satisfy the continuity equation. Therefore, corrections u', v' and p' are proposed in a way that the new velocity field will be divergent free (mass imbalance equal zero) and thus convergence of the numerical model can be measured in terms of the mass imbalance decayment. The new velocity and pressure fields, satisfying the continuity condition, will then be given by:

$$u = u^* + u' \tag{9a}$$

$$v = v^* + v' \tag{9b}$$

$$p = p^* + p' \tag{9c}$$

In the SIMPLEC method, the relations between the velocity components and the pressure-correction are given by:

$$u_P = u_P^* - d_u \frac{(p'_E - p'_W)}{2} \tag{10}$$

$$v_P = v_P^* - d_v \frac{(p'_N - p'_S)}{2} \tag{11}$$

where $d_u = \Delta y/(A_P^V - A_{nb}^V)$ and $d_v = \Delta x/(A_P^V - A_{nb}^V)$ are called SIMPLEC coefficients. Thus, the velocity components are corrected by these expressions, after the pressure-corrections are calculated. The face velocity components are corrected by similar expressions. The face velocity components needed in the pressure-correction equations can be calculated by interpolations involving the momentum equations, Eqs. (5) and (6). More details can be found in Ferziger and Perić (2002). The steps of SIMPLEC algorithm can be seen in algorithm 1. In order to execute the third and last steps (inside the loop) the above mentioned references can be consulted.

Initialize the variables u, v, p e p' ;
while tolerance is not satisfied
    solve x momentum equation, Eq. (5), to obtain intermediate values u^* ;
    solve y momentum equation, Eq. (6), to obtain intermediate values v^* ;
    calculate face velocity components u_e^* e v_n^* using interpolations of Eqs. (5) and (6) ;
    solve the pressure-correction equation, Eq. (7), to obtain p' ;
    correct the pressure field with the relation (9c) ;
    correct velocity components (in nodal point and on faces) using Eqs. (9a) and (9b) and similars ;
end

Algorithm 1: SIMPLEC method for pressure-velocity coupling on Navier-Stokes equations.

2.2 MULTIGRID

Generally, the multigrid approaches used in conjunction with SIMPLE-like methods are based on the so called coarse grid equations, which are equations that are similar to the momentum equations but involve corrections for the velocity components and pressures as unknown variables instead. The solutions on all grid levels are obtained through the same structure of algorithm 1. For more pieces of information, the works Ferziger *et al.* (1989); Yan *et al.* (2007) and Kumar *et al.* (2009) can be viewed.

A survey on multigrid features, including basic schemes for linear (correction scheme - CS) and nonlinear (full approximation scheme - FAS) problems, types of cycles, coarsening strategies, transfer operators, among others, can be done by studying Hackbusch (1985); Briggs *et al.* (2000) and Trottenberg *et al.* (2001). Before presenting the scheme developed in this work, there are some points worth mentioning. The independent (or source) terms of the discretized momentum equations, Eqs. (5) and (6) are composed by the respectives pressure plus deferred correction terms. So, if only these equations were to be solved, as in the Burgers equations (Gonçalves, 2013), for example, the intermediate grids independent terms would not contain explicit pressure terms, only their influence from the initial fine grid. Thus, if the pressure field was fixed at the beginning, an independent V-cycle could be executed to solve the momentum equations.

In the case of the pressure-correction equation, the velocity components are needed in the calculation of the coefficients A^p, but, again, if the fine grid velocity field is fixed prior to solving this equation, it can be restricted for the calculation of the coefficients on the coarser grids. Thereby, an independent V-cycle could also be used for the pressure-correction equation. Furthermore, if the zero gradient boundary condition (Ferziger *et al.*, 1989) is used, this equation is linear and CS can be used instead of FAS.

The scheme developed in this work takes advantages of the above mentioned facts. In the proposed scheme, a modified V-cycle is performed to solve the momentum equations. The steps of this V-cycle can be checked in algorithm 2. First some iterations are performed on the momentum equations in order to produce an initial velocity field to be used in the pressure coefficients. With this initial field, the velocity coefficients A^V are updated because they are need in the calculation of the SIMPLEC coefficients d_u and d_v, Eqs. (10) and (11). The pressure-correction equation can now be solved if the previous fixed velocity field and coefficients, a chained V-cycle is then executed for this. After this cycle, the pressure and velocity fields are corrected and then a V-cycle is performed to solve the momentum equations this time. This process is repeated until convergence.

Initialize the variables u, v, p e p' ;
while tolerance is not satisfied
    perform a few iterations on the momentum equations, Eqs. (5) and (6), to obtain intermediate values u^* and v^* ;
    calculate face velocity components u_e^* e v_n^* using interpolations of Eqs. (5) and (6) ;
    update the velocity coefficients A^V ;
    perform a V-cycle to solve the pressure-correction equation, Eq. (7), with a fixed velocity field;
    correct the velocity components and pressures using Eqs. (10), (11) and (9c), respectively ;
    perform a V-cycle to solve the momentum equations, Eqs. (5) and (6), with a fixed pressure field;
end

Algorithm 2: Modified V-cycle for the multigrid-SIMPLEC scheme.

The cycles executed for the momentum and pressure-correction equations are standard multigrid V-cycles and can be checked in Briggs *et al.* (2000). The only difference is that for the pressure-correction cycle, the velocity components are also restricted in the restriction stage. Figure 2 shows a representation of the modified cycle. In this figure, I_h^{2h} and I_{2h}^h are the restriction and prolongation operators, respectively, and Ω^h, Ω^{2h}, Ω^{4h}, ..., are the sequence of grids traveled by the restriction and prolongation stages. Ω^h is the finest grid, Ω^{2h} is the imediate coarser grid and so on. The smaller green and brown V's in this figure represent the V-cycle performed for the resolution of the pressure-correction equation, Eq. 7. The transfer operators used are the same for both cycles.

The features of the multigrid method used in this work are:

- the restriction operators employed for the variables and residuals are the mean and the sum of correspondent fine volumes (Ferziger and Perić, 2002), respectively;
- the bilinear prolongation operator (Gonçalves, 2013) is used for all variables;
- the standard coarsening ratio (Briggs *et al.*, 2000) 2 is adopted;
- the maximum number of levels (Gonçalves, 2013) is used in the V-cycles, being the coarsest grid with two volumes in each coordinate direction;
- the number of iterations that each inner solver executes always varies between one or two in both restriction and prolongation stages;
- the outer loop is iterated until all residuals are smaller than 10^{-8}, the criterion by which this tolerance is adopted can be checked in Gonçalves (2013);
- all the equations in all the cases were solved using the red-black Gauss-Seidel solver (Zhang, 1996).

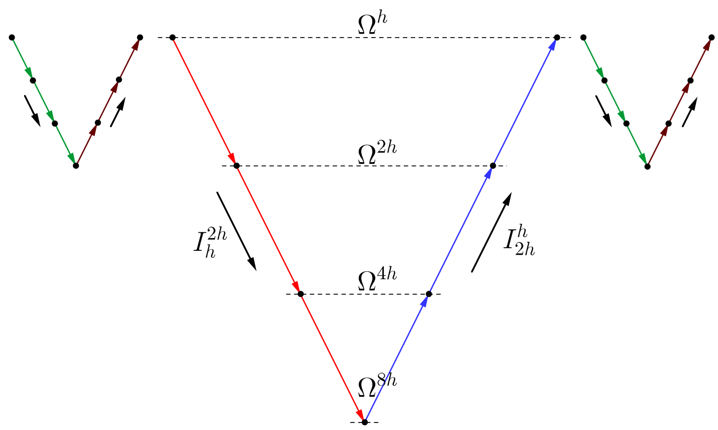

Figure 2: Representation of the modified V-cycle for the multigrid-SIMPLEC scheme.

3. RESULTS

The results obtained in this work are presented in this section. First, the results regarding the quality of the solutions from the multigrid version of the model on the square cavity are presented and compared with the ones from literature. After that, the performances of the singlegrid and multigrid versions of the numerical model are compared by means of speedups and residual decayments. Finally, the performances of the new developed V-cycle and the standard V-cycle (based on the course grid versions of the momentum equations) are compared. The standard V-cycle from the work of Kumar *et al.* (2009) is used in this stage.

The solutions accuracy is evaluated through the central profiles of the velocity components, presented in Figs. 3 and 4 for values of Reynolds number. These profiles are compared with results from Marchi *et al.* (2009), which are indicated by the label "Marchi" while the results from this work are indicated only by the Reynolds numbers. It can be noted that the two works are in well agreement. Marchi *et al.* (2009) also compare their results with those from Ghia *et al.* (1982), Bruneau and Saad (2006) and Botella and Peyret (1998). Thus, the present results are also in agreement with these works. Central nodal point values of the velocity components are also compared with their respectives from Marchi *et al.* (2009) and shown in Tab. 1.

Figure 5 presents comparisons of the velocity components residual decays with the number of iterations from singlegrid and multigrid versions of the numerical model. The term "RES" in the vertical axis actually indicates the ratio between the L1-norm of the residual in the present iteration and the L1-norm of the initial (first iteration) residual. While

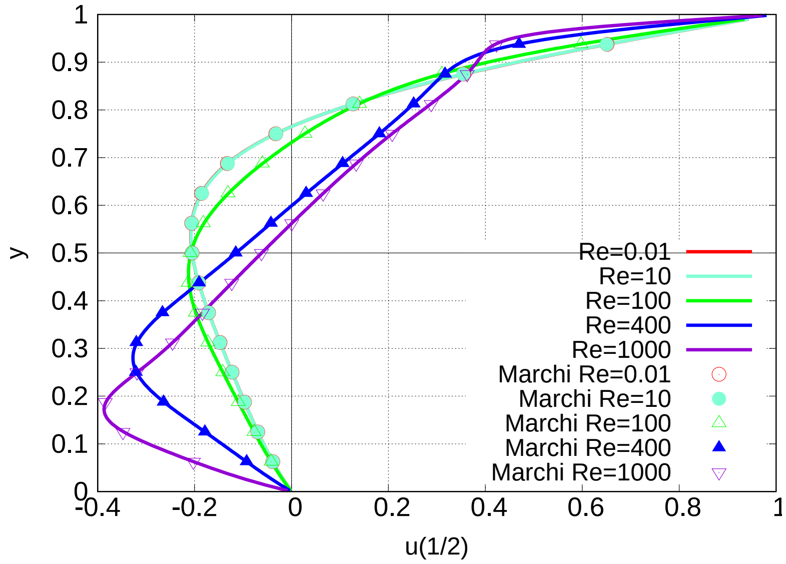

Figure 3: u-velocity profile on the vertical central line $x = 1/2$ for some Reynolds number values.

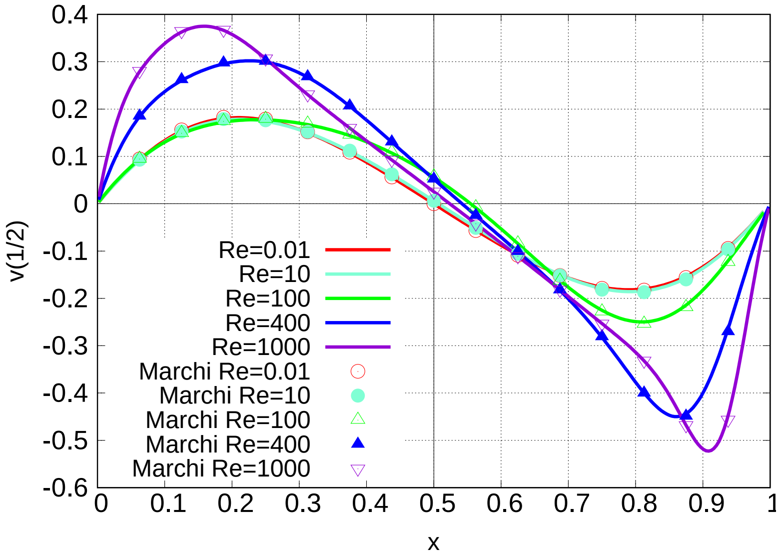

Figure 4: v-velocity profile on the horizontal central line $y = 1/2$ for some Reynolds number values.

the singlegrid residuals need more than nine thousand iterations to reach the convergence criterion, the multigrid residuals converge in around five hundred iterations.

The efficiency gain of the multigrid version of numerical model in relation to the singlegrid version can be evaluated through the execution times (measured in seconds) of singlegrid (SG) and multigrid (MG) versions of the model and the related speedups (S) presented in Tabs. 2 and 4. Table 2 shows the variation of execution times and speedups with Reynolds numbers. It can be noted that the performance the developed multigrid decays with the increase in Reynolds numbers. In Tab. 4, the variation of times and speedups with grid refinement can be observed. Reynolds number equal 1000 is used for these simulations. A graph representing Tab. 4 is presented in Fig. 6 with the two axes, number of variables (N) versus execution times (t - in seconds), being in logarithmic scale. The slope of the line in this graph is 1.5. Ideally, when the multigrid presents its theoretical efficiency, the slope of this kind of graph must be equal unity (Oliveira, 2010). However, for nonlinear systems of equations, this optimal multigrid efficiency is achieved only by the Full Multigrid algorithm (Trottenberg *et al.*, 2001). As can be seen, the gain by using multigrid increases with grid refinement.

The performances of the new developed V-cycle and the standard one are compared in Tab. 4. The data from the standard V-cycle for Navier-Stokes equations are taken from the work of Kumar *et al.* (2009). The multigrid method adopted in that work is based in the method described in detail in the work of Lien and Leschziner (1994). In this method, the course versions of the momentum equations depend on pressure corrections differences instead of pressures. The pressure-correction equation, equivalent to Eq. (7), solves for corrections to pressure corrections and its independent

Table 1: Comparisons of the velocity components at the cavity's centroid.

	variable	**MG**	**Marchi**
Re = 0.01	u(0.5;0.5)	-2.0518524E-01	-2.0519171E-01
	v(0.5;0.5)	6.3685327E-06	6.3677058E-06
Re = 10	u(0.5;0.5)	-2.0489487E-01	-2.05164738E-01
	v(0.5;0.5)	6.2676012E-03	6.3603620E-03
Re = 100	u(0.5;0.5)	-2.0912865E-01	-2.091491418E-01
	v(0.5;0.5)	5.7538960E-02	5.7536559E-02
Re = 400	u(0.5;0.5)	-1.1504821E-01	-1.15053628E-01
	v(0.5;0.5)	5.2073907E-02	5.2058082E-02
Re = 1000	u(0.5;0.5)	-6.2050674E-02	-6.205613E-02
	v(0.5;0.5)	2.5793316E-02	2.579946E-02

Table 2: Execution times (s) and *speedups* (S) for some values of Re in a 512×512 grid.

Re	SG	MG	S
100	2409.769	78.797	30.6
400	2549.1	60.196	42.3
1000	3092.053	62.633	49.4
2000	3410.481	70.502	48.4
3200	3384.133	92.875	36.4
5000	4430.856	165.735	26.7
7500	4521.187	310.548	14.6

term, or mass imbalance, similar to Eq. 8, is based on corrections to face velocities. A similar method is also described in Roy *et al.* (2015).

Table 4 offers a comparison between the speedups obtained in this work and in the work of Kumar *et al.* (2009). A 128×128 grid is used in these simulations. Although the speedups obtained in the present work for these grids are the smaller ones (Tab. 4), they are still bigger than the ones from Kumar *et al.* (2009) by an average factor of 1.67.

In figure 7, the mass residual, Eq. (8), decayment with the advancement of the number of iterations is shown. Similar graphs are presented in Kumar *et al.* (2009). The convergence criterion in that work is based on the mass residual and a tolerance of 10^{-8} is adopted. It can be seen that this tolerance is achieved by the modified V-cycle in about six or seven hundred iterations while in the referred work it takes around 1.2, 7 and 10 thousands iterations to be achieved for Reynolds numbers equal 1000, 5000 and 7500, respectively.

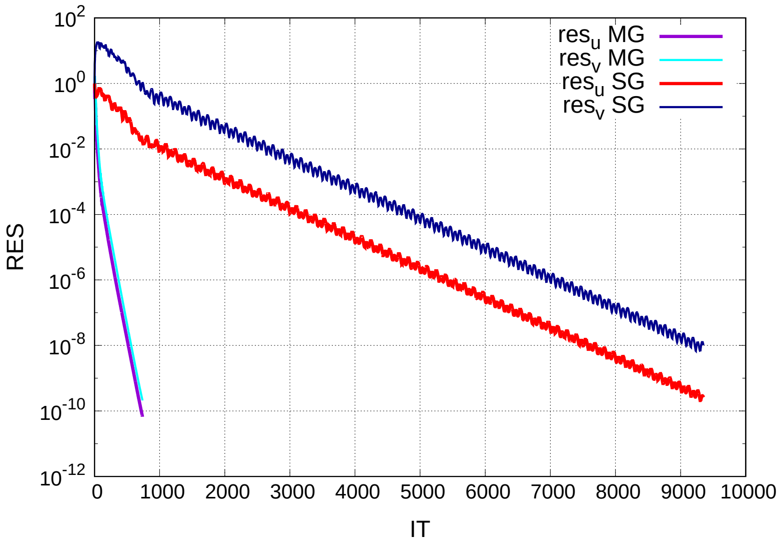

Figure 5: Residual decayments for the velocity components u and v on a 1024×1024 grid with Re $= 1000$.

Table 3: Execution times (s) and *speedups* (S) for some grid sizes with Re $= 1000$.

grid	SG	MG	S
128×128	11.764	0.986	12
256×256	210.12	6.478	32.4
512×512	3092.053	62.633	49.4
1024×1024	30443.298	487.431	62.4
2048×2048	312625.338	3602.751	86,8

Table 4: *Speedups* comparisons for some Reynolds numbers in a 128×128 grid.

Reynolds number	present	Kumar *et al.* (2009)
1000	12	10.41
3200	13.5	7.95
5000	15.5	8.06
7500	17.3	9.01

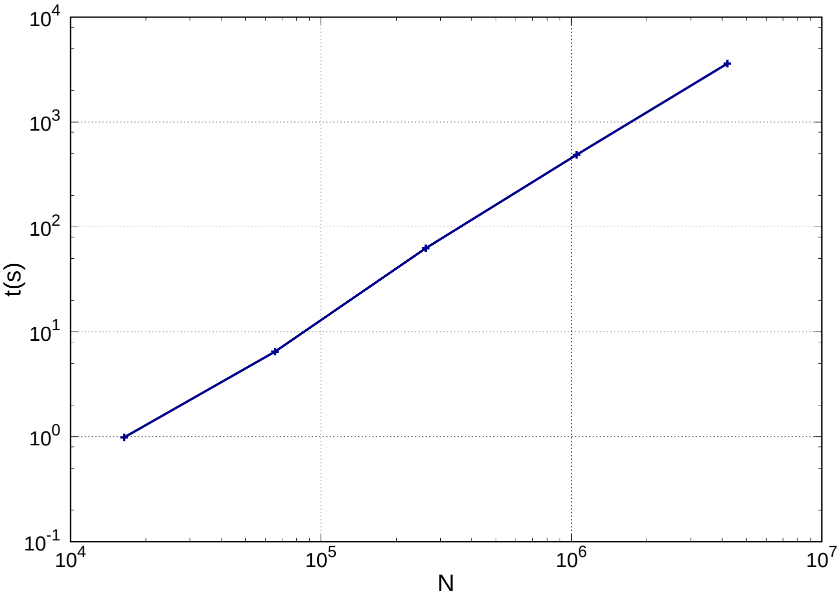

Figure 6: Increase in executions times (s) with numbers of variables for Re $= 1000$.

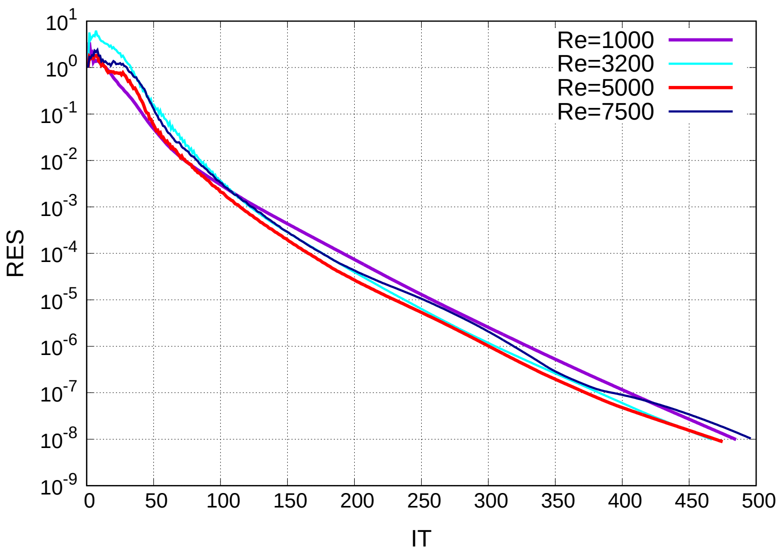

Figure 7: Mass residual decayments for the velocity components u and v on a 1024×1024 grid with Re $= 1000$.

4. CONCLUSIONS

A new developed SIMPLEC multigrid V-cycle scheme for the solution of the discrete Navier-Stokes equations was presented in this work. In this scheme, the structure of the SIMPLEC algorithm is maintained but independent V-cycles are performed in order to solve the momentum and pressure equations. Solutions on the unit lid-driven square cavity were obtained for various some Reynolds numbers and compared with the ones from literature. Good agreement between these solutions was achieved. Performance comparisons between the new developed and the standard SIMPLEC V-cycles show that new scheme coverges faster e and has a better gain (speedup) in relation to singlegrid execution time. These results show that the new scheme can be used as a good V-cycle basis for the fullmultigrid (FMG) algorithm.

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