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A multigrid waveform relaxation method for solving the poroelasticity equations

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Abstract In this work, a multigrid waveform relaxation method is proposed for solving a collocated finite difference discretization of the linear Biot's model. This gives rise to the first space–time multigrid solver for poroelasticity equations in the literature. The waveform relaxation iteration is based on a point-wise Vanka smoother that couples the pressure variable at a grid-point with the displacements around it. A semi-algebraic mode analysis is proposed to theoretically analyze the convergence of the multigrid waveform relaxation algorithm. This analysis is novel since it combines the semi-algebraic analysis, suitable for parabolic problems, with the non-standard analysis for overlapping smoothers. The practical utility of the method is illustrated through several numerical experiments in one and two dimensions.

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1 Introduction

A porous medium is assumed to be composed of a solid matrix containing a pore network. The presence of a moving fluid in such a network can affect the mechanical response of the solid skeleton, and at the same time, the mechanical changes influence the behavior of the fluid inside the pores. Thus, poroelasticity equations mathematically model the interaction between the deformation of an elastic porous material and the fluid flow inside of it. The first one who accounted for the influence of the pore fluid on the deformation of soils was Terzaghi in Terzaghi (1943). His studies, however, were based on one-dimensional experiments. The general three dimensional theory of poroelasticity was formulated by Biot in several pioneering works, (see Biot 1941, 1955). Nowadays, the analysis and numerical simulation of Biot's model has become increasingly popular due to the wide range of applications in different disciplines such as medicine, biomechanics, petroleum engineering, food processing, and other fields of science and engineering.

The poroelasticity problem can be formulated as a time-dependent coupled system of partial differential equations. Depending on which are the quantities of our interest, there exist two-, three- or four-field formulations of the Biot's problem. Here, we will consider the two-field formulation which includes the pressure of the fluid and the displacements of the solid matrix as unknowns.

It is well known that the solution of the large linear systems arising from the discretization of the Biot's model is the most consuming part in real simulations. Hence, the design of efficient solvers for this type of problems has attracted a lot of attention. Moreover, current computer architectures tend to increase the multiprocessing capabilities by having more cores and hardware threads per chip. To exploit these characteristics, one must develop algorithms which are capable to efficiently use large number of cores, appearing the need for greater parallelism. Typical solution algorithms for time-dependent problems are based on a time-marching approach, in which each time step is solved after the other in a sequential manner. These traditional methods, however, do not allow the parallelization of the temporal variable. This leads us to look for the increase of the concurrency by using time-parallel and full space–time methods. Time parallel existing techniques can be classified into four groups (see Gander 2015): methods based on multiple shooting, domain decomposition and waveform relaxation methods, space–time multigrid techniques and direct time parallel methods. Among all these, here we consider a multigrid waveform relaxation algorithm (WRMG) Lubich and Ostermann (1987), Vandewalle (1993), in which the novelty is the waveform relaxation method based on a point-wise Vanka smoother (see Vanka 1986; Molenaar 1991). This Vanka smoother has been successfully applied for the solution of the poroelasticity system when using a time-stepping approach based on the application of multigrid methods on each time-step, (for example see Gaspar et al. 2008). We want to notice that, up to our knowledge, it is the first time that a space–time solver is designed for the Biot's model.

Moreover, we apply a semi-algebraic mode analysis (SAMA) to theoretically support the convergence rates provided by the multigrid waveform relaxation method. This analysis is essentially a generalization of the classical local mode analysis or local Fourier analysis

(LFA) (Brandt 1977, 1994; Wienands and Joppich 2005) that combines standard LFA with algebraic computations. Local Fourier analysis is a commonly used approach, based on the Fourier transform theory, for analyzing the convergence properties of geometric multigrid methods. However, the failure of this analysis for the prediction of the multigrid convergence for convection-dominated or parabolic problems has been observed by different authors as Brandt 1981; Friedhoff and MacLachlan 2015; Oosterlee et al. 1998. To overcome this difficulty, the SAMA was proposed in Friedhoff and MacLachlan (2015). The analysis presented in this work is novel in the sense that combines the semi-algebraic mode analysis with the nonstandard local Fourier analysis of Vanka smoothers (MacLachlan and Oosterlee 2011; Rodrigo et al. 2016b).

The rest of the work is organized as follows: In Sect. 2, the one-dimensional poroelasticity equations are introduced together with their discretization by finite differences. Next, Sect. 3 is devoted to presenting the multigrid waveform relaxation method, which is proposed to efficiently solve the poroelasticity problem. In Sect. 4, a semi-algebraic mode analysis suitable to study space–time multigrid algorithms is introduced and the details to perform this analysis for the multigrid waveform relaxation method based on the Vanka smoother are explained. Section 5 presents two numerical experiments; one of them showing the extension of the proposed strategy to the case of two spatial dimensions. Finally, in Sect. 6 some conclusions are drawn.

2 Poroelasticity equations and their discretization

First we consider the one-dimensional poroelasticity equations on the spatial domain $\Omega = (0, 1)$, given as follows,

$$-\frac{\partial}{\partial x}\left(E\frac{\partial u}{\partial x}\right)+\alpha\frac{\partial p}{\partial x} = f(x,t), \text{ in } \Omega, \tag{1}$$

$$\frac{1}{\beta}\frac{\partial p}{\partial t}+\alpha\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial x}\left(K\frac{\partial p}{\partial x}\right) = g(x,t), \text{ in } \Omega, \tag{2}$$

where the physical parameters E and K denote the Young modulus and the hydraulic conductivity, respectively, α is the Biot–Willis constant, β is the Biot modulus, f is the density of applied body forces, and g represents a forced fluid extraction or injection process. The boundary conditions and the initial condition are given by

$$E\frac{\partial u}{\partial x}(0,t) = \sigma_0, \quad p(0,t) = 0, \tag{3}$$

$$u(1,t) = 0, \quad K\frac{\partial p}{\partial x}(1,t) = 0, \tag{4}$$

$$\frac{1}{\beta}p(x,0)+\frac{\partial u}{\partial x}(x,0) = 0. \tag{5}$$

This problem is commonly approximated by using finite element methods, see Lewis and Schrefler 1998. However, discretizations by finite differences and finite volumes have also been successfully applied (see Gaspar et al. 2003 and Nordbotten 2016, respectively, for example). Here, we consider a standard collocated finite difference scheme, which has to be stabilized in order to remove the non-physical oscillations that may appear in the approximation of the pressure field (see Gaspar et al. 2003; Ferronato et al. 2010; Haga et al. 2012; Favino et al. 2013; Phillips and Wheeler 2009). A perturbation term is added to the flow equation, which consists of the temporal derivative of a diffusion term in pressure

multiplied by a parameter that only depends on the physics of the problem and on the spatial discretization parameter. In particular, such stabilization parameter is given by $\frac{h^2}{4E}$ (see Gaspar et al. 2008). A similar stabilization for linear finite element methods was proposed in Aguilar et al. (2008) and theoretically studied in Rodrigo et al. (2016a).

We consider a uniform space–time grid on $\Omega \times (0, T]$ given by $G_{h,\tau} = G_h \times G_\tau$, where

$$G_h = \{x_i = ih \mid i = 0, \ldots, N+1\}, \tag{6}$$

$$G_\tau = \{t_j = j\tau \mid j = 0, \ldots, M\}, \tag{7}$$

being the discretization parameters $h = 1/(N+1)$ and $\tau = T/M$. The fully discrete problem is obtained by applying the stabilized collocated finite difference scheme, together for example with an implicit Euler scheme. For interior points, and assuming constant coefficients, the equations of the system read,

$$-E\,\frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2} + \alpha\,\frac{p_{i+1}^j - p_{i-1}^j}{2h} = f_i^j, \tag{8}$$

$$\frac{1}{\beta}\,\frac{p_i^j - p_i^{j-1}}{\tau} + \alpha\,\frac{u_{i+1}^j - u_{i-1}^j}{2h\tau} - \left(K + \frac{h^2}{4E\tau}\right)\frac{p_{i-1}^j - 2p_i^j + p_{i+1}^j}{h^2}$$
$$-\alpha\,\frac{u_{i+1}^{j-1} - u_{i-1}^{j-1}}{2h\tau} + \frac{h^2}{4E\tau}\,\frac{p_{i-1}^{j-1} - 2p_i^{j-1} + p_{i+1}^{j-1}}{h^2} = g_i^j, \tag{9}$$

for $i = 1, \ldots, N$, $j = 1, \ldots, M$, and where, for example, u_i^j denotes the approximation of u in the spatial grid-point x_i and the temporal point t_j.
This corresponds to a large system of algebraic equations that needs to be solved efficiently.

3 Multigrid waveform relaxation method

In this section, we describe the solver based on the so-called multigrid waveform relaxation method proposed for the numerical simulation of the poroelasticity equations. We start by introducing the waveform relaxation algorithms, also called dynamic iteration methods. They differ from the standard iterative methods, also referred to as static iterations, in the fact that their iterates are functions in time instead of scalar values. For more details, see Vandewalle 1993 and Gander 2015. These continuous-in-time iterative algorithms were created for numerically solving large systems of ordinary differential equations (ODEs), but they, however, can also be applied for solving time-dependent partial differential equations (PDEs). This can be done by using the numerical method of lines to replace the spatial derivatives by discrete formulas in the discrete spatial domain, obtaining a semi-discretization of the problem. Thus, the PDE is transformed into a large set of ordinary differential equations, and an iterative algorithm can be used to solve this system. In this work, due to the saddle point character of the poroelasticity problem, we consider a Vanka type iteration (see Vanka 1986). In particular, we use a three-point Vanka scheme, consisting of an overlapping block Gauss–Seidel smoother in which three primary unknowns are updated simultaneously. In particular, all displacement unknowns appearing in the divergence operator in the flow equation are relaxed together centred around a pressure point, that is, unknowns $u_{i-1}(t)$, $u_{i+1}(t)$ and $p_i(t)$. In a space–time grid, this iteration is equivalent to a line smoother in which three lines (two corresponding to displacement unknowns and one to pressure unknowns) are simultaneously updated. We will call this procedure as the three-point Vanka waveform relaxation (see Fig. 1).

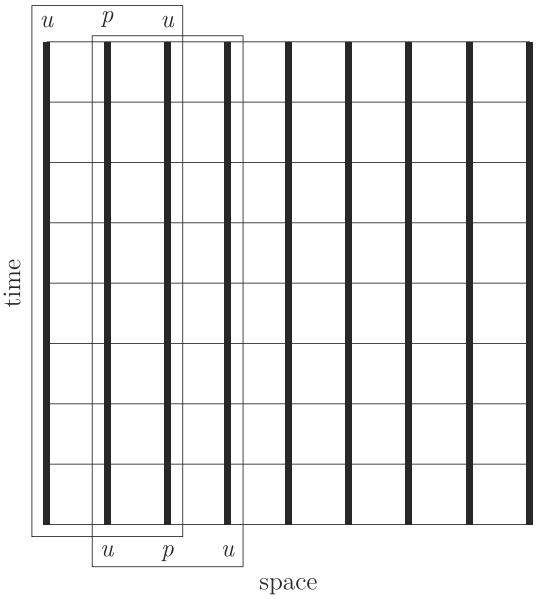

Fig. 1 Unknowns updated together for each grid-point within the three-point Vanka waveform relaxation

Furthermore, a coarse-grid correction procedure in the spatial dimension can be used to accelerate the convergence of the three-point Vanka waveform relaxation, giving rise to the so-called linear multigrid waveform relaxation algorithm. Essentially, this method is based in the application of the standard multigrid algorithm to the system of ODEs obtained after semi-discretization of the PDE problem. The smoother in such a multigrid procedure is the three-point Vanka waveform relaxation, and for the inter-grid transfer operators we consider a standard full-weighting restriction and the linear interpolation. In Algorithm 1, the structure of the WRMG is presented for the solution of system $A_h(t)\mathbf{u}_h(t) = \mathbf{f}_h(t)$.

Algorithm 1 : Multigrid waveform relaxation (WRMG): $\mathbf{u_h^k(t) \rightarrow u_h^{k+1}(t)}$

if we are on the coarsest grid-level (with spatial grid-size given by h_0) **then**

$A_{h_0}(t)\mathbf{u}_{h_0}^{k+1}(t) = \mathbf{f}_{h_0}(t)$ Solve with a direct or fast solver.

else

$\overline{\mathbf{u}}_h^k(t) = S_h^{n_1}(\mathbf{u}_h^k(t))$ **(Pre-smoothing)**
n_1 steps of the **three-point Vanka waveform relaxation**.

$\overline{\mathbf{r}}_h^k(t) = f_h(t) - A_h(t)\,\overline{\mathbf{u}}_h^k(t)$ Compute the defect.

$\overline{\mathbf{r}}_{2h}^k(t) = I_h^{2h}\,\overline{\mathbf{r}}_h^k(t)$ Restrict the defect.

$A_{2h}(t)\widehat{\mathbf{e}}_{2h}^k(t) = \overline{\mathbf{r}}_{2h}^k(t),\ \widehat{\mathbf{e}}_{2h}^k(0) = 0$ Solve the defect equation on G_{2h} by performing $\gamma \geq 1$ cycles of WRMG.

$\widehat{\mathbf{e}}_h^k(t) = I_{2h}^h\,\widehat{\mathbf{e}}_{2h}^k(t)$ Interpolate the correction.

$\overline{\mathbf{u}}_h^{k+1}(t) = \overline{\mathbf{u}}_h^k(t) + \widehat{\mathbf{e}}_h^k(t)$ Compute a new approximation.

$\mathbf{u}_h^{k+1}(t) = S_h^{n_2}(\overline{\mathbf{u}}_h^{k+1}(t))$ **(Post-smoothing)**
n_2 steps of the **three-point Vanka waveform relaxation**.

end if

4 Semi-algebraic mode analysis for the multigrid waveform relaxation based on the Vanka smoother

In this section, practical convergence rate estimates of the multigrid waveform relaxation method are obtained by using a semi-algebraic mode analysis (SAMA). This type of analysis was firstly proposed in Friedhoff and MacLachlan (2015), where the convergence of multigrid methods on space–time grids for parabolic problems was studied. Also in that paper, SAMA was applied to study non-parabolic problems like elliptic diffusion in layered media and convection diffusion problems, and recently, in Gaspar and Rodrigo (2017) and Hu et al. (2017) such analysis has been used to analyze the convergence of the WRMG algorithm for the time-fractional heat equation when uniform or non-uniform temporal meshes are considered, respectively. This analysis can be seen as a generalization of the classical local Fourier analysis (LFA) or local mode analysis (Brandt 1977, 1994; Trottenberg et al. 2001; Wesseling 1992; Wienands and Joppich 2005), which combines the standard exponential LFA only in space with an exact analytical approach in time.

LFA is based on a Fourier decomposition of the error function in terms of the so-called Fourier modes. After that, it analyzes the behavior of each operator involved in the multigrid method on these components, assuming that the involved operations are local processes. By neglecting the effects of the boundary conditions and assuming that the discrete operator has constant coefficients, a basis of complex exponential eigenfunctions of the operator, called Fourier components, can be obtained.

To perform the LFA, a regular infinite grid has to be assumed. In our case, since this analysis is applied only in space, we consider the infinite extension of the spatial grid G_h given in (6), that we denote as $\mathcal{G}_h$. On such a grid, we can define the Fourier modes as $\varphi_h(\theta, x) := \mathrm{e}^{\imath\theta x}$ with $\theta \in (-\pi/h, \pi/h]$ and $x \in \mathcal{G}_h$, which yield the so-called Fourier space

$$\mathcal{F}(\mathcal{G}_h) := \mathrm{span}\{\varphi_h(\theta, x) \,|\, \theta \in (-\pi/h, \pi/h]\}.$$

Under the assumptions of the LFA, the Fourier modes are formal eigenfunctions of the discrete version of the differential operator A_h. More precisely, $A_h\varphi_h(\theta, x) = \widetilde{A}_h(\theta)\varphi_h(\theta, x)$ where $\widetilde{A}_h(\theta)$ is the representation of the discrete operator on the Fourier space, also called symbol of operator A_h. Most classical iterative methods, used as smoothers in the multigrid iteration, satisfy also such invariance property, see Trottenberg et al. 2001; Wesseling 1992; Wienands and Joppich 2005. This is the case of the overlapping block smoothers considered here (MacLachlan and Oosterlee 2011; Rodrigo et al. 2016b). The distinction with respect to other smoothers, however, is that they update some variables more than once because of the overlapping of the local subdomains which are simultaneously solved. This implies that, in addition to the initial and final errors, some intermediate errors appear, which have to be taken into account in the analysis. Thus, a special strategy is required to carry out the local Fourier analysis for overlapping block smoothers.

In particular, the three-point Vanka smoother simultaneously solves the equations corresponding to unknowns $p_i(t)$, $u_{i-1}(t)$ and $u_{i+1}(t)$, which means that pressure unknowns are updated once whereas displacement variables are relaxed twice per relaxation step. More concretely, before solving the local system at grid-point x_i, unknown $u_{i-1}(t)$ has already been updated once, whereas $u_{i+1}(t)$ and $p_i(t)$ have not been relaxed yet. The system to solve on each grid-point x_i can be written in terms of corrections and residuals as follows,

$$\frac{2E}{h^2}\delta u_{i-1}(t) + \frac{1}{2h}\delta p_i(t) = r^u_{i-1}(t), \tag{10}$$

$$\frac{2E}{h^2}\delta u_{i+1}(t) - \frac{1}{2h}\delta p_i(t) = r^u_{i+1}(t), \tag{11}$$

$$\frac{\partial}{\partial t}\left(\delta u_{i+1}(t) - \delta u_{i-1}(t)\right)/2h + \frac{2K}{h^2}\delta p_i(t) + \frac{\partial}{\partial t}\left(\frac{\delta p_i(t)}{2E}\right) = r^p_i(t), \tag{12}$$

where

$$\begin{aligned}
\delta u_{i-1}(t) &= \mathrm{e}_h^{u,k+1}(x_{i-1},t) - \mathrm{e}_h^{u,k+1/2}(x_{i-1},t),\\
\delta u_{i+1}(t) &= \mathrm{e}_h^{u,k+1/2}(x_{i+1},t) - \mathrm{e}_h^{u,k}(x_{i+1},t),\\
\delta p_i(t) &= \mathrm{e}_h^{p,k+1}(x_i,t) - \mathrm{e}_h^{p,k}(x_i,t),
\end{aligned}$$

being $\mathrm{e}_h^{u,k}$, $\mathrm{e}_h^{p,k}$ the initial errors at k-iteration for displacement and pressure, respectively, $\mathrm{e}_h^{u,k+1}$, $\mathrm{e}_h^{p,k+1}$ the final errors and $\mathrm{e}_h^{u,k+1/2}$ the intermediate error that appears after u is updated once.

Following the LFA assumptions, we can write such initial, intermediate and final errors as a single Fourier mode multiplied by a coefficient $\alpha^{(m)}_{u,\theta}(t)$ or $\alpha^{(m)}_{p,\theta}(t)$, denoting by m the number of times that the unknown has been updated in the current iteration ($m = 0, 1, 2$). Notice that the Fourier coefficients depend on t, opposite to the classical analysis. Then, the corrections and the residuals appearing in system (10–12) can be written in terms of these coefficients, and we can rewrite the local system in the following way,

$$\frac{2E}{h^2}\left(\alpha^{(2)}_{u,\theta}(t) - \alpha^{(1)}_{u,\theta}(t)\right)\mathrm{e}^{-\imath\theta} + \frac{1}{2h}\left(\alpha^{(1)}_{p,\theta}(t) - \alpha^{(0)}_{p,\theta}(t)\right) = r^u_{i-1}(t), \tag{13}$$

$$\frac{2E}{h^2}\left(\alpha^{(1)}_{u,\theta}(t) - \alpha^{(0)}_{u,\theta}(t)\right)\mathrm{e}^{\imath\theta} - \frac{1}{2h}\left(\alpha^{(1)}_{p,\theta}(t) - \alpha^{(0)}_{p,\theta}(t)\right) = r^u_{i+1}(t), \tag{14}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\left(\alpha^{(1)}_{u,\theta}(t) - \alpha^{(0)}_{u,\theta}(t)\right)\mathrm{e}^{\imath\theta} - \left(\alpha^{(2)}_{u,\theta}(t) - \alpha^{(1)}_{u,\theta}(t)\right)\mathrm{e}^{-\imath\theta}\right)/2h\\
&\quad + \frac{2K}{h^2}\left(\alpha^{(1)}_{p,\theta}(t) - \alpha^{(0)}_{p,\theta}(t)\right) + \frac{\partial}{\partial t}\left(\frac{\left(\alpha^{(1)}_{p,\theta}(t) - \alpha^{(0)}_{p,\theta}(t)\right)}{2E}\right) = r^p_i(t),
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
r^u_{i-1}(t) &= \frac{E}{h^2}\left(\alpha^{(2)}_{u,\theta}(t)\mathrm{e}^{-\imath 2\theta} + \alpha^{(1)}_{u,\theta}(t) - 2\alpha^{(1)}_{u,\theta}(t)\mathrm{e}^{-\imath\theta}\right) - \frac{1}{2h}\left(\alpha^{(0)}_{p,\theta}(t) - \alpha^{(1)}_{p,\theta}(t)\mathrm{e}^{-\imath 2\theta}\right),\\
r^u_{i+1}(t) &= \frac{E}{h^2}\left(\alpha^{(1)}_{u,\theta}(t) + \alpha^{(0)}_{u,\theta}(t)\mathrm{e}^{\imath 2\theta} - 2\alpha^{(0)}_{u,\theta}(t)\mathrm{e}^{\imath\theta}\right) - \frac{1}{2h}\left(\alpha^{(0)}_{p,\theta}(t)\mathrm{e}^{\imath 2\theta} - \alpha^{(0)}_{p,\theta}(t)\right),\\
r^p_i(t) &= -\frac{\partial}{\partial t}\left(\frac{\alpha^{(0)}_{u,\theta}(t)\mathrm{e}^{\imath\theta} - \alpha^{(1)}_{u,\theta}(t)\mathrm{e}^{-\imath\theta}}{2h}\right) + \frac{K}{h^2}\left(\alpha^{(1)}_{p,\theta}(t)\mathrm{e}^{-\imath\theta} + \alpha^{(0)}_{p,\theta}(t)\mathrm{e}^{\imath\theta} - 2\alpha^{(0)}_{p,\theta}(t)\right)\\
&\quad + \frac{\partial}{\partial t}\left(\frac{\alpha^{(1)}_{p,\theta}(t)\mathrm{e}^{-\imath\theta} + \alpha^{(0)}_{p,\theta}(t)\mathrm{e}^{\imath\theta} - 2\alpha^{(0)}_{p,\theta}(t)}{4E}\right).
\end{aligned}$$

Our aim is to apply SAMA for the analysis of the multigrid waveform relaxation method based on the three-point Vanka smoother and therefore, the next step is to discretize in time and to build the system resulting from (13–15) after substituting $\frac{\partial}{\partial t}\alpha^{(m)}_{u,\theta}(t)$

by $\left(\dfrac{\alpha_{u,\theta}^{(m)}(j)-\alpha_{u,\theta}^{(m)}(j-1)}{\tau}\right)$ where j and $j-1$ denote two consecutive time levels. The obtained system is written in the way that the Fourier coefficients already updated are given in terms of those which have not been updated yet. This yields a system of the form

$$P\begin{pmatrix}\vdots\\ \alpha_{u,\theta}^{(1)}(j-1)\\ \alpha_{u,\theta}^{(2)}(j-1)\\ \alpha_{p,\theta}^{(1)}(j-1)\\ \alpha_{u,\theta}^{(1)}(j)\\ \alpha_{u,\theta}^{(2)}(j)\\ \alpha_{p,\theta}^{(1)}(j)\\ \vdots\end{pmatrix}=Q\begin{pmatrix}\vdots\\ \alpha_{u,\theta}^{(0)}(j-1)\\ \alpha_{p,\theta}^{(0)}(j-1)\\ \alpha_{u,\theta}^{(0)}(j)\\ \alpha_{p,\theta}^{(0)}(j)\\ \vdots\end{pmatrix}, \tag{16}$$

where P and Q are block bi-diagonal matrices with sizes $(3M\times 3M)$ and $(3M\times 2M)$, respectively. Matrix P has the following diagonal blocks

$$\begin{pmatrix}-\frac{E}{h^2} & \frac{2E}{h^2}\mathrm{e}^{-\imath\theta}-\frac{E}{h^2}\mathrm{e}^{-\imath 2\theta} & \frac{1}{2h}-\frac{\mathrm{e}^{-\imath 2\theta}}{2h}\\ \frac{2E}{h^2}\mathrm{e}^{\imath\theta}-\frac{E}{h^2} & 0 & -\frac{1}{2h}\\ \frac{1}{2h\tau}\mathrm{e}^{\imath\theta} & -\frac{1}{2h\tau}\mathrm{e}^{-\imath\theta} & \frac{2K}{h^2}-\frac{K}{h^2}\mathrm{e}^{-\imath\theta}+\frac{1}{2E\tau}-\frac{1}{4E\tau}\mathrm{e}^{-\imath\theta}\end{pmatrix},$$

and the blocks below the diagonal read as

$$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -\frac{1}{2h\tau}\mathrm{e}^{\imath\theta} & \frac{1}{2h\tau}\mathrm{e}^{-\imath\theta} & \frac{1}{4E\tau}\mathrm{e}^{-\imath\theta}-\frac{1}{2E\tau}\end{pmatrix}.$$

The diagonal blocks of matrix Q are given by

$$\begin{pmatrix}0 & 0\\ \frac{E}{h^2}\mathrm{e}^{\imath 2\theta} & -\frac{1}{2h}\mathrm{e}^{\imath 2\theta}\\ 0 & \frac{K}{h^2}\mathrm{e}^{\imath\theta}+\frac{1}{4E\tau}\mathrm{e}^{\imath\theta}\end{pmatrix},$$

whereas the blocks below the diagonal are

$$\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & -\frac{1}{4E\tau}\mathrm{e}^{\imath\theta}\end{pmatrix}.$$

From system (16), we can write,

$$\begin{pmatrix}\vdots\\ \alpha_{u,\theta}^{(1)}(j-1)\\ \alpha_{u,\theta}^{(2)}(j-1)\\ \alpha_{p,\theta}^{(1)}(j-1)\\ \alpha_{u,\theta}^{(1)}(j)\\ \alpha_{u,\theta}^{(2)}(j)\\ \alpha_{p,\theta}^{(1)}(j)\\ \vdots\end{pmatrix}=P^{-1}Q\begin{pmatrix}\vdots\\ \alpha_{u,\theta}^{(0)}(j-1)\\ \alpha_{p,\theta}^{(0)}(j-1)\\ \alpha_{u,\theta}^{(0)}(j)\\ \alpha_{p,\theta}^{(0)}(j)\\ \vdots\end{pmatrix}, \tag{17}$$

and by choosing the rows corresponding to the fully updated Fourier coefficients, we obtain the relation between the initial and the fully corrected errors, that is,

$$\begin{pmatrix} \vdots \\ \alpha^{(2)}_{u,\theta}(j-1) \\ \alpha^{(1)}_{p,\theta}(j-1) \\ \alpha^{(2)}_{u,\theta}(j) \\ \alpha^{(1)}_{p,\theta}(j) \\ \vdots \end{pmatrix} = \widetilde{S}_{h,\tau}(\theta) \begin{pmatrix} \vdots \\ \alpha^{(0)}_{u,\theta}(j-1) \\ \alpha^{(0)}_{p,\theta}(j-1) \\ \alpha^{(0)}_{u,\theta}(j) \\ \alpha^{(0)}_{p,\theta}(j) \\ \vdots \end{pmatrix}, \tag{18}$$

which gives us the matrix representation of the smoothing operator $S_{h,\tau}$, denoted by $\widetilde{S}_{h,\tau}(\theta)$, necessary to perform the SAMA. By using this representation, we can carry out a smoothing or two-grid analysis in order to predict the smoothing factor of the three-point Vanka waveform relaxation or the two-grid convergence factor of the multigrid waveform relaxation method proposed. The smoothing factor gives information about how well the smoother removes the oscillatory modes of the error. However, if we want to know about the interplay between the smoothing and the coarse-grid correction, and consequently about the behavior of the whole multigrid algorithm, a two-grid analysis is needed. Thus, the two-grid convergence factor, usually denoted as ρ, predicts the asymptotic convergence factor of the W-cycle multigrid algorithm, ρ_h. For the details of such an analysis we refer the reader to Friedhoff and MacLachlan 2015; Gaspar and Rodrigo 2017.

Remark 1 The semi-algebraic analysis presented for the one-dimensional case can also be extended to the two-dimensional case. In the one-dimensional case, the computational complexity of this analysis corresponds to the number of operations required to solve, for each frequency, a block bi-diagonal system, with blocks of size 3×3. In the two-dimensional case, the systems to solve are also block bi-diagonal but with blocks of size 5×5. In both cases, these systems can be efficiently solved by performing a forward substitution method. Moreover, the diagonal blocks are the same for all time steps and therefore they need to be inverted only once.

5 Numerical results

In this section, we want to illustrate the efficiency of the proposed multigrid waveform relaxation method for solving the poroelasticity equations. We present two numerical experiments corresponding to the one- and two-dimensional poroelastic problems.

5.1 One-dimensional numerical experiment

In this first numerical experiment, we solve problem (1–5) by using the multigrid waveform relaxation method proposed in Sect. 3. We analyze the convergence properties of such an algorithm by using the semi-algebraic mode analysis described in Sect. 4. In order to study the robustness of the method with respect to the involved parameters, we define parameter $\Lambda = \frac{K\tau}{h^2}$.

In Fig. 2 we display the two-grid convergence factors obtained by the semi-algebraic mode analysis for a wide range of values of the logarithm of parameter Λ, by using only one smoothing step. These results are compared with the asymptotic convergence factors

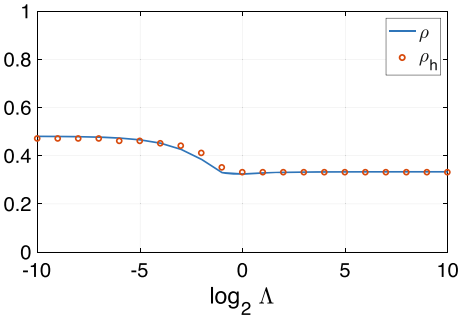

Fig. 2 Comparison between the two-grid convergence factors provided by the SAMA analysis, ρ, and the asymptotic convergence factors computationally obtained by using a W-cycle, ρ_h, for a wide range of values of $\log_2(\Lambda)$ and using only one smoothing step

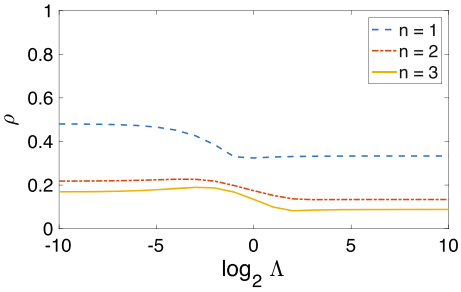

Fig. 3 Two-grid convergence factors provided by the SAMA analysis by using different numbers of smoothing steps n, for a wide range of values of $\log_2(\Lambda)$

experimentally computed by using a W-cycle in the real code. The experimentally obtained results show that the semi-algebraic mode analysis yields very accurate performance predictions. Thus, this analysis becomes an efficient tool for the study of the behavior of the proposed algorithm. We can analyze then the convergence of the proposed multigrid method by using different numbers of smoothing steps, n. In Fig. 3, the two-grid convergence factors predicted by the SAMA for $n = 1, 2, 3$ are shown. We observe that the more smoothing steps, the better the convergence rate becomes. However, it is necessary to take into account the whole efficiency of the multigrid cycle in order to find the most efficient choice.

For this purpose, we compare the CPU times for different numbers of smoothing steps and various values of the hydraulic conductivity. In Fig. 4, we display the CPU times needed to reach the stopping criterion, that is, to reduce the initial residual in a factor of 10^{-10}, by using $N = 2048$, $M = 1024$ and $T = 1$. As can be observed, the performance of $W-$cycle with one or two smoothing steps is very similar, improving that of $W(1, 1)-$cycle when the hydraulic conductivity tends to be small. Since this latter is the most realistic case, we choose a $W(1, 1)-$cycle for the next experiments.

It is well-known that multigrid $V-$cycles are often preferred to $W-$cycles because of their lower computational cost. However, for our problem, we have seen that the use of $V-$cycles does not provide a method robust with respect to the considered parameters. In fact, even for the one-dimensional problem, the $V-$cycle leads to divergence in some cases. This is not surprising, since for strongly coupled systems of PDEs, as the poroelasticity problem is, often $W-$cycles are needed to obtain a robust solver, (see for example Gaspar et al. 2008). For this reason, we will use $W-$cycles in the following tests.

In Table 1, we show the robustness of the method with respect to the physical and discretization parameters. For $E = 10^4$, we display the number of multigrid iterations which are necessary to reduce the maximum initial residual in a factor of 10^{-10}. These results are displayed for different values of the hydraulic conductivity K and for different spatial grids

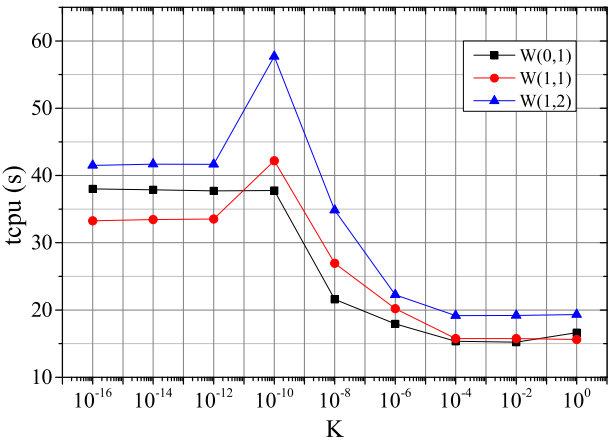

Fig. 4 CPU times (in seconds) necessary to reach the stopping criterion by using $W-$cycle with different smoothing steps, for several values of the hydraulic conductivity

Table 1 Numbers of iterations necessary to reduce the maximum initial residual in a factor of 10^{-10}, for different values of K and different spatial grids characterized by $h = 1/N$

$K \backslash N$	128	256	512	1024	2048	4096	8192
1	8	8	8	7	7	7	7
10^{-2}	8	8	7	7	7	7	7
10^{-4}	10	9	8	8	7	7	7
10^{-6}	14	12	11	10	9	9	8
10^{-8}	18	19	19	16	12	11	10
10^{-10}	15	15	16	18	19	19	17
10^{-12}	15	15	15	15	15	15	17
10^{-14}	15	15	15	15	15	15	15
10^{-16}	15	15	15	15	15	15	15

Table 2 Numbers of iterations necessary to reduce the maximum initial residual in a factor of 10^{-10} for different values of the discretization parameters $h = 1/N$ and $\tau = \frac{T}{M}$

The physical parameters are fixed as $K = 10^{-3}$ and $E = 10^4$

$M \backslash N$	512	1024	2048	4096	8192
512	7	7	8	8	8
1024	7	7	7	8	8
2048	7	7	7	7	8
4096	7	7	7	7	7
8192	7	7	7	7	7

characterized by $h = 1/N$, with $N = 2^k$, $k = 7, \ldots, 13$. We can observe that when K becomes smaller, the number of iterations grows, since parameter Λ is small and then the convergence rates are a bit worse as previously shown in Fig. 3. However, we can see that at some point the number of iterations remains constant for any discretization parameter h, showing the robustness of the proposed multigrid waveform relaxation algorithm.

Finally, to show that the good results obtained are also independent on the temporal resolution of the problem, in Table 2 we display the numbers of iterations necessary to reach the stopping criterium for different values of N and M defining the spatial and temporal discretization parameters h and τ, respectively.

Table 3 Numbers of iterations necessary to reduce the maximum initial residual in a factor of 10^{-10}, for different values of K and different spatial grids characterized by $h = 1/N$, by using a Crank–Nicolson scheme

$K \backslash N$	128	256	512	1024	2018	4096	8192
1	8	8	8	7	7	7	7
10^{-2}	9	9	8	7	7	7	7
10^{-4}	19	15	11	10	9	8	8
10^{-6}	22	21	21	20	17	12	10
10^{-8}	18	20	21	22	22	21	20
10^{-10}	15	15	16	18	19	21	21
10^{-12}	15	15	15	15	15	15	17
10^{-14}	15	15	15	15	15	15	15
10^{-16}	15	15	15	15	15	15	15

We can observe that the proposed space–time method is robust with respect to both spatial and temporal resolutions of the problem. Almost identical results are obtained if the second-order accurate Crank–Nicolson scheme is used for the time discretization instead of the first-order backward Euler. This can be seen in Table 3, in which we show the number of multigrid iterations necessary to fulfil the stopping criterion when Crank–Nicolson is considered. These results are obtained by using the same conditions as in Table 1 for the Euler scheme.

Remark 2 We can conclude that the convergence obtained by using the proposed multigrid waveform relaxation method is similar to that provided by other multigrid methods in the literature for the poroelasticity equations, (see Gaspar et al. 2008 for example). Its advantage, however, is that the space–time solver is parallel-in-time, taking advantage of the use of massively parallel systems with thousands of cores, which permits to reduce drastically the computing time in the numerical simulations.

5.2 Two-dimensional numerical experiment

In this second numerical experiment, we present the extension of the proposed strategy to solve the two-dimensional Biot's model, which can be formulated as a coupled system of PDEs for the unknowns displacements of the solid matrix, $\mathbf{u}$, and pore pressure of the fluid, p. Following this notation, the governing equations read as follows,

$$\nabla(\lambda + \mu)\nabla \cdot \mathbf{u} + \nabla \cdot \mu\nabla\mathbf{u} - \alpha\nabla p = \mathbf{f}(\mathbf{x}, t), \tag{19}$$

$$\frac{1}{\beta}\frac{\partial p}{\partial t} + \alpha\frac{\partial}{\partial t}(\nabla \cdot \mathbf{u}) - \nabla \cdot (K\nabla p) = g(\mathbf{x}, t), \tag{20}$$

where λ and μ are the Lamé coefficients, given in terms of the Young's modulus E and the Poisson ration ν as follows,

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}, \tag{21}$$

β is the Biot modulus, α is the Biot–Willis fluid/solid coupling coefficient, and K is the hydraulic conductivity. Let $\Omega = (0, 1)^2$ be the domain of our test-problem, and for simplicity we impose Dirichlet boundary conditions for all variables. For the discretization of the problem, we consider a uniform space–time grid on $\Omega \times (0, T]$ given by $G_{h,\tau} = G_h \times G_\tau$, where

$$G_h = \{x_{i,j} = (ih, jh) \mid i, j = 0, \ldots, N+1\}, \tag{22}$$

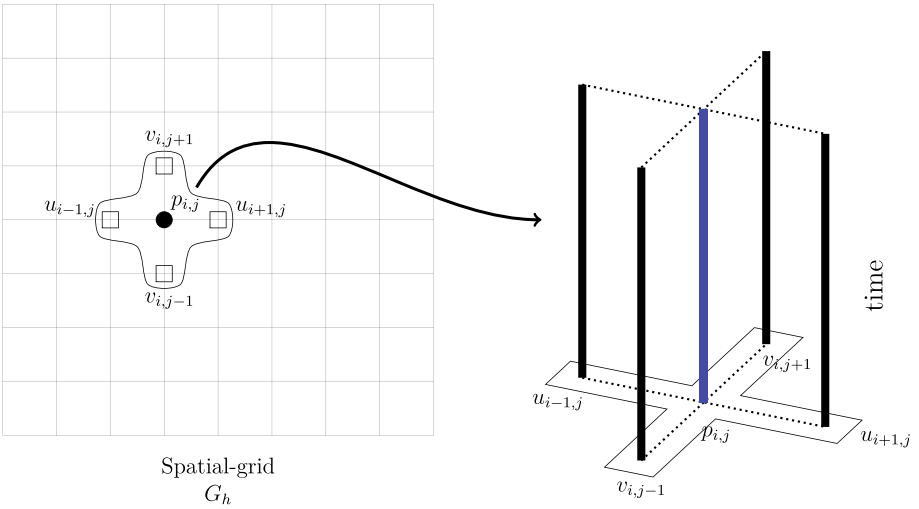

Fig. 5 Five unknowns involved in the point-wise Vanka smoother for 2D and the corresponding unknowns simultaneously updated in the point-wise Vanka waveform relaxation

being the discretization parameter $h = 1/(N+1)$ for both dimensions; and G_τ is defined as in the previous sections with $\tau = T/M$. A standard collocated finite difference scheme on such a grid, together with a stabilization term to avoid unphysical oscillations, is considered. More concretely, the additional term looks like a temporal derivative of a Laplacian of the pressure which is multiplied by the stabilization parameter $\dfrac{h^2}{4(\lambda+2\mu)}$, (see Gaspar et al. 2008; Oosterlee and Gaspar 2008).

The extension of the multigrid waveform relaxation to 2D is straightforward (see Vandewalle 1993). In this case, the method combines a two-dimensional coarsening strategy in the space variables and a line-in-time block smoother which in this case is based on a point-wise Vanka relaxation appropriate to the two-dimensional poroelasticity problem. In particular, a five-point Vanka smoother, which simultaneously updates all unknowns appearing in the discrete divergence operator in the pressure equation, is considered. This approach implies that four unknowns corresponding to displacements and one pressure unknown are relaxed simultaneously (see left picture in Fig. 5). This smoother was successfully applied for the multigrid solution of the poroelasticity problem in a time-stepping approach, (see for example Gaspar et al. 2008; Oosterlee and Gaspar 2008). Here, the waveform relaxation based on this point-wise Vanka smoother solves together those five unknowns for all time steps (see right picture in Fig. 5). Notice that, since an Euler method is considered for the time discretization, this can be easily done by solving consecutively the (5×5)-systems for each time level.

For the two dimensional problem, a multigrid $W(0,1)-$cycle is used in order to make less costly the algorithm. In fact, by comparing the complexity of the multigrid $W-$cycle for different numbers of smoothing steps, it results to be the most efficient choice. This can be seen in Fig. 6, where the CPU times necessary to reach the stopping criterion by using $W(0,1)$, $W(1,1)$ and $W(1,2)$ are shown for different values of the hydraulic conductivity. For this reason, in the rest of the section a $W(0,1)-$cycle is considered to perform the numerical experiments for the two-dimensional example.

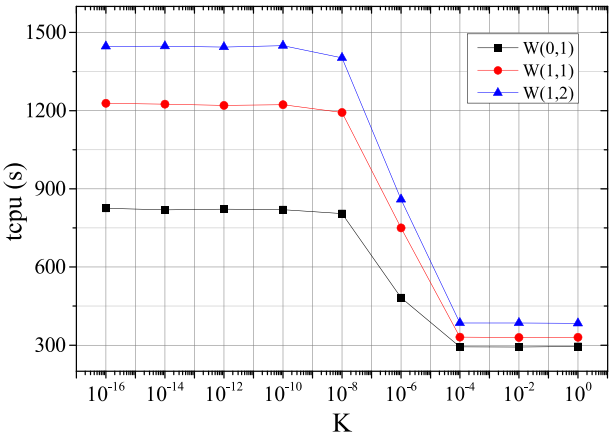

Fig. 6 CPU times (in seconds) necessary to reach the stopping criterion by using W-cycle with different smoothing steps, for several values of the hydraulic conductivity, in the two-dimensional problem. For these results, $N = 256$, $M = 512$ and $T = 1$

Table 4 Numbers of iterations necessary to reduce the maximum initial residual in a factor of 10^{-10} for different values of the discretization parameters

$M \backslash N$	128	256	512
128	17	17	17
256	17	17	17
512	17	17	18
1024	17	17	18

In order to show the robustness of the proposed solver in 2D, in Table 4 we display the numbers of iterations which are necessary to reduce the maximum initial residual in a factor of 10^{-10} for different values of M and N. The physical parameters are chosen as $E = 10^4$, $\nu = 0.2$, and the hydraulic conductivity is fixed to $K = 10^{-3}$. We can observe that the number of iterations remain constant for the different choices of the space–time grids, so that the proposed multigrid waveform relaxation method is robust independently of the discretization parameters.

Remark 3 As in the one-dimensional experiment, a Crank–Nicolson scheme has been implemented for the time discretization. The obtained results, however, are similar to those provided by using the Euler scheme.

Finally, we want to investigate if the behavior of the method is independent on the permeability. The number of iterations necessary to fulfil the stopping criterion is depicted in Fig. 7 for different values of parameter K. We can observe that when the permeability is small the number of iterations increases. This number, however, stays robust as the permeability becomes smaller and smaller. Therefore, the proposed solution method is robust with respect to K when it is small.

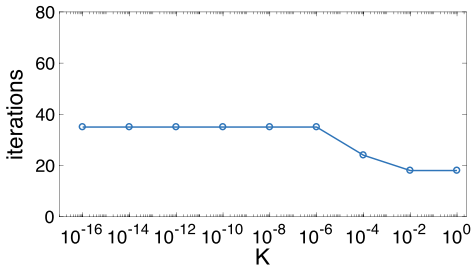

Fig. 7 Number of iterations necessary to reduce the maximum initial residual in a factor of 10^{-10} for different values of the hydraulic conductivity K

6 Conclusion

An efficient multigrid method on space–time grids has been proposed for solving the collocated finite difference discretization of the linear Biot's consolidation model. This multigrid solver is built on a waveform relaxation iteration based on a Vanka smoother, which is the key for the good performance of the method. The good convergence rates obtained are confirmed by performing a semi-algebraic mode analysis. This analysis is special since it combines the SAMA with the non-standard treatment of the Vanka smoother. Both one- and two-dimensional numerical experiments are presented, showing the good performance of the multigrid waveform relaxation algorithm for the poroelasticity problem.

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