Chapter 6

EIGENVALUE AND TIME-DEPENDENT PROBLEMS

6.1 EIGENVALUE PROBLEMS

6.1.1 Introduction

An *eigenvalue problem* is defined to be one in which we seek the values of the parameter λ such that the equation

$$A(u) = \lambda B(u) \tag{6.1.1}$$

is satisfied for nontrivial values of u. Here A and B denote either matrix operators or differential operators, and values of λ for which Eq. (6.1.1) is satisfied are called *eigenvalues*. For each value of λ there is a vector u, called an *eigenvector* or *eigenfunction*. For example, the equation

$$-\frac{d^2u}{dx^2} = \lambda u(x), \quad \text{with } A = -\frac{d^2}{dx^2}, \quad B = 1$$

which arises in connection with natural axial vibrations of a bar or the transverse vibration of a cable, constitutes an eigenvalue problem. Here λ denotes the square of the frequency of vibration, ω.

In general, the determination of the eigenvalues is of engineering as well as mathematical importance. In structural problems, the eigenvalues denote either natural frequencies or buckling loads. In fluid mechanics and heat transfer, eigenvalue problems arise in connection with the determination of the homogeneous parts of the transient solution, as will be shown shortly. In these cases, eigenvalues often denote amplitudes of the Fourier components making up the solution. Eigenvalues are also useful in determining the stability characteristics of temporal schemes, as discussed in Section 6.2.

In this section, we develop finite element models of eigenvalue problems described by differential equations. In view of the close similarity between the differential equations governing eigenvalue and boundary value problems, the steps involved in the construction of their finite element models are entirely analogous. Differential eigenvalue problems are reduced to algebraic eigenvalue problems (i.e., $[A]\{X\} = \lambda[B]\{X\}$) by means of the finite element approximation. The methods of solution of algebraic eigenvalue problems are then used to solve for the eigenvalues and eigenvectors.

6.1.2 Formulation of Eigenvalue Problems

Parabolic Equation

Consider the partial differential equation

$$\rho c A \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(kA\frac{\partial u}{\partial x}\right) = q(x,t) \tag{6.1.2}$$

which arises in connection with transient heat transfer in one-dimensional systems (e.g., a plane wall or a fin). Here u denotes the temperature, k the thermal conductivity, ρ the density, A the cross-sectional area, c the specific heat, and q the heat generation per unit length. Equations involving the first-order time derivative are called *parabolic equations*.

The homogenous solution (i.e., the solution when $q=0$) of Eq. (6.1.2) is often sought in the form of a product of a function of x and a function of t (i.e., through the *separation-of-variables* technique):

$$u^h(x,t) = U(x)T(t) \tag{6.1.3}$$

Substitution of this assumed form of solution into the homogeneous form of (6.1.2) gives

$$\rho c A U \frac{dT}{dt} - \frac{d}{dx}\left(kA\frac{dU}{dx}\right)T = 0$$

Separating variables of t and x (assuming that ρcA and kA are functions of x only), we arrive at

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{\rho c A}\frac{1}{U}\frac{d}{dx}\left(kA\frac{dU}{dx}\right) \tag{6.1.4}$$

Note that the left-hand side of this equation is a function of t only while the right-hand side is a function of x only. For two functions of two independent variables to be equal for all values of the independent variables, both functions must be equal to the same constant, say $-\lambda$ $(\lambda > 0)$:

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{\rho c A}\frac{1}{U}\frac{d}{dx}\left(kA\frac{dU}{dx}\right) = -\lambda$$

or

$$\frac{dT}{dt} = -\lambda T \tag{6.1.5}$$

$$-\frac{d}{dx}\left(kA\frac{dU}{dx}\right) - \lambda \rho c A U = 0 \tag{6.1.6}$$

The negative sign of the constant λ is based on the physical requirement that the solution $U(x)$ be harmonic in x while $T(t)$ decay exponentially with increasing t.

The solution of (6.1.5) is

$$T(t) = Ke^{-\lambda t} \tag{6.1.7}$$

where K is a constant of integration. The values of λ are determined by solving (6.1.6), which also gives $U(x)$. With $T(t)$ and $U(x)$ known, we have the complete homogeneous solution (6.1.3) of Eq. (6.1.2). The problem of solving (6.1.6) for λ and $U(x)$ is termed an *eigenvalue problem*, and λ is called the eigenvalue and $U(x)$ the eigenfunction. When k,

A, ρ, and c are constants, the solution of Eq. (6.1.6) is

$$U(x) = C\sin\alpha x + D\cos\alpha x, \quad \alpha^2 = \frac{\rho c}{k}\lambda \tag{6.1.8}$$

where C and D are constants of integration. Boundary conditions of the problem are used to find algebraic relations among C and D. The algebraic relations can be expressed in matrix form as $([A] - \alpha[B])\{V\} = \{0\}$, where $[A]$ and $[B]$ depend, in general, on k, c, and ρ, and $\{V\}$ is the vector of integration constants C and D. Since C and D both cannot be zero (otherwise, we obtain the trivial solution), we require the determinant of the coefficient matrix $[A] - \alpha[B]$ to be zero. This results in an algebraic eigenvalue problem whose solution yields α (hence, λ) and $\{V\} = (C, D)$.

To fix the ideas, consider Eq. (6.1.6) subject to the boundary conditions (e.g., a fin with specified temperature at $x = 0$ and insulated at $x = L$)

$$U(0) = 0, \quad \left[kA\frac{dU}{dx}\right]_{x=L} = 0 \tag{6.1.9}$$

We note that nonhomogeneous boundary conditions can be converted to homogeneous boundary conditions by a change of variables. Using the above boundary conditions in (6.1.8), we obtain

$$0 = C\cdot 0 + D\cdot 1, \quad 0 = \alpha\,(C\cos\alpha L - D\sin\alpha L)$$

or

$$\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \alpha\begin{bmatrix} 0 & 0 \\ \cos\alpha L & -\sin\alpha L \end{bmatrix}\right)\begin{Bmatrix} C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6.1.10}$$

For nontrivial solution (i.e., not both C and D are equal to zero), we set the determinant of the coefficient matrix in (6.1.10) to zero and obtain (since α cannot be zero)

$$\cos\alpha L = 0 \;\rightarrow\; \alpha_n L = \frac{(2n-1)\pi}{2}$$

Hence, the homogeneous solution becomes [note that the constant K of Eq. (6.1.7) is absorbed into C_n]

$$u^h(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n t}\sin\alpha_n x, \quad \lambda_n = \alpha_n^2\left(\frac{k}{\rho c}\right), \quad \alpha_n = \frac{(2n-1)\pi}{2L} \tag{6.1.11}$$

The constants C_n are determined using the initial condition of the problem, $u(x,0) = u_0(x)$:

$$u^h(x,0) = \sum_{n=1}^{\infty} C_n \sin\alpha_n x = u_0(x)$$

Multiplying both sides with $\sin\alpha_m x$, integrating over the interval $(0, L)$, and making use of the orthogonality condition

$$\int_0^L \sin\alpha_n x\,\sin\alpha_m x\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \dfrac{L}{2}, & \text{if } m = n \end{cases} \tag{6.1.12}$$

we obtain

$$C_n = \frac{2}{L}\int_0^L u_0(x)\sin\alpha_n x\,dx \tag{6.1.13}$$

The complete solution of Eq. (6.1.2) is given by the sum of the homogeneous solution and the particular solution $u(x,t) = u^h(x,t) + u^p(x,t)$.

The example discussed above provides the need for determining eigenvalues (α_n) in the context of finding the transient response of a parabolic equation. Next, we consider the transient response of a second-order equation in time, known as a *hyperbolic equation*.

Hyperbolic Equation

The axial motion of a bar, for example, is described by the equation [see Reddy (2002), pp. 185–187]

$$\rho A \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = f(x,t) \tag{6.1.14}$$

where u denotes the axial displacement, E the modulus of elasticity, A the cross-sectional area, ρ the density, and f the axial force per unit length. The solution of (6.1.14) consists of two parts: homogeneous solution u^h (i.e., when $f = 0$) and particular solution u^p. The homogeneous part is determined by the separation-of-variables technique, as we discussed for the parabolic equation.

The homogenous solution of Eq. (6.1.14) is also assumed to be of the form in (6.1.3). Substitution of (6.1.3) into the homogeneous form of (6.1.14) gives

$$\rho A U \frac{d^2 T}{dt^2} - \frac{d}{dx}\left(EA\frac{dU}{dx}\right) T = 0$$

Assuming that ρA and EA are functions of x only, we arrive at

$$\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{\rho A}\frac{1}{U}\frac{d}{dx}\left(kA\frac{dU}{dx}\right) = -\alpha^2$$

or

$$\frac{d^2T}{dt^2} + \alpha^2 T = 0 \tag{6.1.15}$$

$$-\frac{d}{dx}\left(EA\frac{dU}{dx}\right) - \alpha^2 \rho A U = 0 \tag{6.1.16}$$

The negative sign of the constant α is based on the physical requirement that the solution $u(x,t)$ be harmonic in x and t.

The solution of (6.1.15) is

$$T(t) = Ke^{-i\alpha t} = K_1 \cos\alpha t + K_2 \sin\alpha t \tag{6.1.17}$$

where K_1 and K_2 are constants of integration. The solution of (6.1.16), when E, A, and ρ, are constants, is

$$U(x) = C\sin\bar{\alpha}x + D\cos\bar{\alpha}x, \quad \bar{\alpha}^2 = \frac{\rho}{E}\alpha^2 \tag{6.1.18}$$

where again C and D are constants of integration. In the process of determining the constants C and D using the boundary conditions of the problem, once again we are required to solve an eigenvalue problem (the steps are analogous to those described for a parabolic equation). Unlike in the case of heat conduction problem, the quantities α_n in the case of bars have direct physical meaning, namely, they represent natural frequencies of the system. Thus, in structural mechanics we may be interested in determining only the natural frequencies of the system but not the transient response.

When we are interested only in natural frequencies of the system, the eigenvalue problem may be formulated directly from the equation of motion (6.1.14) by assuming a solution form that is periodic in time t

$$u(x,t) = U(x)e^{-i\omega t} \tag{6.1.19}$$

where ω denotes the frequency of natural vibration, i equals $\sqrt{-1}$, and $U(x)$ denotes the configuration of the structure at that frequency, called the *mode shape* (i.e., for each value of ω, there is an associated mode shape). Substitution of (6.1.19) into the homogeneous form of (6.1.14) gives

$$\left[-\rho A\omega^2 U - \frac{d}{dx}\left(EA\frac{dU}{dx}\right)\right]e^{-i\omega t} = 0$$

or

$$-\frac{d}{dx}\left(EA\frac{dU}{dx}\right) - \omega^2\rho AU = 0$$

which is identical to Eq. (6.1.16) with $\alpha = \omega$. Similar equations can be formulated for natural vibration of beams. Another eigenvalue problem that arises directly from the governing equilibrium equation is that of buckling of beam-columns. We shall return to this topic shortly.

In summary, eigenvalue problems associated with parabolic equations are obtained from the corresponding equations of motion by assuming solution of the form

$$u(x,t) = U(x)e^{-\alpha t}, \quad \lambda = \alpha \tag{6.1.20a}$$

whereas those associated with hyperbolic equations are obtained by assuming solution of the form

$$u(x,t) = U(x)e^{-i\omega t}, \quad \lambda = \omega^2 \tag{6.1.20b}$$

where λ denotes the eigenvalue. Finite element formulations of both problems are presented next.

6.1.3 Finite Element Formulation

Comparison of Eqs. (6.1.6) and (6.1.16) with the model equation (3.2.1) reveals that the equations governing eigenvalue problems are special cases of the model equations studied in Chapters 3 and 5. Here we summarize the steps in the finite element formulation of eigenvalue problems for the sake of completeness and ready reference. We will consider eigenvalue problems described by (a) a single equation in a single unknown (e.g., heat transfer, bar, and Euler–Bernoulli beam problems), and (b) a pair of equations in two variables (e.g., Timoshenko beam theory).

Heat Transfer and Bar-Like Problems

Consider the problem of solving the equation

$$-\frac{d}{dx}\left[a(x)\frac{dU}{dx}\right] + c(x)U(x) = \lambda c_0(x)U(x) \tag{6.1.21}$$

for λ and $U(x)$. Here a, c, and c_0 are known quantities that depend on the physical problem (i.e., data), λ is the eigenvalue, and U is the eigenfunction. Special cases of Eq. (6.1.21) are given below.

$$\text{Heat transfer:}\quad a = kA,\quad c = P\beta,\quad c_0 = \rho c A \tag{6.1.22}$$

$$\text{Bars:}\quad a = EA,\quad c = 0,\quad c_0 = \rho A \tag{6.1.23}$$

Over a typical element Ω_e, we seek a finite element approximation of U in the form

$$U_h^e(x) = \sum_{j=1}^{n} u_j^e \psi_j^e(x) \tag{6.1.24}$$

The weak form of (6.1.21) is

$$0 = \int_{x_a}^{x_b} \left(a\frac{dw}{dx}\frac{dU}{dx} + cwU(x) - \lambda c_0 w U \right) dx - Q_1^e w(x_a) - Q_n^e w(x_b) \tag{6.1.25}$$

where w is the weight function, and Q_1^e and Q_n^e are the secondary variables at node 1 and node n, respectively (assume that $Q_i^e = 0$ when $i \neq 1$ and $i \neq n$)

$$Q_1^e = -\left[a\frac{dU}{dx}\right]_{x_a},\quad Q_n^e = \left[a\frac{dU}{dx}\right]_{x_b} \tag{6.1.26}$$

Substitution of the finite element approximation into the weak form gives the finite element model of the eigenvalue equation (6.1.21):

$$[K^e]\{u^e\} - \lambda[M^e]\{u^e\} = \{Q^e\} \tag{6.1.27a}$$

where

$$K_{ij}^e = \int_{x_a}^{x_b} \left[a(x)\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} + c(x)\psi_i^e\psi_j^e \right] dx,\quad M_{ij}^e = \int_{x_a}^{x_b} c_0(x)\psi_i^e\psi_j^e\, dx \tag{6.1.27b}$$

Equation (6.1.27a) contains the finite element models of the eigenvalue equations (6.1.6) and (6.1.16) as special cases.

The assembly of element equations and imposition of boundary conditions on the assembled equations remain the same as in static problems of Chapter 3. However, the solution of the condensed equations for the unknown primary nodal variables is reduced to an algebraic eigenvalue problem in which the determinant of the coefficient matrix is set to zero to determine the values of λ and subsequently the nodal values of the eigenfunction $U(x)$. These ideas are illustrated through examples.

Example 6.1.1

Consider a plane wall, initially at a uniform temperature T_0, which has both surfaces suddenly exposed to a fluid at temperature T_∞. The governing differential equation is (for constant data)

$$k\frac{\partial^2 T}{\partial x^2} = \rho c_0 \frac{\partial T}{\partial t} \tag{6.1.28a}$$

and the initial condition is

$$T(x, 0) = T_0 \tag{6.1.28b}$$

where k is the thermal conductivity, ρ the density, and c_0 the specific heat at constant pressure. Equation (6.1.28a) is also known as the *diffusion equation* with diffusion coefficient $\alpha = k/\rho c_0$.

We consider two sets of boundary conditions, each being representative of a different scenario for $x = L$. It amounts to solving Eq. (6.1.28a) and (6.1.28b) for two different sets of boundary conditions.

Set 1. If the heat transfer coefficient at the surfaces of the wall is assumed to be infinite, the boundary conditions can be expressed as

$$T(0,t) = T_\infty, \quad T(L,t) = T_\infty \quad \text{for } t > 0 \tag{6.1.29a}$$

Set 2. If we assume that the wall at $x = L$ is subjected to ambient temperature, we have

$$T(0,t) = T_\infty, \quad \left[k\frac{\partial T}{\partial x} + \beta(T - T_\infty)\right]\Bigg|_{x=L} = 0 \tag{6.1.29b}$$

Equation (6.1.28a) can be normalized to make the boundary conditions homogeneous. Let

$$\alpha = \frac{k}{\rho c_0}, \quad \bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{\alpha t}{L^2}, \quad u = \frac{T - T_\infty}{T_0 - T_\infty}$$

The differential equation (6.1.28a), initial condition (6.1.28b), and boundary conditions (6.1.29a) and (6.1.29b) become

$$-\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0 \tag{6.1.30a}$$

$$u(0,t) = 0, \quad u(1,t) = 0 \quad u(x,0) = 1 \tag{6.1.30b}$$

$$u(0,t) = 0, \quad \left(\frac{\partial u}{\partial x} + Hu\right)\Bigg|_{x=1} = 0, \quad H = \frac{\beta L}{k} \tag{6.1.30c}$$

where the bars over x and t are omitted in the interest of brevity.

Solution of (6.1.30a) and (6.1.30b) by separation-of-variables technique (or substitute $u = Ue^{-\lambda t}$) leads to the solution of the eigenvalue problem

$$-\frac{d^2U}{dx^2} - \lambda U = 0, \quad U(0) = 0, \quad U(1) = 0 \tag{6.1.31a}$$

whereas solution of (6.1.30a) and (6.1.30c) results in

$$-\frac{d^2U}{dx^2} - \lambda U = 0, \quad U(0) = 0, \quad \left(\frac{dU}{dx} + HU\right)\Bigg|_{x=1} = 0 \tag{6.1.31b}$$

This differential equation is a special case of Eq. (6.1.21) with $a = 1$, $c = 0$, and $c_0 = 1$.

For a linear element, the element equations (6.1.27a) have the explicit form [see Eq. (3.2.34) for element matrices]

$$\left(\frac{1}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\frac{h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{6.1.32a}$$

For a quadratic element, we have [see Eq. (3.2.37a) for element matrices]

$$\left(\frac{1}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} - \lambda\frac{h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}\right)\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ 0 \\ Q_3^e \end{Bmatrix} \tag{6.1.32b}$$

Solution for Set 1. For a mesh of two linear elements (the minimum number needed for Set 1 boundary conditions), with $h_1 = h_2 = 0.5$, the assembled equations are

$$\left(2\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \lambda\frac{1}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

The boundary conditions $U(0) = 0$, $Q_2^1 + Q_1^2 = 0$, and $U(1) = 0$ require $U_1 = U_3 = 0$. Hence, the eigenvalue problem reduces to the single equation

$$\left(4 - \lambda\frac{4}{12}\right)U_2 = 0, \quad \text{or} \quad \lambda_1 = 12.0, \quad U_2 \neq 0$$

The mode shape is given (within an arbitrary constant; take $U_2 = 1$) by

$$U(x) = U_2\Phi_2(x) = \begin{cases} U_2\psi_2^1(x) = x/h = 2x & 0 \le x \le 0.5 \\ U_2\psi_1^2(x) = (2h - x)/h = 2(1 - x) & 0.5 \le x \le 1.0 \end{cases}$$

For a mesh of one quadratic element, we have ($h = 1.0$)

$$\frac{16}{3} - \lambda\frac{16}{30} = 0, \quad \text{or} \quad \lambda_1 = 10.0, \quad U_2 \neq 0$$

The corresponding eigenfunction is

$$U(x) = U_2\Phi_2(x) = U_2\psi_2^1 = 4\frac{x}{h}\left(1 - \frac{x}{h}\right), \quad 0 \le x \le 1.0$$

The exact eigenvalues are $\lambda_n = (n\pi)^2$ and $\lambda_1 = \pi^2 = 9.8696$. The exact eigenfunctions for Set 1 boundary conditions are $U_n(x) = \sin n\pi x$ and $U_1(x) = \sin \pi x$. Clearly, one quadratic element gives more accurate solution than two linear elements.

The eigenvalues and eigenfunctions can be used to construct the solution of the transient problem. For example, the solution of (6.1.30*a*) and (6.1.30*b*) is

$$u(x,t) = \sum_{n=1}^{\infty} C_n U_n(x) e^{-\lambda_n t} = \sum_{n=1}^{\infty} C_n \sin n\pi x \, e^{-(n\pi)^2 t} \tag{6.1.33}$$

where C_n are constants to be determined using the initial condition of the problem [see Eq. (6.1.13)]. The finite element solution of the same problem, when one quadratic element is used, is given by

$$u_h(x,t) = 4x(1 - x)e^{-10t}T_0$$

which is very close to the one-term solution in (6.1.33).

Solution for Set 2. For a mesh of two linear elements, the Set 2 boundary conditions translate into $U_1 = 0$ and $Q_2^2 + HU_3 = 0$. The condensed equations are

$$\left(\frac{1}{h}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\frac{h}{6}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -HU_3 \end{Bmatrix}$$

or

$$\left(\begin{bmatrix} 4 & -2 \\ -2 & 2 + H \end{bmatrix} - \frac{\lambda}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6.1.34}$$

For nontrivial solution (i.e., at least one of the U_i is nonzero), the determinant of the coefficient matrix should be zero:

$$\begin{vmatrix} 4-4\bar{\lambda} & -(2+\bar{\lambda}) \\ -(2+\bar{\lambda}) & 2+H-2\bar{\lambda} \end{vmatrix} = 0, \quad \bar{\lambda} = \frac{\lambda}{12}$$

or

$$7\bar{\lambda}^2 - 4(5+H)\bar{\lambda} + 4(1+H) = 0$$

The above equation is known as the *characteristic equation* of the eigenvalue problem. The two roots of this quadratic equation for $H = 1$ are

$$\bar{\lambda}_{1,2} = \frac{12 \pm \sqrt{(12)^2 - 7\times 8}}{7} \rightarrow \bar{\lambda}_1 = 0.374167, \quad \bar{\lambda}_2 = 3.0544$$

and the eigenvalues are ($\lambda = 12\bar{\lambda}$)

$$\lambda_1 = 4.49, \quad \lambda_2 = 36.6529$$

The eigenvectors associated with each eigenvalue can be computed from equations (6.1.34)

$$\begin{bmatrix} 4-4\bar{\lambda}_i & -(2+\bar{\lambda}_i) \\ -(2+\bar{\lambda}_i) & 3-2\bar{\lambda}_i \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}^{(i)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad i = 1, 2$$

For example, for $\bar{\lambda}_1 = 0.374167$ ($\lambda_1 = 4.4900$), we have

$$2.5033U_2^{(1)} - 2.3742U_3^{(1)} = 0$$

Taking $U_2^{(1)} = 1$, we obtain

$$\{U^{(1)}\} = \begin{Bmatrix} U_2^{(1)} \\ U_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1.0000 \\ 1.0544 \end{Bmatrix}$$

On the other hand, if we normalize the components, $(U_2^{(1)})^2 + (U_3^{(1)})^2 = 1$, we obtain

$$\{U^{(1)}\} = \begin{Bmatrix} U_2^{(1)} \\ U_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0.6881 \\ 0.7256 \end{Bmatrix}$$

In other words, we obtain a mode shape whose amplitude is not unique. Hence, the eigenfunction corresponding to $\lambda_1 = 4.4900$ ($h = 0.5$) is

$$U^{(1)}(x) = \begin{cases} 0.6881\, x/h & \text{for } 0 \le x \le 0.5 \\ 0.6881\,(1-x)/h + 0.7256(2x-1)/2h & \text{for } 0.5 \le x \le 1.0 \end{cases}$$

Similarly, the eigenvector components corresponding to $\lambda_2 = 36.6529$ is (not normalized)

$$\{U^{(2)}\} = \begin{Bmatrix} U_2^{(2)} \\ U_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1.0000 \\ -1.6258 \end{Bmatrix}$$

For a mesh of one quadratic element, the condensed equations are

$$\left(\frac{1}{3}\begin{bmatrix} 16 & -8 \\ -8 & 7+3H \end{bmatrix} - \lambda \frac{1}{30}\begin{bmatrix} 16 & 2 \\ 2 & 4 \end{bmatrix} \right) \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

and the characteristic equation is

$$15\bar{\lambda}^2 - 4(13+3H)\bar{\lambda} + 12(1+H) = 0, \quad \bar{\lambda} = \frac{\lambda}{10}$$

The two roots of this quadratic equation for $H = 1$ are

$$\bar{\lambda}_{1,2} = \frac{32 \pm \sqrt{(32)^2 - 15 \times 24}}{15} \rightarrow \bar{\lambda}_1 = 0.41545, \quad \bar{\lambda}_2 = 3.85121$$

and the eigenvalues are ($\lambda = 12\bar{\lambda}$)

$$\lambda_1 = 4.1545, \quad \lambda_2 = 38.5121$$

The corresponding eigenvector components are (not normalized)

$$\{U^{(1)}\} = \begin{Bmatrix} U_2^{(1)} \\ U_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1.0000 \\ 1.0591 \end{Bmatrix}$$

$$\{U^{(2)}\} = \begin{Bmatrix} U_2^{(2)} \\ U_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1.0000 \\ -2.9052 \end{Bmatrix}$$

The exact eigenfunctions for Set 2 boundary conditions are

$$U_n(x) = \sin\sqrt{\lambda_n}\,x \tag{6.1.35a}$$

and the eigenvalues λ_n are computed from the equation

$$H \sin\sqrt{\lambda_n} + \sqrt{\lambda_n}\cos\sqrt{\lambda_n} = 0 \tag{6.1.35b}$$

The first two roots of the transcendental equation (6.1.35*b*) are (for $H = 1$)

$$\sqrt{\lambda_1} = 2.0288 \rightarrow \lambda_1 = 4.1160; \quad \sqrt{\lambda_2} = 4.9132 \rightarrow \lambda_2 = 24.1393$$

A comparison of the eigenvalues obtained using meshes of linear and quadratic elements with the exact values is presented in Table 6.1.1. Note that the number of eigenvalues obtained in

Table 6.1.1 Eigenvalues of the heat conduction equations (6.1.31*a*) and (6.1.31*b*) for two sets of boundary conditions.

Mesh	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
2L	12.0000						
	(4.4900)*	(36.6529)					
4L	10.3866	48.0000	126.756				
	(4.2054)	(27.3318)	(85.7864)	(177.604)			
8L	9.9971	41.5466	99.4855	192.000	328.291	507.025	686.512
	(4.1380)	(24.9088)	(69.1036)	(143.530)	(257.580)	(417.701)	(607.022)
1Q	10.000						
	(4.1545)	(38.5121)					
2Q	9.9439	40.000	128.723				
	(4.1196)	(24.8995)	(81.4446)	(207.653)			
4Q	9.8747	39.7754	91.7847	160.000	308.253	514.891	794.794
	(4.1161)	(24.2040)	(64.7704)	(129.261)	(240.540)	(405.254)	(658.133)
Exact	9.8696	39.4784	88.8264	157.9137	246.740	355.306	483.611
	(4.1160)	(24.1393)	(63.6597)	(122.889)	(201.851)	(300.550)	(418.987)

* The second line for each mesh corresponds to set 2 boundary conditions.

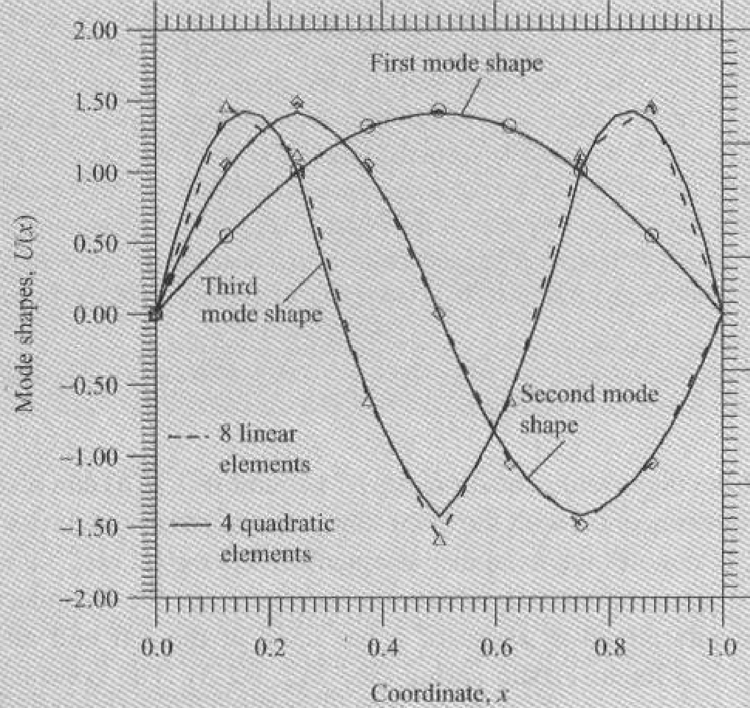

Figure 6.1.1 The first three mode shapes as predicted by a mesh of eight linear elements and a mesh of four quadratic elements for the heat transfer problem of Example 6.1.1 (Set 1 boundary conditions).

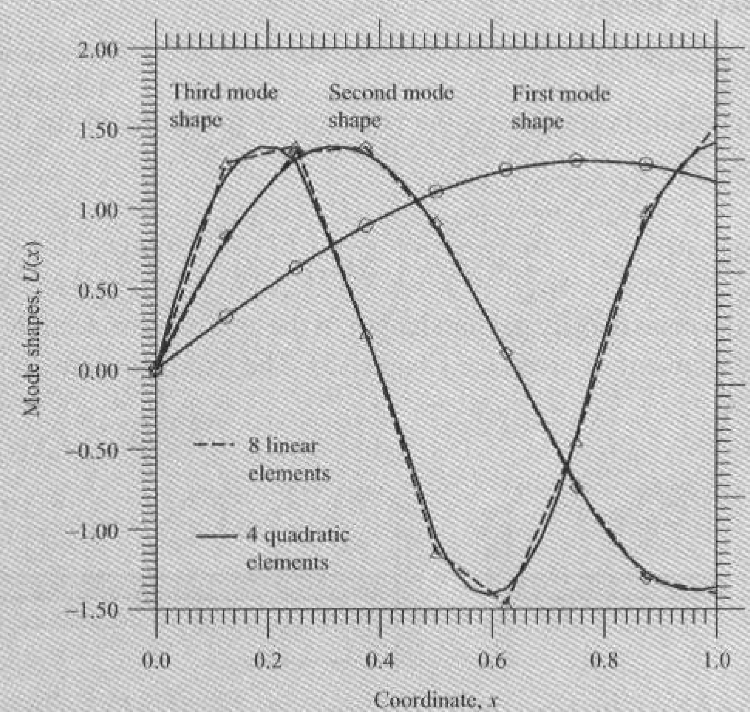

Figure 6.1.2 The first three mode shapes of the heat transfer problem of Example 6.1.1 (Set 2 boundary conditions).

the finite element method is always equal to the number of unknown nodal values. As the mesh is refined, not only do we increase the number of eigenvalues but we also improve the accuracy of the preceding eigenvalues. Note also that the convergence of the numerical eigenvalues to the exact ones is from the above, i.e., the finite element solution provides an upper bound to the exact eigenvalues. For structural systems, this can be interpreted as follows: According to the principle of minimum total potential energy, any approximate displacement field would overestimate the total potential energy of the system. This is equivalent to approximating the stiffness of the system with a larger value than the actual one. A stiffer system will have larger eigenvalues (or frequencies). The first three mode shapes of the system are shown in Figs. 6.1.1 and 6.1.2.

We note that the eigenvalue equations (6.1.31*a*) and (6.1.31*b*) can also be interpreted as those arising in connection with the axial vibrations of a constant cross sectional member. In that case, U denotes the axial displacement $[u(x,t) = U(x)e^{-i\omega t}]$ and $\lambda = \omega^2\rho/E$, ω being the frequency of natural vibration. For example, the boundary conditions in (6.1.31*b*) can be interpreted as those of a bar fixed at the left end and the right end being connected to a linear elastic spring (see Fig. 6.1.3). The constant H is equal to k/EA, k being the spring constant. Thus, the results presented in Table 6.1.1 can be interpreted as $\omega^2\rho/E$ of an uniform bar for the two different boundary conditions shown in Fig. 6.1.3.

Natural Vibration of Beams

Euler–Bernoulli Beam Theory

For the Euler–Bernoulli beam theory, the equation of motion is of the form [see Reddy (2002), pp. 193–196]

$$\rho A\frac{\partial^2 w}{\partial t^2} - \rho I\frac{\partial^4 w}{\partial t^2 \partial x^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = q(x,t) \tag{6.1.36}$$

where ρ denotes the mass density per unit length, A the area of cross section, E the modulus, and I the second moment of area (see Chapter 5). The expression involving ρI is called

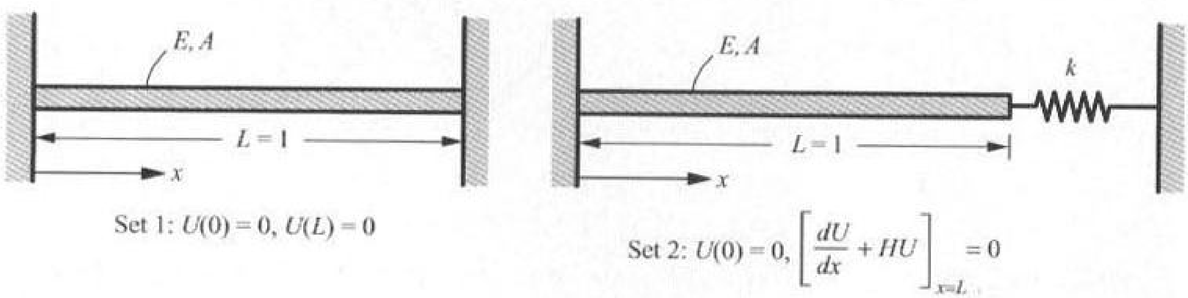

Figure 6.1.3 Uniform bar with two sets of boundary conditions.

rotary inertia term. Equation (6.1.36) can be formulated as an eigenvalue problem in the interest of finding the frequency of natural vibration by assuming periodic motion

$$w(x,t) = W(x)e^{-i\omega t} \tag{6.1.37}$$

where ω is the frequency of natural transverse motion and $W(x)$ is the mode shape of the transverse motion. Substitution of Eq. (6.1.37) into (6.1.36) yields

$$\frac{d^2}{dx^2}\left(EI\frac{d^2W}{dx^2}\right) - \lambda\left(\rho AW - \rho I\frac{d^2W}{dx^2}\right) = 0 \tag{6.1.38}$$

where $\lambda = \omega^2$.

The weak form of Eq. (6.1.38) is given by

$$\begin{aligned} 0 &= \int_{x_a}^{x_b}\left(EI\frac{d^2v}{dx^2}\frac{d^2W}{dx^2} - \lambda\rho AvW - \lambda\rho I\frac{dv}{dx}\frac{dW}{dx}\right)dx \\ &\quad + \left\{v\left[\frac{d}{dx}\left(EI\frac{d^2W}{dx^2}\right) + \lambda\rho I\frac{dW}{dx}\right]\right\}_{x_a}^{x_b} + \left[\left(-\frac{dv}{dx}\right)EI\frac{d^2W}{dx^2}\right]_{x_a}^{x_b} \end{aligned} \tag{6.1.39}$$

where v is the weight function. Note that the rotary inertia term contributes to the shear force term, giving rise to an effective shear force that must be known at a boundary point when the deflection is unknown at the point.

To obtain the finite element model of Eq. (6.1.39), assume finite element approximation of the form

$$W(x) = \sum_{j=1}^{4}\Delta_j^e\phi_j^e(x) \tag{6.1.40}$$

where ϕ_j^e are the Hermite cubic polynomials [see Eqs. (5.2.11) and (5.2.12)], and obtain the finite element model

$$\left([K^e] - \omega^2[M^e]\right)\{\Delta^e\} = \{Q^e\} \tag{6.1.41a}$$

where

$$K_{ij}^e = \int_{x_a}^{x_b} EI\frac{d^2\phi_i^e}{dx^2}\frac{d^2\phi_j^e}{dx^2}\,dx, \quad M_{ij}^e = \int_{x_a}^{x_b}\left(\rho A\,\phi_i^e\,\phi_j^e + \rho I\,\frac{d\phi_i^e}{dx}\frac{d\phi_j^e}{dx}\right)dx \tag{6.1.41b}$$

$$\begin{aligned} Q_1^e &\equiv \left[\frac{d}{dx}\left(EI\frac{d^2W}{dx^2}\right) + \lambda\rho I\frac{dW}{dx}\right]\Bigg|_{x_a}, \quad Q_2^e \equiv \left(EI\frac{d^2W}{dx^2}\right)\Bigg|_{x_a} \\ Q_3^e &\equiv -\left[\frac{d}{dx}\left(EI\frac{d^2W}{dx^2}\right) + \lambda\rho I\frac{dW}{dx}\right]\Bigg|_{x_b}, \quad Q_4^e \equiv -\left(EI\frac{d^2W}{dx^2}\right)\Bigg|_{x_b} \end{aligned} \tag{6.1.41c}$$

For constant values of EI and ρA, the stiffness matrix $[K^e]$ and mass matrix $[M^e]$ are

$$[K^e] = \frac{2E_eI_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}$$

$$[M^e] = \frac{\rho^e A_e h_e}{420}\begin{bmatrix} 156 & -22h_e & 54 & 13h_e \\ -22h_e & 4h_e^2 & -13h_e & -3h_e^2 \\ 54 & -13h_e & 156 & 22h_e \\ 13h & -3h_e^2 & 22h_e & 4h_e^2 \end{bmatrix}$$
$$+ \frac{\rho^e I_e}{30h_e}\begin{bmatrix} 36 & -3h_e & -36 & -3h_e \\ -3h & 4h_e^2 & 3h_e & -h_e^2 \\ -36 & 3h_e & 36 & 3h_e \\ -3h_e & -h_e^2 & 3h_e & 4h_e^2 \end{bmatrix} \tag{6.1.42}$$

When rotary inertia is neglected, we omit the second part of the mass matrix in (6.1.42).

Timoshenko Beam Theory

The equations of motion of the Timoshenko beam theory are [see Reddy (2002), pp. 193–196]

$$\rho A \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left[GAK_s\left(\frac{\partial w}{\partial x} + \Psi\right)\right] = 0 \tag{6.1.43a}$$

$$\rho I \frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial}{\partial x}\left(EI\frac{\partial \Psi}{\partial x}\right) + GAK_s\left(\frac{\partial w}{\partial x} + \Psi\right) = 0 \tag{6.1.43b}$$

where G is the shear modulus ($G = E/[2(1+\nu)]$) and K_s is the shear correction factor ($K_s = 5/6$). Note that Eq. (6.1.43*b*) contains the rotary inertia term. Once again, we assume periodic motion and write

$$w(x,t) = W(x)e^{-i\omega t}, \quad \Psi(x,t) = S(x)e^{-i\omega t} \tag{6.1.44}$$

and obtain the eigenvalue problem from Eqs. (6.1.43*a*) and (6.1.43*b*)

$$-\frac{d}{dx}\left[GAK_s\left(\frac{dW}{dx} + S\right)\right] - \omega^2 \rho A W = 0 \tag{6.1.45a}$$

$$-\frac{d}{dx}\left(EI\frac{dS}{dx}\right) + GAK_s\left(\frac{dW}{dx} + S\right) - \omega^2 \rho I\, S = 0 \tag{6.1.45b}$$

For equal interpolation of $W(x)$ and $S(x)$

$$W(x) = \sum_{j=1}^{n} W_j^e \psi_j^e(x), \quad S(x) = \sum_{j=1}^{n} S_j^e \psi_j^e(x) \tag{6.1.46}$$

where ψ_j^e are the $(n-1)$ order Lagrange polynomials, the finite element model is given by

$$\left(\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} - \omega^2 \begin{bmatrix} [M^{11}] & [0] \\ [0] & [M^{22}] \end{bmatrix}\right)\begin{Bmatrix} \{W\} \\ \{S\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix} \tag{6.1.47}$$

where $[K^e]$ is the stiffness matrix and $[M^e]$ is the mass matrix

$$K_{ij}^{11} = \int_{x_a}^{x_b} GAK_s \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\,dx$$

$$K_{ij}^{12} = \int_{x_a}^{x_b} GAK_s \frac{d\psi_i^e}{dx}\psi_j^e\, dx = K_{ji}^{21}$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left(EI \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + GAK_s \psi_i^e \psi_j^e \right) dx$$

$$M_{ij}^{11} = \int_{x_a}^{x_b} \rho A \psi_i^e \psi_j^e \, dx, \quad M_{ij}^{22} = \int_{x_a}^{x_b} \rho I \psi_i^e \psi_j^e \, dx$$

$$F_j^1 = Q_{2i-1}, \quad F_i^2 = Q_{2i} \tag{6.1.48a}$$

$$Q_1^e \equiv -\left[GAK_s \left(S + \frac{dW}{dx} \right) \right] \bigg|_{x_a}, \quad Q_2^e \equiv -\left(EI \frac{dS}{dx} \right) \bigg|_{x_a}$$

$$Q_3^e \equiv \left[GAK_s \left(S + \frac{dW}{dx} \right) \right] \bigg|_{x_b}, \quad Q_4^e \equiv \left(EI \frac{dS}{dx} \right) \bigg|_{x_b} \tag{6.1.48b}$$

For the choice of linear interpolation functions, Eq. (6.1.47) has the explicit form (after rearranging the nodal variables)

$$[K^e] = \left(\frac{2E_e I_e}{\mu_0 h_e^3} \right) \begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & h_e^2(1.5 + 6\Lambda_e) & 3h_e & h_e^2(1.5 - 6\Lambda_e) \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2(1.5 - 6\Lambda_e) & 3h_e & h_e^2(1.5 + 6\Lambda_e) \end{bmatrix}$$

$$[M^e] = \frac{\rho^e A_e}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2r_e & 0 & r_e \\ 1 & 0 & 2 & 0 \\ 0 & r_e & 0 & 2r_e \end{bmatrix}, \quad r_e = \frac{I_e}{A_e} \tag{6.1.49a}$$

$$\Lambda_e = \frac{E_e I_e}{G_e A_e K_s h_e^2}, \quad \mu_0 = 12\Lambda_e \tag{6.1.49b}$$

For Hermite cubic interpolation of $W(x)$ and related quadratic approximation of $S(x)$ [i.e., *interdependent interpolation element* (IIE)], the resulting mass matrix is cumbersome and depends on the stiffness parameters (E and G). We will not consider it here [see Reddy (1999*b*, 2000)].

Example 6.1.2

Consider a uniform beam of rectangular cross section ($B \times H$), fixed at $x = 0$ and free at $x = L$ (i.e., a cantilever beam). We wish to determine the first four flexural frequencies of the beam.

The boundary conditions of the structure are

$$W(0) = 0, \quad \frac{dW}{dx} = 0, \quad \left[EI \frac{d^2 W}{dx^2} \right]_{x=L} = 0, \quad \left[EI \frac{d^3 W}{dx^3} \right]_{x=L} = 0 \tag{6.1.50a}$$

in the Euler–Bernoulli beam theory and

$$W(0) = 0, \quad S(0) = 0, \quad \left[EI \frac{dS}{dx} \right]_{x=L} = 0, \quad \left[GAK_s \left(\frac{dW}{dx} + S \right) \right]_{x=L} = 0 \tag{6.1.50b}$$

in the Timoshenko beam theory. The first two terms in each case are the essential (or geometric) boundary conditions and the last two are the natural (or force) boundary conditions.

The number of eigenvalues we wish to determine dictates the minimum number of elements to be used to analyze the structure. First, consider the Euler–Bernoulli beam element. Since two primary degrees of freedom are specified ($U_1 = U_2 = 0$), a one-element mesh will have only two unknown degrees of freedom (U_3 and U_4). Hence, we can determine only two natural frequencies. Thus, a minimum of two Euler–Bernoulli beam elements are needed to determine four natural frequencies. If we use the reduced integration Timoshenko beam element, a mesh of two linear elements can only represent the first two mode shapes of the cantilever beam (see Fig. 6.1.4) because there are only two independent deflection degrees of freedom that are unspecified. In order to represent the first four mode shapes using the *reduced integration element* (RIE), we must use at least four linear elements or two quadratic elements. However, the four computed frequencies may not be the lowest four.

For illustrative purpose, first we consider the one-element mesh of the Euler–Bernoulli beam element. We have

$$\left(\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix} - \omega^2\frac{\rho AL}{420}\begin{bmatrix} 156 & -22L & 54 & 13L \\ -22L & 4L^2 & -13L & -3L^2 \\ 54 & -13L & 156 & 22L \\ 13L & -3L^2 & 22L & 4L^2 \end{bmatrix}\right.$$

$$\left. -\omega^2\frac{\rho I}{30L}\begin{bmatrix} 36 & -3L & -36 & -3L \\ -3h & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix}\right)\begin{Bmatrix} W_1 \\ \Theta_1 \\ W_2 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

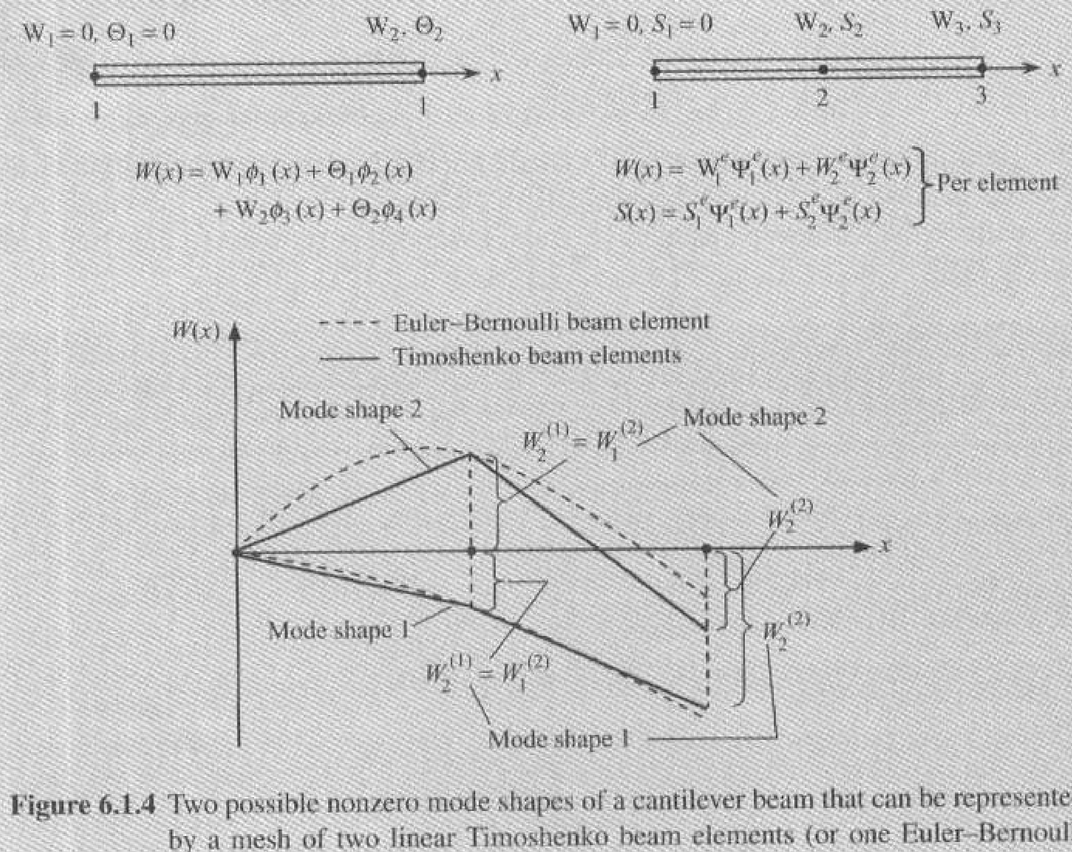

Figure 6.1.4 Two possible nonzero mode shapes of a cantilever beam that can be represented by a mesh of two linear Timoshenko beam elements (or one Euler–Bernoulli beam element).

The specified displacement and force degrees of freedom are $W_1=0$, $\Theta_1=0$, $Q_3=0$, and $Q_4=0$. Hence, the condensed equations are

$$\left(\frac{2EI}{L^3}\begin{bmatrix}6 & 3L\\ 3L & 2L^2\end{bmatrix}-\omega^2\frac{\rho AL}{420}\begin{bmatrix}156 & 22L\\ 22L & 4L^2\end{bmatrix}-\omega^2\frac{\rho I}{30L}\begin{bmatrix}36 & 3L\\ 3L & 4L^2\end{bmatrix}\right)\begin{Bmatrix}W_2\\ \Theta_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Setting the determinant of the coefficient matrix in the above equation to zero, we obtain a quadratic characteristic equation in ω^2.

First, consider the case in which the rotary inertia is neglected. The characteristic equation for this case is

$$15120-1224\lambda+\lambda^2=0,\quad \lambda=\frac{\rho AL^4}{EI}\omega^2$$

whose roots are

$$\lambda_1=12.48\ \lambda_2=1211.52,\quad \text{or}\ \omega_1=\frac{3.5327}{L^2}\sqrt{\frac{EI}{\rho A}},\quad \omega_2=\frac{34.8069}{L^2}\sqrt{\frac{EI}{\rho A}}$$

The exact frequencies are $\bar{\omega}_1=3.5160$ and $\bar{\omega}_2=22.0345$, where $\bar{\omega}_i=\omega_i(L^2\sqrt{\rho A/EI})$. The eigenvector components are computed from the first of the condensed equations

$$W_2^{(i)}=-L\left(\frac{1260-11\lambda_i}{420-13\lambda_i}\right)\Theta_2^{(i)},\quad i=1,2$$

Hence, the eigenvectors are

$$\begin{Bmatrix}W_2\\ \Theta_2\end{Bmatrix}^{(1)}=\begin{Bmatrix}0.7259L\\ -1.0000\end{Bmatrix},\quad \begin{Bmatrix}W_2\\ \Theta_2\end{Bmatrix}^{(2)}=\begin{Bmatrix}0.1312L\\ -1.0000\end{Bmatrix}$$

For the case in which rotary inertia is not neglected, we write $\rho I=\rho A\,(H^2/12)$ and take $H/L=0.01$. The characteristic equation is

$$15112-1223.406\lambda+\lambda^2=0,\quad \lambda=\frac{\rho AL^4}{EI}\omega^2$$

whose roots are

$$\lambda_1=12.4797,\quad \lambda_2=1210.926\ \text{or}\ \omega_1=\frac{3.5327}{L^2}\sqrt{\frac{EI}{\rho A}},\quad \omega_2=\frac{34.7984}{L^2}\sqrt{\frac{EI}{\rho A}}$$

The exact frequencies are $\bar{\omega}_1=3.5158$ and $\bar{\omega}_2=22.0226$, where $\bar{\omega}_i=\omega_i(L^2\sqrt{\rho A/EI})$.

Next, we consider the one-element mesh of the Timoshenko beam element (RIE). We have

$$\left(\frac{2EI}{\mu_0L^3}\begin{bmatrix}6 & -3L & -6 & -3L\\ -3L & L^2\Xi_1 & 3L & L^2\Xi_2\\ -6 & 3L & 6 & 3L\\ -3L & L^2\Xi_2 & 3L & L^2\Xi_1\end{bmatrix}-\omega^2\frac{\rho AL}{6}\begin{bmatrix}2 & 0 & 1 & 0\\ 0 & 2r & 0 & r\\ 1 & 0 & 2 & 0\\ 0 & r & 0 & 2r\end{bmatrix}\right)\begin{Bmatrix}W_1\\ S_1\\ W_2\\ S_2\end{Bmatrix}=\begin{Bmatrix}Q_1\\ Q_2\\ Q_3\\ Q_4\end{Bmatrix}$$

where $K_s=5/6$, $r=I/A=H^2/12$, and

$$\Xi_1=1.5+6\Lambda,\quad \Xi_2=1.5-6\Lambda,\quad \Lambda=\frac{EI}{GAK_sL^2}=\frac{1+\nu}{5}\frac{H^2}{L^2},\quad \mu_0=12\Lambda$$

The specified displacement and force degrees of freedom are $W_1=0$, $S_1=0$, $Q_3=0$, and $Q_4=0$; and the condensed equations are

$$\left(\frac{2EI}{\mu_0L^3}\begin{bmatrix}6 & 3L\\ 3L & L^2(1.5+6\Lambda)\end{bmatrix}-\omega^2\frac{\rho AL}{6}\begin{bmatrix}2 & 0\\ 0 & 2r\end{bmatrix}\right)\begin{Bmatrix}W_2\\ S_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Setting the determinant of the coefficient matrix to zero, we obtain the characteristic equation

$$\bar{\lambda}^2 - 3\left(1 + 3s^2 + 12s^2\Lambda\right)\bar{\lambda} + 108s^2 = 0, \quad \bar{\lambda} = \frac{\rho A L^4 \Lambda}{EI}\omega^2$$

where s is the length-to-height ratio, $s = L/H$.

When the rotary inertia is neglected, we have

$$\bar{\lambda} = \frac{12\Lambda}{1 + 4\Lambda}$$

To further simplify the expression, we assume $\nu = 0.25$ and obtain

$$\omega^2 = \frac{12}{(1 + H^2/L^2)}\frac{EI}{\rho A L^4}$$

and

$$\frac{L}{H} = 100: \quad \omega = \frac{3.4639}{L^2}\sqrt{\frac{EI}{\rho A}}; \qquad \frac{L}{H} = 10: \quad \omega = \frac{3.4469}{L^2}\sqrt{\frac{EI}{\rho A}}$$

The exact frequency is $\bar{\omega} = 3.5158$ for $L/H = 100$ and $\bar{\omega} = 3.5092$ for $L/H = 10$, where $\bar{\omega}_i = \omega_i(L^2\sqrt{\rho A/EI})$. If the rotary inertia is included, we obtain

$$\frac{L}{H} = 100: \quad \omega = \frac{3.4639}{L^2}\sqrt{\frac{EI}{\rho A}}; \qquad \frac{L}{H} = 10: \quad \omega = \frac{3.4413}{L^2}\sqrt{\frac{EI}{\rho A}}$$

The frequencies obtained by the Euler–Bernoulli beam elements and Timoshenko beam elements (RIE) are shown in Table 6.1.2 for two different values of the length-to-height ratio L/H. The value $L/H = 100$ makes the effect of shear deformation negligible, and we obtain essentially the same result as in the Euler–Bernoulli beam theory. Since the frequencies are normalized, $\bar{\omega} = \omega L^2\sqrt{(\rho A/EI)}$, it is necessary only to select the values of ν and L/H. For

Table 6.1.2 Natural frequencies of a cantilever beam according to Timoshenko beam theory (TBT) and Euler–Bernoulli beam theory (EBT) $[\bar{\omega} = \omega L^2(\rho A/EI)^{1/2}]$.

	$L/H = 100$				$L/H = 10$			
Mesh	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$
4L	3.5406	25.6726	98.3953	417.1330	3.5137	24.1345	80.2244	189.9288
8L	3.5223	22.8851	68.8937	151.8431	3.4956	21.7004	60.6297	119.2798
16L	3.5174	22.2350	63.3413	127.5434	3.4908	21.1257	56.4714	104.6799
2Q	3.5214	23.3226	78.3115	328.3250	3.4947	22.0762	67.0884	181.0682
4Q	3.5161	22.1054	63.3271	133.9828	3.4895	21.0103	56.4572	108.6060
8Q	3.5158	22.0280	61.7325	121.4458	3.4892	20.4421	55.2405	100.7496
EBT†	3.5158	22.0226	61.6179	120.6152	3.4892	20.9374	55.1530	100.2116
TBT†	3.5158	22.0315	61.6774	120.8300	3.5092	21.7425	59.8013	114.2898
EBT‡	3.5160	22.0345	61.6972	120.9019	3.5160	22.0345	61.6972	120.9019

† Rotary inertia is included (Exact).

‡ Rotary inertia is neglected; the results are independent of the ratio L/H (Exact).

computational purposes, we take $\nu = 0.25$; also, note that

$$GAK = \frac{E}{2(1+\nu)} BH\frac{5}{6} = \frac{4EI}{H^2}, \quad \rho I = \frac{H^2 \rho A}{12}$$

where B and H are the width and height, respectively, of the beam.

We note from Table 6.1.2 that the finite element results converge with h-refinement (i.e., when more of the same kind of elements are used) and also with p-refinement (i.e., when higher-order elements are used). The p-refinement shows more rapid convergence of the fundamental (i.e., first) frequency. Note that the effect of shear deformation is to reduce the magnitude of natural frequency when compared to the EBT. In other words, the assumed infinite shear rigidity makes the EBT over-predict the magnitude of frequencies. The first four mode shapes of the cantilever beam, as obtained using the 16-element mesh of linear elements, are shown in Fig. 6.1.5.

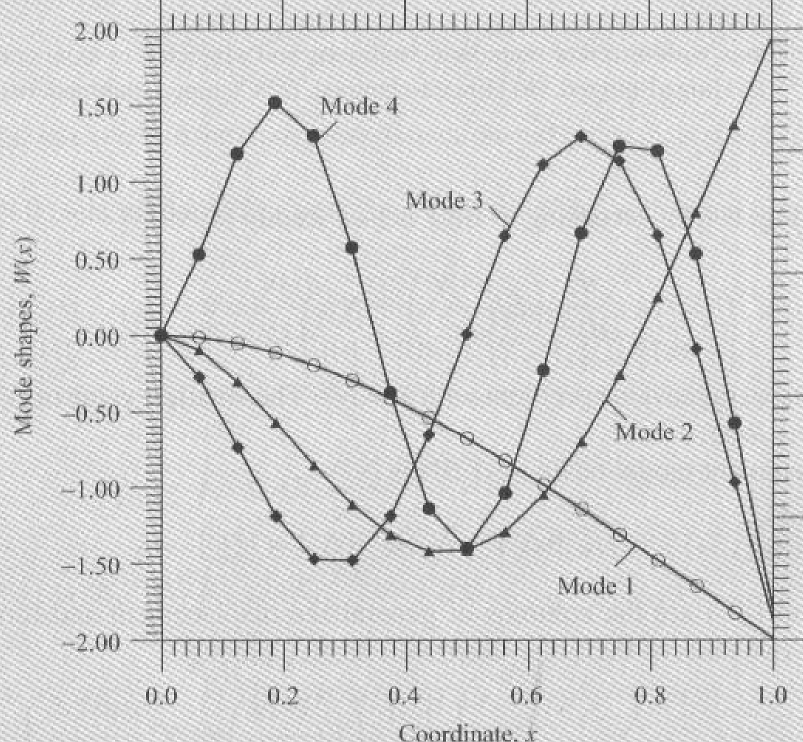

Figure 6.1.5 First four natural mode shapes of a cantilever beam, as predicted using a 16-element mesh of linear Timoshenko beam elements ($L/H = 10$).

In closing the discussion on natural vibration, it is noted that when the symmetry of the system is used to model the problem, only symmetric modes are predicted. It is necessary to model the whole system in order to obtain all the modes of vibration.

Stability (Buckling) of Beams

EBT

The study of buckling of beam-columns also leads to an eigenvalue problem. For example, the equation governing onset of buckling of a column subjected to an axial compressive

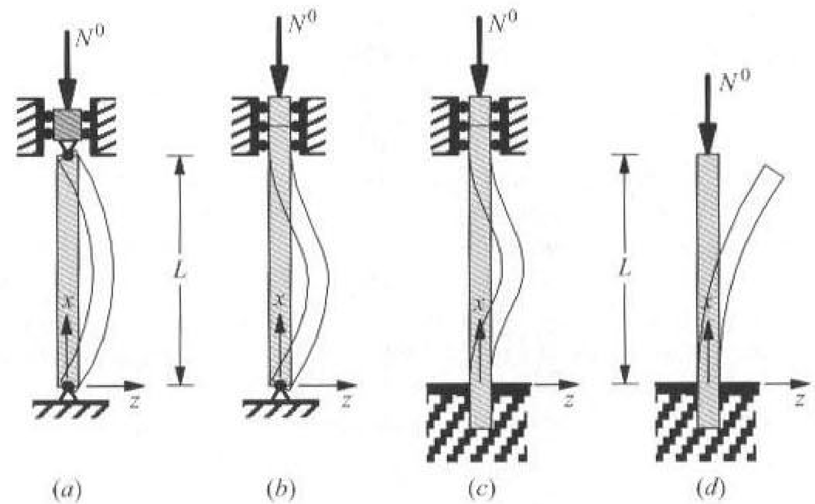

Figure 6.1.6 Columns with different boundary conditions and subjected to axial compressive load, N^0. (*a*) Hinged-hinged. (*b*) Hinged-clamped. (*c*) Clamped-clamped. (*d*) Clamped-free.

force N^0 (see Fig. 6.1.6), according to EBT is [see Reddy (1999*b*)]

$$\frac{d^2}{dx^2}\left(EI\frac{d^2W}{dx^2}\right) + N^0\frac{d^2W}{dx^2} = 0 \tag{6.1.51}$$

which describes an eigenvalue problem with $\lambda = N^0$. The smallest value of N^0 is called the *critical buckling load*.

The finite element model of Eq. (6.1.51) is

$$[K^e]\{\Delta^e\} - N^0[G^e]\{\Delta^e\} = \{Q^e\} \tag{6.1.52a}$$

where $\{\Delta^e\}$ and $\{Q^e\}$ are the columns of generalized displacement and force degrees of freedom at the two ends of the Euler–Bernoulli beam element:

$$\{\Delta\}^e = \begin{Bmatrix} W_1 \\ \left(-\dfrac{dW}{dx}\right)_1 \\ W_2 \\ \left(-\dfrac{dW}{dx}\right)_2 \end{Bmatrix}^e, \quad \{Q\}^e = \begin{Bmatrix} \left[\dfrac{d}{dx}\left(EI\dfrac{d^2W}{dx^2}\right) + N^0\dfrac{dW}{dx}\right]_1 \\ \left(EI\dfrac{d^2W}{dx^2}\right)_1 \\ \left[-\dfrac{d}{dx}\left(EI\dfrac{d^2W}{dx^2}\right) - N^0\dfrac{dW}{dx}\right]_2 \\ \left(-EI\dfrac{d^2W}{dx^2}\right)_2 \end{Bmatrix}^e \tag{6.1.52b}$$

where the subscripts 1 and 2 refer to element nodes 1 and 2 (at $x = x_a$ and $x = x_b$, respectively). The coefficients of the stiffness matrix $[K^e]$ and the *stability matrix* $[G^e]$ are

$$K_{ij}^e = \int_{x_a}^{x_b} EI\frac{d^2\phi_i^e}{dx^2}\frac{d^2\phi_j^e}{dx^2}\,dx, \quad G_{ij}^e = \int_{x_a}^{x_b} \frac{d\phi_i^e}{dx}\frac{d\phi_j^e}{dx}\,dx \tag{6.1.52c}$$

where ϕ_i^e are the Hermite cubic interpolation functions. The explicit form of $[G^e]$ is [see the second part of the mass matrix in Eq. (6.1.42)]

$$[G^e] = \frac{1}{30h_e}\begin{bmatrix} 36 & -3h_e & -36 & -3h_e \\ -3h_e & 4h_e^2 & 3h_e & -h_e^2 \\ -36 & 3h_e & 36 & 3h_e \\ -3h_e & -h_e^2 & 3h_e & 4h_e^2 \end{bmatrix} \tag{6.1.52d}$$

TBT

For TBT, the equations governing buckling of beams are [see Reddy (1999*b*)]

$$-\frac{d}{dx}\left[GAK_s\left(\frac{dW}{dx}+S\right)\right] + N^0\frac{d^2W}{dx^2} = 0 \tag{6.1.53a}$$

$$-\frac{d}{dx}\left(EI\frac{dS}{dx}\right) + GAK_s\left(\frac{dW}{dx}+S\right) = 0 \tag{6.1.53b}$$

Here $W(x)$ and $S(x)$ denote the transverse deflection and rotation, respectively, at the onset of buckling.

The finite element model of Eqs. (6.1.53*a*) and (6.1.53*b*), with equal interpolation of W and S, is

$$\left(\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{12}]^{\mathrm{T}} & [K^{22}] \end{bmatrix} - N^0\begin{bmatrix} [G] & [0] \\ [0] & [0] \end{bmatrix}\right)\begin{Bmatrix} \{W^e\} \\ \{S^e\} \end{Bmatrix} = \begin{Bmatrix} \{V^e\} \\ \{M^e\} \end{Bmatrix} \tag{6.1.54a}$$

where

$$K_{ij}^{11} = \int_{x_a}^{x_b} GAK_s \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\,dx$$

$$K_{ij}^{12} = \int_{x_a}^{x_b} GAK_s \frac{d\psi_i^e}{dx}\psi_j^e\,dx$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left(GAK_s\psi_i^e\psi_j^e + EI\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\right)dx$$

$$G_{ij} = \int_{x_a}^{x_b} \psi_i^e\psi_j^e\,dx$$

$$V_1^e \equiv Q_1^e = \left[-GAK_s\left(\frac{dW}{dx}+S\right) + N^0\frac{dW}{dx}\right]_{x_a}$$

$$V_n^e \equiv Q_3^e = \left[GAK_s\left(\frac{dW}{dx}+S\right) - N^0\frac{dW}{dx}\right]_{x_b}$$

$$M_1^e \equiv Q_2^e = \left(-EI\frac{dS}{dx}\right)_{x_a}$$

$$M_n^e \equiv Q_4^e = \left(-EI\frac{dS}{dx}\right)_{x_b} \tag{6.1.54b}$$

For the linear RIE, the stiffness matrix $[K^e]$ is given in Eq. (6.1.49a). The stability matrix is given by

$$[G^e] = \frac{1}{h_e}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.1.55}$$

Example 6.1.3

Consider a uniform column (L, A, I, E, and ν) fixed at one end and pinned at the other, as shown in Fig. 6.1.7. We wish to determine the critical buckling load.

Let us consider a mesh of two Euler–Bernoulli beam elements. The assembled system of finite element equations is ($h = L/2$)

$$\left(\frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h & 0 & 0 \\ -3h & 2h^2 & 3h & h^2 & 0 & 0 \\ -6 & 3h & 6+6 & 3h-3h & -6 & -3h \\ -3h & h^2 & 3h-3h & 2h^2+2h^2 & 3h & h^2 \\ 0 & 0 & -6 & 3h & 6 & 3h \\ 0 & 0 & -3h & h^2 & 3h & 2h^2 \end{bmatrix} - \frac{N^0}{30h}\begin{bmatrix} 36 & -3h & -36 & -3h & 0 & 0 \\ -3h & 4h^2 & 3h & -h^2 & 0 & 0 \\ -36 & 3h & 36+36 & 3h-3h & -36 & -3h \\ -3h & -h^2 & 3h-3h & 4h^2+4h^2 & 3h & -h^2 \\ 0 & 0 & -36 & 3h & 36 & 3h \\ 0 & 0 & -3h & -h^2 & 3h & 4h^2 \end{bmatrix}\right)\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

where U_I denote the generalized displacements of the global nodes.

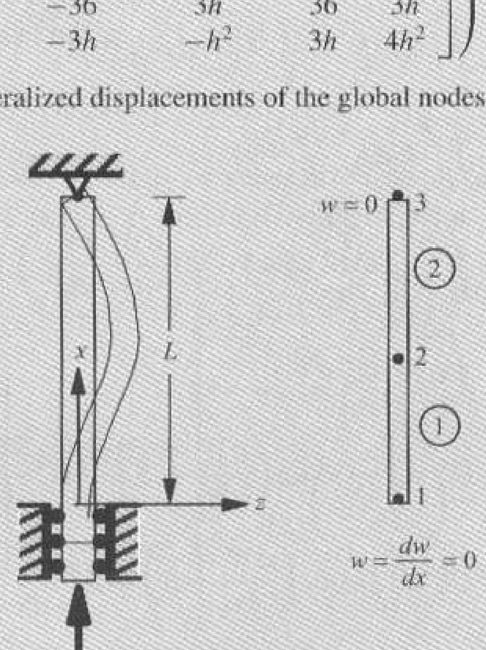

Figure 6.1.7 A column fixed at $x = 0$ and hinged at $x = L$, and subjected to axial compressive load N^0.

The geometric supports and equilibrium conditions of the column require that (note that the axial compressive force N^0 enters the equilibrium equation and the boundary condition through Q_3^2)

$$U_1 = U_2 = U_5 = 0, \quad Q_3^1 + Q_1^2 = 0, \quad Q_4^1 + Q_2^2 = 0, \quad Q_4^2 = 0$$

Hence, the condensed equations become

$$\left(\frac{2EI}{h^3} \begin{bmatrix} 12 & 0 & -3h \\ 0 & 4h^2 & h^2 \\ -3h & h^2 & 2h^2 \end{bmatrix} - \frac{N^0}{30h} \begin{bmatrix} 72 & 0 & -3h \\ 0 & 8h^2 & -h^2 \\ -3h & -h^2 & 4h^2 \end{bmatrix} \right) \begin{Bmatrix} U_3 \\ U_4 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

which defines the eigenvalue problem for the determination of N^0 and U_I.

Setting the determinant of the coefficient matrix to zero, we obtain a cubic equation for N^0. The smallest value of N^0 is the critical buckling load, which is (obtained using FEM1D)

$$N_{\text{cri}}^0 = 20.7088 \frac{EI}{L^2}$$

whereas the "exact" solution is $20.187\, EI/L^2$ [see Reddy (1999*b*), pp. 155–166].

Next, consider a two-element mesh of the RIE. The assembled equations are

$$\left(\frac{2EI}{\mu_0 h^3} \begin{bmatrix} 6 & -3h & -6 & -3h & 0 & 0 \\ -3h & h^2\Xi_1 & 3h & h^2\Xi_2 & 0 & 0 \\ -6 & 3h & 12 & 0 & -6 & -3h \\ -3h & h^2\Xi_2 & 0 & 2h^2\Xi_1 & 3h & h^2\Xi_2 \\ 0 & 0 & -6 & 3h & 6 & 3h \\ 0 & 0 & -3h & h^2\Xi_2 & 3h & h^2\Xi_1 \end{bmatrix} \right.$$

$$\left. - \frac{N^0}{h} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

where $K_s = 5/6$, and

$$\Xi_1 = 1.5 + 6\Lambda, \quad \Xi_2 = 1.5 - 6\Lambda, \quad \mu_0 = 12\Lambda, \quad \Lambda = \frac{EI}{GAK_s h^2} = \frac{1+\nu}{5} \frac{H^2}{h^2}$$

The condensed equations are

$$\left(\frac{2EI}{\mu_0 h^3} \begin{bmatrix} 12 & 0 & -3h \\ 0 & 2h^2\Xi_1 & h^2\Xi_2 \\ -3h & h^2\Xi_2 & h^2\Xi_1 \end{bmatrix} - \frac{N^0}{h} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} U_3 \\ U_4 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Note that because of the zero diagonal element in the stability matrix, we obtain a linear algebraic equation for N^0 ($\nu = 0.25$ and $\Lambda = H^2/L^2$)

$$N^0 = \frac{180 + 144\Lambda}{2.25 + 54\Lambda + 36\Lambda^2} \frac{EI}{L^2}$$

which is almost four times the correct value! A mesh of eight linear RIEs yields $21.348\,EI/L^2$ and four quadratic elements yields $20.267\,EI/L^2$ for $L/H = 100$. Clearly, the RIE has slow convergence rate for buckling problems.

We close this section with a note that eigenvalue problems (for natural vibration and buckling analysis) of frame structures can be formulated using the ideas discussed in Section 5.4. The transformed element equations are of the form

$$[K^e]\{\Delta^e\} - \omega^2[M^e]\{\Delta^e\} = \{Q^e\}, \quad [K^e]\{\Delta^e\} - N^0[G^e]\{\Delta^e\} = \{Q^e\}$$
$$[K^e] = [T^e]^{\mathrm{T}}[\bar{K}^e][T^e], \quad [M^e] = [T^e]^{\mathrm{T}}[\bar{M}^e][T^e], \quad [G^e] = [T^e]^{\mathrm{T}}[\bar{G}^e][T^e] \tag{6.1.56}$$

6.2 TIME-DEPENDENT PROBLEMS

6.2.1 Introduction

In this section, we develop the finite element models of one-dimensional time-dependent problems and describe time approximation schemes to convert ordinary differential equations in time to algebraic equations. We consider finite element models of the time-dependent version of the differential equations studied in Chapters 3 and 5. These include the second-order (in space) parabolic (i.e., first time derivative) and hyperbolic (i.e., second time derivative) equations and fourth-order hyperbolic equations arising in connection with the bending of beams [see Eqs. (6.1.2), (6.1.14), (6.1.36), (6.1.43*a*), and (6.1.43*b*)].

Finite element models of time-dependent problems can be developed in two alternative ways: (*a*) coupled formulation in which the time t is treated as an additional coordinate along with the spatial coordinate x and (*b*) decoupled formulation where time and spatial variations are assumed to be separable. Thus, the approximation in the two formulations takes the form

$$u(x,t) \approx u_h^e(x,t) = \sum_{j=1}^{n} \hat{u}_j^e \hat{\psi}_j^e(x,t) \quad \text{(coupled formulation)} \tag{6.2.1a}$$

$$u(x,t) \approx u_h^e(x,t) = \sum_{j=1}^{n} u_j^e(t) \psi_j^e(x) \quad \text{(decoupled formulation)} \tag{6.2.1b}$$

where $\hat{\psi}_j^e(x,t)$ are time-space (two-dimensional) interpolation functions and $\hat{u}_j$ are the nodal values that are independent of x and t, where $\psi_j^e(x)$ are the usual one-dimensional interpolation functions in spatial coordinate x only and the nodal values $u_j^e(t)$ are functions of time t only. Derivation of the interpolation functions in two coordinates will be discussed in Chapters 8 and 9 in connection with the finite element analysis of two-dimensional problems. Space-time coupled finite element formulations are not common, and they have not been adequately studied. In this section, we consider the space-time decoupled formulation only.

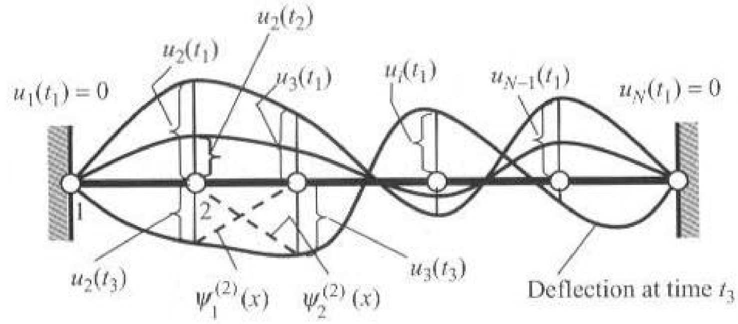

Figure 6.2.1 Deflections of a cable at different times.

The space-time decoupled finite element formulation of time-dependent problems involves two steps:

1. *Spatial approximation,* where the solution u of the equation under consideration is approximated by expressions of the form (6.2.1b), and the spatial finite element model of the equation is developed using the procedures of static or steady-state problems while carrying all time-dependent terms in the formulation. This step results in a set of ordinary differential equations (i.e., a semidiscrete system of equations) in time for the nodal variables $u_j^e(t)$ of the element. Equation (6.2.1b) represents the spatial approximation of u for any time t. When the solution is separable into functions of time only and space only, $u(x, t) = T(t)X(x)$, the approximation (6.2.1b) is clearly justified. Even when the solution is not separable, (6.2.1b) can represent a good approximation of the actual solution provided a sufficiently small time step is used.
2. *Temporal approximation,* where the system of ordinary differential equations are further approximated in time, often using finite difference formulae for the time derivatives. This step allows conversion of the system of ordinary differential equations into a set of algebraic equations among u_j^e at time $t_{s+1} = (s+1)\Delta t$, where Δt is the time increment and s is a nonnegative integer.

All time approximation schemes seek to find u_j at time t_{s+1} using the known values of u_j from previous times:

$$\text{compute} \quad \{u\}_{s+1} \quad \text{using} \quad \{u\}_s, \ \{u\}_{s-1}, \ \ldots$$

Thus, at the end of the two-stage approximation, we have a continuous spatial solution at discrete intervals of time:

$$u(x, t_s) \approx u_h^e(x, t_s) = \sum_{j=1}^{n} u_j^e(t_s)\psi_j^e(x) \quad (s = 0, 1, \ldots) \tag{6.2.2}$$

Note that the approximate solution (6.2.2) has the same form as that in the separation-of-variables technique used to solve boundary value and initial value problems. By taking nodal values to be functions of time, we see that the spatial points in an element take on different values for different times (see Fig. 6.2.1).

We study the details of the two steps by considering a model differential equation that contains both second- and fourth-order spatial derivatives and first- and second-order time derivatives:

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 u}{\partial x^2}\right)+c_0 u+c_1\frac{\partial u}{\partial t}+c_2\frac{\partial^2 u}{\partial t^2}=f(x,t) \tag{6.2.3a}$$

The above equation is subject to appropriate boundary and initial conditions. The boundary conditions are of the form

$$\text{specify} \quad u(x,t) \quad \text{or} \quad -a\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\left(b\frac{\partial^2 u}{\partial x^2}\right)$$

$$\text{specify} \quad \frac{\partial u}{\partial x} \quad \text{or} \quad b\frac{\partial^2 u}{\partial x^2} \tag{6.2.3b}$$

at $x=0$, L, and the initial conditions involve specifying

$$c_2 u(x,0) \quad \text{and} \quad c_2\dot{u}(x,0)+c_1 u(x,0) \tag{6.2.3c}$$

where $\dot{u}\equiv\partial u/\partial t$. Equation (6.2.3$a$) describes, for example, the following physical problems:

(a) Heat transfer and fluid flow: $c_2=0$ and $b=0$
(b) Transverse motion of a cable: $a=T$, $c_0=0$, $b=0$, $c_1=\rho$, $c_2=0$
(c) The longitudinal motion of a rod: $a=EA$, $b=0$; if damping is not considered, $c_1=0$, $c_2=\rho A$
(d) The transverse motion of an Euler–Bernoulli beam: $a=0$, $b=EI$, $c_0=k$, $c_1=0$, $c_2=\rho A$

We will consider these special cases through examples.

6.2.2 Semidiscrete Finite Element Models

The semidiscrete formulation involves approximation of the spatial variation of the dependent variable. The formulation follows essentially the same steps as described in Section 3.2. The first step involves the construction of the weak form of the equation over a typical element. In the second step, we develop the finite element model by seeking approximation of the form in (6.2.1b).

Following the three-step procedure of constructing the weak form of a differential equation, we can develop the weak form of (6.2.3a) over an element. Integration by parts is used once on the first term and twice on the second term to distribute the spatial derivatives equally between the weight function w and the dependent variable u:

$$0=\int_{x_a}^{x_b} w\left[-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 u}{\partial x^2}\right)+c_0 u+c_1\frac{\partial u}{\partial t}+c_2\frac{\partial^2 u}{\partial t^2}-f\right]dx$$

$$=\int_{x_a}^{x_b}\left[\frac{\partial w}{\partial x}a\frac{\partial u}{\partial x}+\frac{\partial^2 w}{\partial x^2}b\frac{\partial^2 u}{\partial x^2}+c_0 wu+c_1 w\frac{\partial u}{\partial t}+c_2 w\frac{\partial^2 u}{\partial t^2}-wf\right]dx$$

$$+\left[w\left[\left(-a\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(b\frac{\partial^2 u}{\partial x^2}\right)\right]+\frac{\partial w}{\partial x}\left(-b\frac{\partial^2 u}{\partial x^2}\right)\right]_{x_a}^{x_b}$$

$$= \int_{x_a}^{x_b} \left(a\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + b\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 u}{\partial x^2} + c_0 wu + c_1 w\frac{\partial u}{\partial t} + c_2 w\frac{\partial^2 u}{\partial t^2} - wf \right) dx$$

$$-\hat{Q}_1 w(x_a) - \hat{Q}_3 w(x_b) - \hat{Q}_2\left(-\frac{\partial w}{\partial x}\right)\bigg|_{x_a} - \hat{Q}_4\left(-\frac{\partial w}{\partial x}\right)\bigg|_{x_b} \tag{6.2.4a}$$

where

$$\hat{Q}_1 = \left[-a\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(b\frac{\partial^2 u}{\partial x^2}\right)\right]_{x_a}, \qquad \hat{Q}_2 = \left[b\frac{\partial^2 u}{\partial x^2}\right]_{x_a}$$

$$\hat{Q}_3 = -\left[-a\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(b\frac{\partial^2 u}{\partial x^2}\right)\right]_{x_b}, \quad \hat{Q}_4 = -\left[b\frac{\partial^2 u}{\partial x^2}\right]_{x_b} \tag{6.2.4b}$$

Next, we assume that u is interpolated by an expression of the form (6.2.1b). Equation (6.2.1b) implies that, at any arbitrarily fixed time $t > 0$, the function u can be approximated by a linear combination of the ψ_j^e and $u_j^e(t)$, with $u_j^e(t)$ being the value of u at time t at the jth node of the element Ω_e. In other words, the time and spatial variations of u are separable. This assumption is not valid, in general, because it may not be possible to write the solution $u(x, t)$ as the product of a function of time only and a function of space only. However, with sufficiently small time steps, it is possible to obtain accurate solutions to even those problems for which the solution is not separable in time and space. The finite element solution that we obtain at the end of the analysis is continuous in space but not in time. We only obtain the finite element solution in the form

$$u(x, t_s) = \sum_{j=1}^{n} u_j^e(t_s)\psi_j^e(x) = \sum_{j=1}^{n} (u_j^s)^e \psi_j^e(x) \quad (s = 1, 2, \ldots) \tag{6.2.5}$$

where $(u_j^s)^e$ is the value of $u(x, t)$ at time $t = t_s$ and node j of the element Ω_e.

Substituting $w = \psi_i(x)$ (to obtain the ith equation of the system) and (6.2.1b) into (6.2.4a), we obtain

$$0 = \int_{x_a}^{x_b} \left[a\frac{d\psi_i}{dx}\left(\sum_{j=1}^{n} u_j\frac{d\psi_j}{dx}\right) + b\frac{d^2\psi_i}{dx^2}\left(\sum_{j=1}^{n} u_j\frac{d^2\psi_j}{dx^2}\right) + c_0\psi_i\left(\sum_{j=1}^{n} u_j\psi_j\right)\right.$$

$$\left. + c_1\psi_i\left(\sum_{j=1}^{n}\frac{du_j}{dt}\psi_j\right) + c_2\psi_i\left(\sum_{j=1}^{n}\frac{d^2u_j}{dt^2}\psi_j\right) - \psi_i f\right] dx$$

$$- \hat{Q}_1\psi_i(x_a) - \hat{Q}_3\psi_i(x_b) - \hat{Q}_2\left(-\frac{d\psi_i}{dx}\right)\bigg|_{x_a} - \hat{Q}_4\left(-\frac{d\psi_i}{dx}\right)\bigg|_{x_b}$$

$$= \sum_{j=1}^{n}\left[(K_{ij}^1 + K_{ij}^2)u_j + M_{ij}^1\frac{du_j}{dt} + M_{ij}^2\frac{d^2u_j}{dt^2}\right] - F_i \tag{6.2.6}$$

In matrix form, we have

$$[K]\{u\} + [M^1]\{\dot{u}\} + [M^2]\{\ddot{u}\} = \{F\} \tag{6.2.7a}$$

where

$$[K]=[K^1]+[K^2]+[M^0] \tag{6.2.7b}$$

$$\begin{aligned} M_{ij}^0 &= \int_{x_a}^{x_b} c_0\, \psi_i \psi_j\, dx, & M_{ij}^1 &= \int_{x_a}^{x_b} c_1\, \psi_i \psi_j\, dx \\ M_{ij}^2 &= \int_{x_a}^{x_b} c_2\, \psi_i \psi_j\, dx, & K_{ij}^1 &= \int_{x_a}^{x_b} a \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\, dx \\ K_{ij}^2 &= \int_{x_a}^{x_b} b \frac{d^2\psi_i}{dx^2}\frac{d^2\psi_j}{dx^2}\, dx, & F_i &= \int_{x_a}^{x_b} \psi_i f\, dx + \hat{Q}_i \end{aligned} \tag{6.2.7c}$$

Equation (6.2.7*a*) is a hyperbolic equation, and it contains the parabolic equation as a special case (set $[M^2]=[0]$). The time approximation of (6.2.7*a*) for these two cases will be considered separately. We begin with the parabolic equation.

6.2.3 Parabolic Equations

Time Approximation

The time approximation is discussed with the help of a single first-order differential equation. Suppose that we wish to determine $u(t)$ for $t>0$ such that $u(t)$ satisfies

$$a\frac{du}{dt}+bu=f(t),\quad 0<t<T \quad \text{and } u(0)=u_0 \tag{6.2.8}$$

where $a\neq 0$, b, and u_0 are constants, and f is a function of time t. The exact solution of the problem consists of two parts: the homogeneous and particular solutions. The homogeneous solution is

$$u^h(t)=Ae^{-kt},\quad k=\frac{b}{a}$$

The particular solution is given by

$$u^p(t)=\frac{1}{a}e^{-kt}\left(\int_0^t e^{k\tau} f(\tau)\, d\tau\right)$$

The complete solution is given by

$$u(t)=e^{-kt}\left(A+\frac{1}{a}\int_0^t e^{k\tau} f(\tau)\, d\tau\right) \tag{6.2.9}$$

In the finite difference solution of (6.2.8), we replace the derivatives with their finite difference approximation. The most commonly used scheme for solving (6.2.8) is the *α-family of approximation* in which a weighted average of the time derivatives at two consecutive time steps is approximated by linear interpolation of the values of the variable at the two steps (see Fig. 6.2.2):

$$(1-\alpha)\dot{u}_s+\alpha\,\dot{u}_{s+1}=\frac{u_{s+1}-u_s}{\Delta t_{s+1}} \quad \text{for } 0\leq\alpha\leq 1 \tag{6.2.10a}$$

where u_s denotes the value of $u(t)$ at time $t=t_s=\sum_{i=1}^{s}\Delta t_i$, and $\Delta t_s=t_s-t_{s-1}$ is the sth time step. If the total time $[0,\ T]$ is divided into equal time steps, then $t_s=s\Delta t$. Equation (6.2.10*a*) can be expressed as

$$\begin{aligned} u_{s+1} &= u_s+\Delta t\,\dot{u}_{s+\alpha} \\ \dot{u}_{s+\alpha} &= (1-\alpha)\dot{u}_s+\alpha\,\dot{u}_{s+1} \quad \text{for } 0\leq\alpha\leq 1 \end{aligned} \tag{6.2.10b}$$

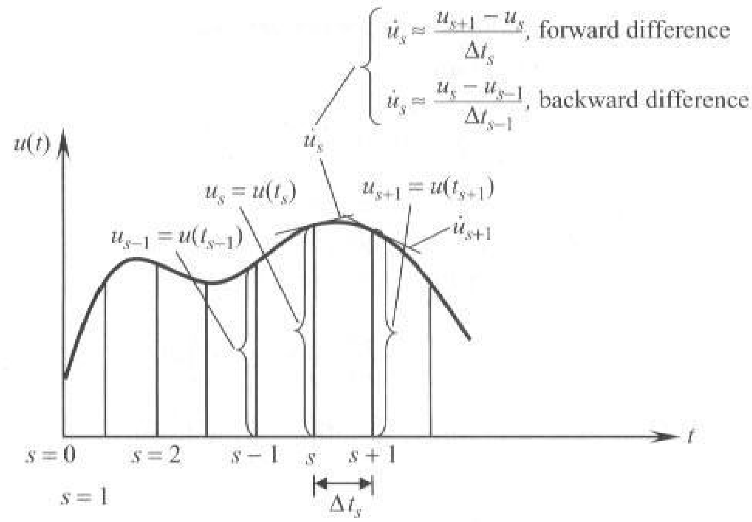

Figure 6.2.2 Approximation of the derivative of a function.

When $\alpha = 0$, Eq. (6.2.10*a*) gives

$$\dot{u}_s = \frac{u_{s+1} - u_s}{\Delta t_{s+1}}$$

which is nothing but the slope of the function $u(t)$ at time $t = t_s$ based on the values of the function at time t_s and t_{s+1}. Since the value of the function from a step in front is used, it is termed a *forward difference* approximation. When $\alpha = 1$, we obtain

$$\dot{u}_{s+1} = \frac{u_{s+1} - u_s}{\Delta t_{s+1}} \quad \rightarrow \quad \dot{u}_s = \frac{u_s - u_{s-1}}{\Delta t_s}$$

which is termed, for obvious reason, the *backward difference* approximation.

Returning to Eq. (6.2.8), we note that it is valid for all times $t > 0$. In particular, it is valid at times $t = t_s$ and $t = t_{s+1}$. Hence, from Eq. (6.2.8) we have

$$\dot{u}_s = \frac{1}{a}(f_s - bu_s), \quad \dot{u}_{s+1} = \frac{1}{a}(f_{s+1} - bu_{s+1})$$

Substituting the above expressions into Eq. (6.2.10*a*), we arrive at

$$(1-\alpha)(f_s - bu_s) + \alpha(f_{s+1} - bu_{s+1}) = a\left(\frac{u_{s+1} - u_s}{\Delta t_{s+1}}\right)$$

Solving for u_{s+1}, we obtain

$$[a + \alpha\, \Delta t_{s+1}\, b]\, u_{s+1} = [a - (1-\alpha)\, \Delta t_{s+1}\, b]\, u_s + \Delta t_{s+1}\, [\alpha\, f_{s+1} + (1-\alpha) f_s] \tag{6.2.11a}$$

or

$$u_{s+1} = \frac{a - (1-\alpha)\, \Delta t_{s+1}\, b}{a + \alpha\, \Delta t_{s+1}\, b} u_s + \Delta t_{s+1} \frac{[\alpha\, f_{s+1} + (1-\alpha) f_s]}{a + \alpha\, \Delta t_{s+1}\, b} \tag{6.2.11b}$$

Thus, Eq. (6.2.11) can be used repeatedly to march in time and obtain the solution at times $t = t_{s+1}, t_{s+2}, \ldots, t_N$, where Ntime is the number of time steps required to reach the final time T (or the solution reaches a steady state). At the very beginning, i.e., $s=0$, the solution u_1 is calculated using the initial value u_0:

$$u_1 = \frac{a-(1-\alpha)\,\Delta t_1\, b}{a+\alpha\,\Delta t_1\, b}u_0 + \Delta t_1\frac{[\alpha\, f_1+(1-\alpha)\, f_0]}{a+\alpha\,\Delta t_1\, b} \tag{6.2.12}$$

We may also develop a time approximation scheme using the finite element method. To this end, we consider the problem in (6.2.8). We wish to determine u_{s+1} in terms of u_s. The weighted-integral form of (6.2.8) over the time interval (t_s, t_{s+1}) is

$$0=\int_{t_s}^{t_{s+1}} v(t)\left(a\frac{du}{dt}+bu-f\right)dt \tag{6.2.13}$$

where v is the weight function. Assuming a solution of the form

$$u(t)\approx\sum_{j=1}^{n} u_j\psi_j(t)$$

where $\psi_j(t)$ are interpolation functions of order $(n-1)$. The Galerkin finite element model is obtained by substituting the above approximation for u and $v=\psi_i$ into (6.2.13). We obtain

$$[A]\{u\}=\{F\} \tag{6.2.14}$$

where

$$A_{ij}=\int_{t_s}^{t_{s+1}}\psi_i(t)\left(a\frac{d\psi_j}{dt}+b\psi_j\right)dt,\quad F_i=\int_{t_s}^{t_{s+1}}\psi_i(t)f(t)\,dt \tag{6.2.15}$$

Equation (6.2.14) is valid with the time interval (t_s, t_{s+1}), and it represents a relationship between the values $u_1, u_2, \ldots, u_n$, which are the values of u at time $t_s, t_{s+\Delta t/(n-1)}, t_{s+2\Delta t/(n-1)}, \ldots, t_{s+1}$, respectively. This would yield a multistep approximation scheme.

To obtain a single-step approximation scheme, i.e., write u_{s+1} in terms of u_s only, we assume linear approximation (i.e., $n=2$)

$$u(t)=u_s\psi_1(t)+u_{s+1}\psi_2(t)$$

where $\psi_1(t)=\frac{t_{s+1}-t}{\Delta t}$ and $\psi_2(t)=\frac{t-t_s}{\Delta t}$. In the same fashion, $f(t)$ can be represented in terms of its values at t_s and t_{s+1}:

$$f(t)=f_s\psi_1(t)+f_{s+1}\psi_2(t)$$

For this choice of approximation, Eq. (6.2.15) becomes

$$\left(\frac{a}{2}\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}+\frac{b\Delta t}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}u_s\\ u_{s+1}\end{Bmatrix}=\frac{\Delta t}{6}\begin{Bmatrix}f_s\\ 2f_{s+1}\end{Bmatrix} \tag{6.2.16}$$

Assuming that u_s is known, we solve for u_{s+1} from the second equation in (6.2.16)

$$\left(a+\frac{2b\Delta t}{3}\right)u_{s+1}=\left(a-\frac{b\Delta t}{3}\right)u_s+\Delta t\left(\frac{f_s}{3}+\frac{2f_{s+1}}{3}\right) \tag{6.2.17}$$

Comparing Eq. (6.2.17) with Eq. (6.2.11a), we find that the Galerkin scheme is a special case of the α-family of approximation, with $\alpha=2/3$.

Stable and Conditionally Stable Schemes

Equation (6.2.11b) can be written in the form

$$u_{s+1} = A(u_s) + F_{s,s+1}, \quad A = \frac{a - (1-\alpha)\,\Delta t_{s+1}\, b}{a + \alpha\,\Delta t_{s+1}\, b}$$

$$F_{s,s+1} = \Delta t_{s+1} \frac{[\alpha\, f_{s+1} + (1-\alpha) f_s]}{a + \alpha\,\Delta t_{s+1}\, b} \tag{6.2.18}$$

The operator A is known as the *amplification operator*. Since u_s is an approximate solution, the error $E_s = u_a(t_s) - u_s$ at time t_s (where u_a is the exact solution) will influence the solution at t_{s+1}. The error will grow (i.e., E_s will be amplified) as we march in time if the magnitude of the operator is greater than 1, $|A| > 1$. When the error grows without bound, the computational scheme (6.2.11b) becomes unstable (i.e., solution u_{s+1} becomes unbounded with time). Therefore, in order for the scheme to be stable, it is necessary that $|A| \leq 1$:

$$\left| \frac{a - (1-\alpha)\,\Delta t_{s+1}\, b}{a + \alpha\,\Delta t_{s+1}\, b} \right| \leq 1 \tag{6.2.19}$$

The above equation places a restriction on the magnitude of the time step for certain values of α. When the error remains bounded for any time step [i.e., condition (6.2.19) is trivially satisfied for any value of Δt], the scheme is *stable*. If the error remains bounded only when the time step remains below certain value [in order to satisfy (6.2.19)], the scheme is said to be *conditionally stable*.

For different values of α, the time approximation scheme in (6.2.11b) yields a different scheme. The following well-known time-approximation schemes along with their order of accuracy and stability should be noted:

$$\alpha = \begin{cases} 0, & \text{the forward difference (or Euler) scheme (conditionally} \\ & \text{stable); order of accuracy } = O(\Delta t) \\ \frac{1}{2}, & \text{the Crank–Nicolson scheme (stable); } O(\Delta t)^2 \\ \frac{2}{3}, & \text{the Galerkin method (stable); } O(\Delta t)^2 \\ 1, & \text{the backward difference scheme (stable); } O(\Delta t) \end{cases} \tag{6.2.20}$$

Of these, the Crank–Nicolson method is the most commonly used scheme.

Fully Discretized Finite Element Equations

We now have the tools necessary to convert the set of ordinary differential equations (6.2.7a) to a set of algebraic equations in much the same way we converted a single differential equation (6.2.8) to an algebraic equation (6.2.11b). Here, we work with matrix equation [i.e., $[M^1] = [M]$ and $[M^2] = [0]$ in Eq. (6.2.7a)]

$$[M]\{\dot{u}\} + [K]\{u\} = \{F\} \tag{6.2.21a}$$

subject to the initial conditions

$$\{u\}_0 = \{u_0\} \tag{6.2.21b}$$

where $\{u\}_0$ denotes the vector of nodal values of u at time $t=0$, whereas $\{u_0\}$ denotes the column of nodal values u_{j0}. As applied to a vector of time derivatives of the nodal values, the α-family of approximation reads as

$$\Delta t_{s+1}\left[(1-\alpha)\{\dot u\}_s+\alpha\{\dot u\}_{s+1}\right]=\{u\}_{s+1}-\{u\}_s \quad \text{for } 0\leq\alpha\leq 1 \tag{6.2.22a}$$

or

$$\begin{aligned}\{u\}_{s+1}&=\{u\}_s+\Delta t\{\dot u\}_{s+\alpha}\\ \{\dot u\}_{s+\alpha}&=(1-\alpha)\{\dot u\}_s+\alpha\{\dot u\}_{s+1}\end{aligned} \tag{6.2.22b}$$

for $0\leq\alpha\leq 1$. Equation (6.2.22a) can be used to reduce the ordinary differential equations (6.2.21a) to algebraic equations among the u_j at time t_{s+1}. Since (6.2.21a) is valid for any $t>0$, we can write it for times $t=t_s$ and $t=t_{s+1}$:

$$[M]\{\dot u\}_s+[K]_s\{u\}_s=\{F\}_s \tag{6.2.23a}$$

$$[M]\{\dot u\}_{s+1}+[K]_{s+1}\{u\}_{s+1}=\{F\}_{s+1} \tag{6.2.23b}$$

where it is assumed that the matrix $[M]$ is independent of time. Premultiplying both sides of (6.2.22a) with $[M]$ we obtain

$$\Delta t_{s+1}\alpha[M]\{\dot u\}_{s+1}+\Delta t_{s+1}(1-\alpha)[M]\{\dot u\}_s=[M](\{u\}_{s+1}-\{u\}_s)$$

Substituting for $[M]\{\dot u\}_{s+1}$ and $[M]\{\dot u\}_s$ from (6.2.23a) and (6.2.23b), respectively, we arrive at

$$\Delta t_{s+1}\alpha(\{F\}_{s+1}-[K]_{s+1}\{u\}_{s+1})+\Delta t_{s+1}(1-\alpha)(\{F\}_s-[K]_s\{u\}_s)=[M](\{u\}_{s+1}-\{u\}_s)$$

Solving for the vector $\{u\}_{s+1}$, we obtain

$$[\hat K]_{s+1}\{u\}_{s+1}=[\bar K]_s\{u\}_s+\{\bar F\}_{s,s+1} \tag{6.2.24a}$$

where

$$\begin{aligned}&[\hat K]_{s+1}=[M]+a_1[K]_{s+1},\quad [\bar K]_s=[M]-a_2[K]_s\\ &\{\bar F\}_{s,s+1}=\Delta t_{s+1}[\alpha\{F\}_{s+1}+(1-\alpha)\{F\}_s]\\ &a_1=\alpha\Delta t_{s+1},\quad a_2=(1-\alpha)\Delta t_{s+1}\end{aligned} \tag{6.2.24b}$$

Note that, in deriving (6.2.24a) and (6.2.24b), it has been assumed that $[M]$ is independent of time and that the time step is nonuniform.

Equations (6.2.24a) and (6.2.24b) are valid for a typical finite element whose semidiscretized equations are of the form (6.2.21a). In other words, Eqs. (6.2.24a) and (6.2.24b) hold for any problem, independent of the dimension and method of spatial approximation, as long as the end result is Eq. (6.2.21a). The assembly, imposition of boundary conditions, and solution of the assembled equations are the same as described before for steady-state problems. Calculation of $[\hat K]$ and $\{\bar F\}$ at time $t=0$ requires knowledge of the initial conditions $\{u\}_0$ and the time variation of $\{F\}$. Note that for $\alpha=0$ (the forward difference scheme), we obtain $[\hat K]=[M]$. If matrix $[M]$ is diagonal, (6.2.24a) becomes

$$u_i^{s+1}=\frac{1}{M_{(ii)}}\left(\sum_{j=1}^{n}\bar K_{ij}^s u_j^s+\hat F_i^{s,s+1}\right),\quad \text{(no sum on } i\text{)} \tag{6.2.25}$$

Thus, no inversion of the coefficient matrix is required in solving for $\{u\}^{s+1}$. Such a scheme is called *explicit*. A scheme is said to be *implicit* when it is not explicit (i.e., an implicit scheme requires the inversion of a coefficient matrix). Thus, explicit schemes are computationally less expensive compared to implicit schemes; implicit schemes are more accurate and have larger critical time steps. In real-world problems, the cost of computation precludes the use of implicit schemes.

In conventional finite element formulations, $[M]$ is seldom diagonal. Therefore, explicit schemes in finite element analysis can exist only if the time-approximation scheme is such that $[\hat{K}] = [M]$ and $[M]$ is *diagonal*. The matrix $[M]$ computed according to the definition (6.2.7*c*) is called the *consistent (mass) matrix,* and it is not diagonal unless ψ_i are orthogonal functions over the element domain. There are several ways to diagonalize mass matrices; these will be discussed in Section 6.2.5.

Consistency, Accuracy, and Stability

Since (6.2.22*a*) represents an approximation, which is used to derive (6.2.24*a*), error is introduced into the solution $\{u\}_{s+1}$ at each time step. In addition to the truncation error introduced in approximating the derivative, round-off errors can be introduced because of the finite arithmetic used in the computations. Since the solution at time t_{s+1} depends on the solution at time t_s, the error can grow with time. As discussed earlier, if the error is bounded, the solution scheme is said to be stable. The numerical scheme (6.2.24*a*) is said to be *consistent* with the continuous problem (6.2.7*a*) if the round-off and truncation errors go to zero as $\Delta t \to 0$. *Accuracy* of a numerical scheme is a measure of the closeness between the approximate solution and the exact solution whereas *stability* of a solution is a measure of the boundedness of the approximate solution with time. As we might expect, the size of the time step can influence both accuracy and stability. When we construct an approximate solution, we like it to converge to the true solution when the number of elements or the degree of approximation is increased and the time step Δt is decreased. A time-approximation scheme is said to be *convergent* if, for fixed t_s, the numerical value $\{u\}_s$ converges to its true value $\{u(t_s)\}$ as $\Delta t \to 0$. Accuracy is measured in terms of the rate at which the approximate solution converges. If a numerical scheme is stable and consistent, it is also convergent.

For all numerical schemes in which $\alpha < \frac{1}{2}$, the α-family of approximations is stable only if the time step satisfies the following (stability) condition [follows from Eq. (6.2.19)]:

$$\Delta t < \Delta t_{\text{cri}} \equiv \frac{2}{(1-2\alpha)\lambda} \tag{6.2.26}$$

where λ is the largest eigenvalue associated with the problem and Δt_{cri} is (see below)

$$([M] - \lambda[K])\,\{u\} = \{Q\} \tag{6.2.27}$$

Note that the same mesh as that used for the transient analysis must be used to calculate the eigenvalues.

6.2.4 Hyperbolic Equations

Time Approximation

Consider matrix equations of the form [set $[M^1]=[C]$ and $[M^2]=[M]$ in Eq. (6.2.7*a*)]

$$[K]\{u\}+[C]\{\dot{u}\}+[M]\{\ddot{u}\}=\{F\} \tag{6.2.28a}$$

subjected to initial conditions

$$\{u(0)\}=\{u_0\}, \quad \{\dot{u}(0)\}=\{v_0\} \tag{6.2.28b}$$

Such equations arise in structural dynamics, where $[M]$ denotes mass matrix, $[C]$ the damping matrix, and $[K]$ the stiffness matrix. The damping matrix $[C]$ is often taken to be a linear combination of the mass and stiffness matrices, $[C]=\beta_1[M]+\beta_2[K]$, where β_1 and β_2 are determined from physical experiments. In the present study, we will not consider damping (i.e., $[C]=[0]$) in the numerical examples, although the theoretical developments will account for it. Transient analysis of both bars and beams lead to equations of the type given in (6.2.28*a*) and (6.2.28*b*). The mass and stiffness matrices for these problems are discussed below.

Axial Motion of Bars

The equation of motion is given by Eq. (6.1.14). The semidiscretization results in Eq. (6.2.28*a*), with $[M^e]$ and $[K^e]$ given by Eq. (6.1.27*b*) and $[C]=[0]$.

Transverse Motion of Euler–Bernoulli Beams

The equation of motion is given by Eq. (6.1.36). The semidiscretized equation is the same as Eq. (6.2.28*a*), with $[M^e]$ and $[K^e]$ given by Eq. (6.1.41*b*) and $[C]=[0]$.

Transverse Motion of Timoshenko Beams

The equations of motion are given by Eqs. (6.1.43*a*) and (6.1.43*b*). The semidiscretization results in Eq. (6.2.28*a*), with $[M^e]$ and $[K^e]$ given by Eq. (6.1.48*a*). Note that $[M^e]$ and $[K^e]$ are the matrices given in Eq. (6.1.47).

There are several numerical methods available to approximate the second-order time derivatives and convert differential equations to algebraic equations. Among these, the Newmark family of time-approximation schemes is widely used in structural dynamics [Newmark (1959)]. Other methods, such as the Wilson method [Bathe and Wilson (1973)] and the Houbolt method [Houbolt (1950)], can also be used to develop the algebraic equations from the second-order differential equations (6.2.28*a*). Here we consider the Newmark family of approximation schemes.

Newmark's Scheme

In the Newmark method, the function and its first time derivative are approximated according to

$$\{u\}_{s+1}=\{u\}_s+\Delta t\{\dot{u}\}_s+\frac{1}{2}(\Delta t)^2\{\ddot{u}\}_{s+\gamma} \tag{6.2.29a}$$

$$\{\dot{u}\}_{s+1}=\{\dot{u}\}_s+\{\ddot{u}\}_{s+\alpha}\Delta t \tag{6.2.29b}$$

where

$$\{\ddot{u}\}_{s+\theta} = (1-\theta)\{\ddot{u}\}_s + \theta\{\ddot{u}\}_{s+1} \tag{6.2.29c}$$

and α and γ $(=2\beta)$ are parameters that determine the stability and accuracy of the scheme. Equations (6.2.29*a*) and (6.2.29*b*) can be viewed as Taylor's series expansions of u and $\dot{u}$. The following schemes are special cases of (6.2.29*a*) and (6.2.29*b*):

$$\begin{aligned}
&\alpha = \tfrac{1}{2}, \quad \gamma = 2\beta = \tfrac{1}{2} && \text{Constant-average acceleration method (stable)}\\
&\alpha = \tfrac{1}{2}, \quad \gamma = 2\beta = \tfrac{1}{3} && \text{Linear acceleration method (conditionally stable)}\\
&\alpha = \tfrac{1}{2}, \quad \gamma = 2\beta = 0 && \text{Central difference method (conditionally stable)}\\
&\alpha = \tfrac{3}{2}, \quad \gamma = 2\beta = \tfrac{8}{5} && \text{Galerkin method (stable)}\\
&\alpha = \tfrac{3}{2}, \quad \gamma = 2\beta = 2 && \text{Backward difference method (stable)}
\end{aligned} \tag{6.2.30}$$

For all schemes in which $\gamma < \alpha$ and $\alpha \geq \frac{1}{2}$, the stability requirement is

$$\Delta t \leq \Delta t_{\text{cri}} = \left[\frac{1}{2}\omega_{\max}^2(\alpha - \gamma)\right]^{-1/2} \tag{6.2.31}$$

where $\omega_{\max}$ is the maximum natural frequency of the system (6.2.28*a*) without $[C]$:

$$\left([M] - \omega^2[K]\right)\{u\} = \{F\} \tag{6.2.32}$$

Fully Discretized Finite Element Equations

Eliminating $\{\ddot{u}\}_{s+1}$ from Eqs. (6.2.29*a*) and (6.2.29*b*) and writing the result for $\{\dot{u}\}_{s+1}$, we obtain

$$\{\dot{u}\}_{s+1} = a_6\left(\{u\}_{s+1} - \{u\}_s\right) - a_7\{\dot{u}\}_s - a_8\{\ddot{u}\}_s \tag{6.2.33}$$

where

$$a_6 = \frac{\alpha}{\beta\Delta t}, \quad a_7 = \frac{\alpha}{\beta} - 1, \quad a_8 = \left(\frac{\alpha}{\gamma} - 1\right)\Delta t \tag{6.2.34}$$

Now premultiplying Eq. (6.2.29*a*) with $[M]_{s+1}$ and substituting for $[M]_{s+1}\{\ddot{u}\}_{s+1}$ from Eq. (6.2.28*a*), we obtain

$$\begin{aligned}
\left([M]_{s+1} + \beta(\Delta t)^2[K]_{s+1}\right)\{u\}_{s+1} &= [M]_{s+1}\{b\}_s + \beta(\Delta t)^2\{F\}_{s+1}\\
&\quad - \beta(\Delta t)^2[C]_{s+1}\{\dot{u}\}_{s+1}
\end{aligned} \tag{6.2.35}$$

where

$$\{b\}_s = \{u\}_s + \Delta t\{\dot{u}\}_s + \frac{1}{2}(1-\gamma)(\Delta t)^2\{\ddot{u}\}_s \tag{6.2.36}$$

Now, multiplying throughout with $a_3 = 1/[\beta(\Delta t)^2]$, we arrive at

$$\left(a_3[M]_{s+1} + [K]_{s+1}\right)\{u\}_{s+1} = a_3[M]_{s+1}\{b\}_s + \{F\}_{s+1} - [C]_{s+1}\{\dot{u}\}_{s+1} \tag{6.2.37}$$

Using Eq. (6.2.33) for $\{\dot{u}\}_{s+1}$ in Eq. (6.2.37) and collecting terms, we obtain the final result

$$[\hat{K}]_{s+1}\{u\}_{s+1} = \{\hat{F}\}_{s,s+1} \tag{6.2.38}$$

where

$$\begin{aligned}
[\hat{K}]_{s+1} &= [K]_{s+1} + a_3[M]_{s+1} + a_6[C]_{s+1} \\
\{\hat{F}\}_{s,s+1} &= \{F\}_{s+1} + [M]_{s+1}\{A\}_s + [C]_{s+1}\{B\}_s \\
\{A\}_s &= a_3\{b\}_s = a_3\{u\}_s + a_4\{\dot{u}\}_s + a_5\{\ddot{u}\}_s \\
\{B\}_s &= a_6\{u\}_s + a_7\{\dot{u}\}_s + a_8\{\ddot{u}\}_s
\end{aligned} \tag{6.2.39}$$

and

$$a_3 = \frac{1}{\beta(\Delta t)^2}, \quad a_4 = a_3\Delta t\,, \quad a_5 = \frac{1}{\gamma} - 1 \tag{6.2.40}$$

Note that the calculation of $[\hat{K}]$ and $\{\hat{F}\}$ requires knowledge of the initial conditions $\{u\}_0$, $\{\dot{u}\}_0$, and $\{\ddot{u}\}_0$. In practice, we do not know $\{\ddot{u}\}_0$. As an approximation, it can be calculated from (6.2.28*a*) (we often assume that the applied force is zero at $t = 0$):

$$\{\ddot{u}\}_0 = [M]^{-1}\left(\{F\}_0 - [K]\{u\}_0 - [C]\{\dot{u}\}_0\right) \tag{6.2.41}$$

At the end of each time step, the new velocity vector $\{\dot{u}\}_{s+1}$ and acceleration vector $\{\ddot{u}\}_{s+1}$ are computed using [from Eqs. (6.2.29*a*) and (6.2.29*b*)]

$$\begin{aligned}
\{\ddot{u}\}_{s+1} &= a_3(\{u\}_{s+1} - \{u\}_s) - a_4\{\dot{u}\}_s - a_5\{\ddot{u}\}_s \\
\{\dot{u}\}_{s+1} &= \{\dot{u}\}_s + a_2\{\ddot{u}\}_s + a_1\{\ddot{u}\}_{s+1} \\
a_1 &= \alpha\Delta t, \quad a_2 = (1-\alpha)\Delta t
\end{aligned} \tag{6.2.42}$$

The remaining procedure stays the same as in static problems.

The fully discretized model in (6.2.38) is based on the assumption that $\gamma \neq 0$. Obviously, for centered difference scheme ($\gamma = 0$), the formulation must be modified (see Problem 6.26).

6.2.5 Mass Lumping

Recall from the time approximation of parabolic equations that use of the forward difference scheme (i.e., $\alpha = 0$) results in the following time marching scheme [see (6.2.24*a*) and (6.2.24*b*)]:

$$[M^e]\{u\}_{s+1} = ([M^e] - \Delta t[K^e])\{u\}_s + \Delta t\{F^e\}_s \tag{6.2.43}$$

The mass matrix $[M^e]$ derived from the weighted-integral formulations of the governing equation is called the *consistent mass matrix*, and it is symmetric positive-definite and nondiagonal. Solution of the global equations associated with (6.2.43) requires inversion of the assembled mass matrix. If the mass matrix is diagonal, then the assembled equations can be solved directly (i.e., without inverting a matrix)

$$(U_I)_{s+1} = M_{II}^{-1}\left[M_{II}(U_I)_s - \Delta t\sum_{J=1}^{N} K_{IJ}(U_J)_s + \Delta t(F_I)_s\right] \tag{6.2.44}$$

and thus saving computational time. The explicit nature of (6.2.44) has motivated analysts to find rational ways of diagonalizing the mass matrix.

There are several ways of constructing diagonal mass matrices, also known as *lumped mass matrices*. The *row-sum* and *proportional lumping* techniques are discussed here.

Row-Sum Lumping

The sum of the elements of each row of the consistent mass matrix is used as the diagonal element:

$$M_{ii}^e = \sum_{j=1}^{n} \int_{x_a}^{x_b} \rho \psi_i^e \psi_j^e \, dx = \int_{x_a}^{x_b} \rho \psi_i^e \, dx \tag{6.2.45}$$

where the property $\sum_{j=1}^{n} \psi_j^e = 1$ of the interpolation functions is used. When ρ is constant, (6.2.45) gives

$$[M^e]_L = \frac{\rho h_e}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(linear element)}$$

$$[M^e]_L = \frac{\rho h_e}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(quadratic element)} \tag{6.2.46}$$

Compare these lumped mass matrices with the consistent mass matrices

$$[M^e]_C = \frac{\rho h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{(linear element)}$$

$$[M^e]_C = \frac{\rho h_e}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \quad \text{(quadratic element)} \tag{6.2.47}$$

Here subscripts L and C refer to lumped and consistent mass matrices, respectively.

Proportional Lumping

Here the diagonal elements of the lumped mass matrix are computed to be proportional to the diagonal elements of the consistent mass matrix while conserving the total mass of the element:

$$M_{ii}^e = \alpha \int_{x_a}^{x_b} \rho \psi_i^e \psi_i^e \, dx, \quad \alpha = \int_{x_a}^{x_b} \rho \, dx \Big/ \sum_{i=1}^{n} \int_{x_a}^{x_b} \rho \psi_i^e \psi_i^e \, dx \tag{6.2.48}$$

For constant ρ, proportional lumping gives the same lumped mass matrices as those obtained in the row-sum technique for the Lagrange linear and quadratic elements.

The use of a lumped mass matrix in transient analyses can save computational time in two ways. First, for forward difference schemes, lumped mass matrices result in explicit algebraic equations not requiring matrix inversions. Second, the critical time step required for conditionally stable schemes is larger, and hence less computational time is required when lumped mass matrices are used. To see this, consider the stability criterion in (6.2.31) for the case $\alpha = \frac{1}{2}$ and $\beta = 0$. For a one linear element model of a uniform bar of stiffness

EA and mass ρA, fixed at the left end, the eigenvalue problem with a consistent mass matrix is

$$\left(\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \omega^2\frac{\rho Ah}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}U_1\\ U_2\end{Bmatrix} = \begin{Bmatrix}Q_1^1\\ Q_2^1\end{Bmatrix} \tag{6.2.49}$$

Since $U_1 = 0$ and $Q_2^1 = 0$, we have

$$\omega^2 = \frac{EA}{h} \Big/ \frac{\rho Ah}{3} = \frac{3E}{\rho h^2}$$

Substituting this into the critical time step relation (6.2.31), we have,

$$(\Delta t_{\text{cri}})_C = 2/\omega_{\max} = h(4\rho/3E)^{1/2}$$

If we use the lumped matrix, ω is given by

$$\omega = (2E/\rho)^{1/2}/h$$

and the critical time step is

$$(\Delta t_{\text{cri}})_L = h(2\rho/E)^{1/2} > (\Delta t_{\text{cri}})_C \tag{6.2.50}$$

Thus, explicit schemes require larger time steps than implicit schemes.

6.2.6 Applications

Here we consider two examples of applications of finite element models of one-dimensional problems. Example 6.2.1 is taken from transient heat transfer (parabolic equation), and Example 6.2.2 is taken from solid mechanics (hyperbolic equation).

Example 6.2.1

Consider the transient heat conduction problem described by the differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } 0 < x < 1 \tag{6.2.51a}$$

with boundary conditions

$$u(0, t) = 0, \quad \frac{\partial u}{\partial x}(1, t) = 0 \tag{6.2.51b}$$

and initial condition

$$u(x, 0) = 1.0 \tag{6.2.51c}$$

where u is the nondimensionalized temperature. The problem at hand is a special case of (6.2.3a) with $a = 1$, $b = 0$, $c_0 = 0$, $c_1 = 1$, $c_2 = 0$, and $f = 0$ [or Eq. (6.1.2) with $kA = 1$, $\rho cA = 1$, and $q = 0$]. The finite element model is

$$[M^e]\{\dot{u}\} + [K^e]\{u\} = \{Q^e\}$$

where

$$M_{ij}^e = \int_{x_a}^{x_b} \psi_i^e \psi_j^e \, dx, \quad K_{ij}^e = \int_{x_a}^{x_b} \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} \, dx$$

For the choice of linear interpolation functions, the semidiscrete equations of a typical element are

$$\frac{h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \dot{u}_1^e \\ \dot{u}_2^e \end{Bmatrix} + \frac{1}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

where h_e is the length of the element. Use of the α-family of approximation (6.2.10a) results in the equation [see (6.2.24a)]

$$([M^e] + \Delta t\, \alpha [K^e])\{u^e\}_{s+1} = ([M^e] - \Delta t(1-\alpha)[K^e])\{u^e\}_s + \Delta t(\alpha\{Q^e\}_{s+1} + (1-\alpha)\{Q^e\}_s)$$

where Δt is the time step.

First, consider a one-element mesh. We have ($h = 1$)

$$\begin{bmatrix} \frac{1}{3}h + \alpha\frac{\Delta t}{h} & \frac{1}{6}h - \alpha\frac{\Delta t}{h} \\ \frac{1}{6}h - \alpha\frac{\Delta t}{h} & \frac{1}{3}h + \alpha\frac{\Delta t}{h} \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}_{s+1} = \begin{bmatrix} \frac{1}{3}h - (1-\alpha)\frac{\Delta t}{h} & \frac{1}{6}h + (1-\alpha)\frac{\Delta t}{h} \\ \frac{1}{6}h + (1-\alpha)\frac{\Delta t}{h} & \frac{1}{3}h - (1-\alpha)\frac{\Delta t}{h} \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}_s + \Delta t\begin{Bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{Bmatrix}$$

where $\bar{Q}_i = \alpha(Q_i^1)_{s+1} + (1-\alpha)(Q_i^1)_s$. The boundary conditions of the problem require

$$(U_1)_s = 0, \quad (Q_2^1)_s = 0 \quad \text{for all } s > 0 \text{ (i.e., } t > 0)$$

However, the initial condition (6.2.51c) requires

$$U_1(0)\psi_1(x) + U_2(0)\psi_2(x) = 1$$

Since the initial condition should be consistent with the boundary conditions, we take $(U_1)_0 = 0$. Then it follows that $(U_2)_0 = 1$.

Using the boundary conditions, we can write for the one-element model ($h = 1.0$)

$$\left(\frac{1}{3}h + \alpha\frac{\Delta t}{h}\right)(U_2)_{s+1} = \left[\frac{1}{3}h - (1-\alpha)\frac{\Delta t}{h}\right](U_2)_s \tag{6.2.52}$$

which can be solved repeatedly for U_2 at different times, $s = 0, 1, \ldots$.

Repeated use of (6.2.52) can cause the temporal approximation error to grow with time, depending on the value of α. As noted earlier, the forward difference scheme ($\alpha = 0$) is a conditionally stable scheme. To determine the critical time step for the one-element mesh, we first calculate the maximum eigenvalue of the associated system

$$\left(-\lambda\frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}, \quad U_1 = 0, \; Q_2^1 = 0$$

The eigenvalue is (there is only one)

$$-\frac{1}{3}\lambda h U_2 + h^{-1}U_2 = 0, \quad \text{or} \quad \lambda = 3/h^2 = 3$$

Hence, from Eq. (6.2.26) we have $\Delta t_{cri} = 2/\lambda = 0.66667$. Thus, in order for the forward difference scheme (6.2.52)

$$\frac{1}{3}h(U_2)_{s+1} = \left(\frac{1}{3}h - \frac{\Delta t}{h}\right)(U_2)_s$$

to be stable, the time step should be smaller than $\Delta t_{cri} = 0.6667$; otherwise, the solution will be unstable, as shown in Fig. 6.2.3.

For a two-element mesh, we have ($h_1 = h_2 = h = 0.5$); the condensed equations of the time-marching scheme are given by

$$\begin{bmatrix} \frac{2}{3}h + 2\alpha\frac{\Delta t}{h} & \frac{1}{6}h - \alpha\frac{\Delta t}{h} \\ \frac{1}{6}h - \alpha\frac{\Delta t}{h} & \frac{1}{3}h + \alpha\frac{\Delta t}{h} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}_{s+1} = \begin{bmatrix} \frac{2}{3}h - 2(1-\alpha)\frac{\Delta t}{h} & \frac{1}{6}h + (1-\alpha)\frac{\Delta t}{h} \\ \frac{1}{6}h + (1-\alpha)\frac{\Delta t}{h} & \frac{1}{3}h - (1-\alpha)\frac{\Delta t}{h} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}_s$$

with $(U_2)_0$ and $(U_3)_0$. The forward difference scheme yields

$$\frac{h}{6}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}_{s+1} = \frac{h}{6}\begin{bmatrix} 4-2\mu & 1+\mu \\ 1+\mu & 2-\mu \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}_s, \quad \mu = \frac{6\Delta t}{h^2}$$

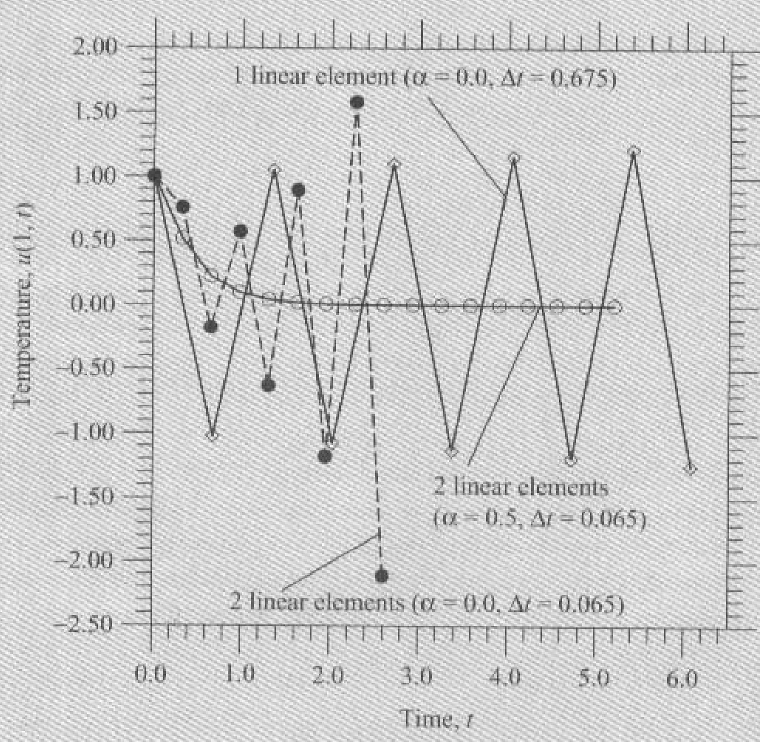

Figure 6.2.3 Stability of the forward difference ($\alpha = 0.0$) and Crank–Nicolson ($\alpha = 0.5$) schemes as applied to a parabolic equation.

The associated eigenvalue problem is

$$\left(-\lambda\frac{h}{6}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}+\frac{1}{h}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\right)\begin{Bmatrix}U_2\\ U_3\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

The characteristic equation is

$$7\bar{\lambda}^2-10\bar{\lambda}+1=0,\quad \bar{\lambda}=\frac{\lambda h^2}{6}$$

whose roots are $\lambda_1=2.5967$ and $\lambda_2=31.6891$. Hence, the critical time step becomes $\Delta t_{cri}=2/31.6891=0.0631$. A time step of $\Delta t=0.065$ results in an unstable solution, as shown in Fig. 6.2.3.

For (unconditionally) stable schemes ($\alpha\geq\frac{1}{2}$), there is no restriction on the time step (e.g., Crank–Nicolson method with two linear elements and $\Delta t=0.065$ yield very smooth and stable solution, as shown in Fig. 6.2.3). However, to obtain a sufficiently accurate solution, the time step must be taken as a fraction of Δt_{cri}. Of course, the accuracy of the solution also depends on the mesh size h. As this is decreased (i.e., the number of elements is increased), Δt_{cri} decreases.

Plots of $u(1,t)$ versus time for $\alpha=0$ and $\alpha=0.5$ with $\Delta t=0.05$ are shown in Fig. 6.2.4. Solutions predicted by meshes of one, two, or four linear (L) or quadratic (Q) elements are compared. The convergence of the solution with increasing number of elements is clear. The finite element solutions obtained with different methods, time steps, and meshes are compared with the exact solution in Table 6.2.1.

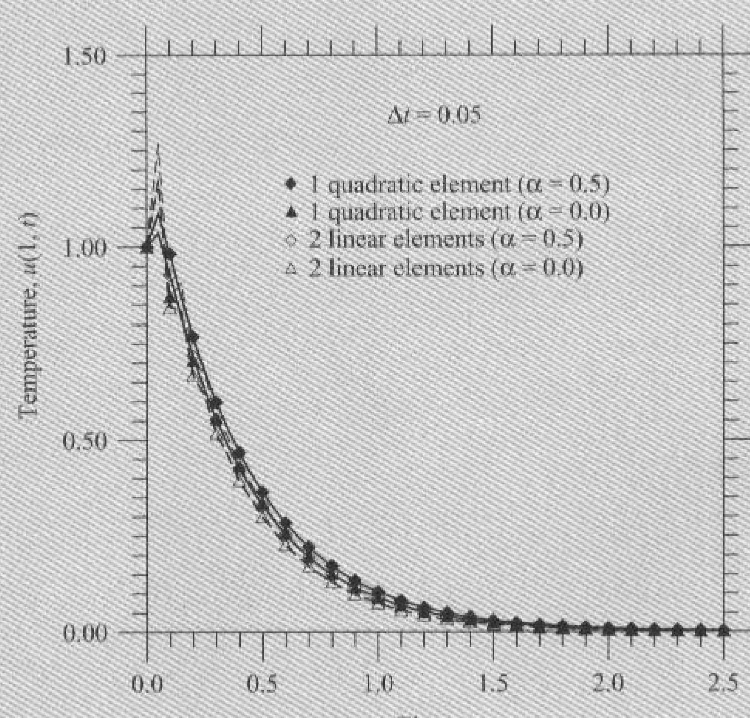

Figure 6.2.4 Transient solution of a parabolic equation according to linear and quadratic finite elements.

Table 6.2.1 A comparison of the finite element solutions obtained using various time approximation schemes and meshes with the analytical solution of a parabolic equation ($\Delta t = 0.05$).

	$\alpha = 0$	$\alpha = 1$	$\alpha = 0.5$							
t	1L	1L	1L	2L	4L	4L	1Q	2Q	4Q	Exact
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.05	0.8500	0.8696	0.8605	1.0359	0.9951	0.9933	1.0870	0.9942	0.9928	0.9969
0.10	0.7225	0.7561	0.7404	0.9279	0.9588	0.9554	0.9819	0.9550	0.9549	0.9493
0.15	0.6141	0.6575	0.6371	0.8169	0.8639	0.8707	0.8693	0.8831	0.8725	0.8642
0.20	0.5220	0.5718	0.5482	0.7176	0.7557	0.7694	0.7679	0.7633	0.7731	0.7723
0.25	0.4437	0.4972	0.4717	0.6300	0.6759	0.6824	0.6780	0.6933	0.6855	0.6854
0.30	0.3771	0.4323	0.4059	0.5533	0.5906	0.6037	0.5987	0.6006	0.6070	0.6068
0.35	0.3206	0.3759	0.3492	0.4858	0.5250	0.5325	0.5286	0.5394	0.5358	0.5367
0.40	0.2725	0.3269	0.3005	0.4266	0.4608	0.4713	0.4668	0.4710	0.4741	0.4745
0.45	0.2316	0.2843	0.2586	0.3746	0.4083	0.4158	0.4121	0.4201	0.4188	0.4194
0.50	0.1969	0.2472	0.2225	0.3289	0.3592	0.3676	0.3639	0.3687	0.3701	0.3708
0.55	0.1673	0.2149	0.1914	0.2888	0.3176	0.3247	0.3213	0.3275	0.3273	0.3277
0.60	0.1422	0.1869	0.1647	0.2536	0.2798	0.2868	0.2837	0.2883	0.2890	0.2897
0.65	0.1209	0.1625	0.1418	0.2227	0.2472	0.2535	0.2505	0.2556	0.2556	0.2561
0.70	0.1028	0.1413	0.1220	0.1955	0.2180	0.2238	0.2212	0.2253	0.2258	0.2264
0.75	0.0874	0.1229	0.1050	0.1717	0.1924	0.1979	0.1953	0.1995	0.1996	0.2001
0.80	0.0743	0.1069	0.0903	0.1508	0.1697	0.1747	0.1725	0.1761	0.1764	0.1769
0.85	0.0631	0.0929	0.0777	0.1324	0.1498	0.1544	0.1523	0.1557	0.1559	0.1563
0.90	0.0536	0.0808	0.0669	0.1162	0.1322	0.1363	0.1345	0.1375	0.1378	0.1382
0.95	0.0456	0.0703	0.0575	0.1020	0.1166	0.1205	0.1187	0.1216	0.1218	0.1222
1.00	0.0388	0.0611	0.0495	0.0896	0.1029	0.1065	0.1048	0.1074	0.1076	0.1080

Example 6.2.2

We wish to determine the transverse motion of a beam clamped at both ends and subjected to initial deflection using EBT and TBT. The governing equations are

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0 \quad \text{for } 0 < x < 1 \tag{6.2.53a}$$

$$w(0,t) = 0, \quad \frac{\partial w}{\partial x}(0,t) = 0, \quad w(1,t) = 0, \quad \frac{\partial w}{\partial x}(1,t) = 0 \tag{6.2.53b}$$

$$w(x,0) = \sin \pi x - \pi x(1-x), \quad \frac{\partial w}{\partial t}(x,0) = 0 \tag{6.2.53c}$$

Note that the initial deflection of the beam is consistent with the boundary conditions. The initial slope is given by

$$\theta(x,0) = -\left(\frac{\partial w}{\partial x}\right)_{t=0} = -\pi \cos \pi x + \pi(1-2x) \tag{6.2.53d}$$

Because of symmetry about $x = 0.5$ (center of the beam), we consider only a half span of the beam for finite element modeling. Here we use the first half of the beam, $0 \le x \le 0.5$, as the computational domain. The boundary condition at $x = 0.5$ is $\theta(0.5, t) = -(\partial w/\partial x)(0.5, t) = 0$. The semidiscretized finite element model of a typical element is

$$\frac{h_e}{420}\begin{bmatrix} 156 & -22h_e & 54 & 13h_e \\ -22h_e & 4h_e^2 & -13h_e & -3h_e^2 \\ 54 & -13h_e & 156 & 22h_e \\ 13h_e & -3h_e^2 & 22h_e & 4h_e^2 \end{bmatrix}\begin{Bmatrix} \ddot{\Delta}_1^e \\ \ddot{\Delta}_2^e \\ \ddot{\Delta}_3^e \\ \ddot{\Delta}_4^e \end{Bmatrix} + \frac{2}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}\begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

We begin with a one-element mesh with the Euler–Bernoulli beam element. The semidiscretized model is

$$\frac{h}{420}\begin{bmatrix} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{bmatrix}\begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{Bmatrix} + \frac{2}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3h \\ -3h & h^2 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix}$$

The boundary conditions for the one-element model translate into

$$U_1 = 0, \quad U_2 = 0, \quad U_4 = 0, \quad Q_3^1 = 0 \quad \text{for all } t > 0$$

The initial conditions can be computed from (6.2.53c) and (6.2.53d)

$$\left.\begin{matrix} U_1 = 0, & U_2 = 0, & U_3 = 0.2146, & U_4 = 0 \\ \dot{U}_1 = 0, & \dot{U}_2 = 0, & \dot{U}_3 = 0 & \dot{U}_4 = 0 \end{matrix}\right\} \quad \text{for } t = 0$$

The condensed equation of the time marching scheme for this case takes the form

$$(K_{33} + a_3 M_{33})(U_3)_{s+1} = (\hat{F}_3)_{s,s+1} \equiv M_{33}(a_3 U_3 + a_4 \dot{U}_3 + a_5 \ddot{U}_3)_s, \quad s = 0, 1, \ldots$$

where a_3, a_4, and a_5 are defined in (6.2.40). The second derivative $\ddot{U}_3$ for time $t = 0$ (i.e., when $s = 0$) is computed from the equation of motion:

$$(\ddot{U}_3)_0 = -\frac{K_{33}(U_3)_0}{M_{33}} = -\left(\frac{12}{h^3} \times 0.2146\right)\frac{420}{156h} = -110.932$$

For $\gamma < \frac{1}{2}$, we must compute the critical time step Δt_{cri}, which depends on the square of maximum natural frequency of the beam [see Eq. (6.2.31)]. For the present model, $\omega_{\max}$ is computed from the eigenvalue problem

$$(K_{33} - \omega^2 M_{33})U_3 = 0 \quad \text{or } \omega^2 = K_{33}/M_{33} = 516.923$$

Hence, the critical time step for $\alpha = 0.5$ and $\gamma = \frac{1}{3}$ (i.e., the linear acceleration scheme) is

$$\Delta t_{\text{cri}} = \sqrt{12}/\omega_{\max} = 0.15236$$

Although there is no restriction on time integration schemes with $\alpha = 0.5$ and $\gamma > 0.5$, the critical time step provides an estimate of the time step to be used to obtain transient solution.

Figure 6.2.5 shows plots of $w(0.5, t)$ versus time for the scheme with $\alpha = 0.5$ and $\gamma = \frac{1}{3}$. Three different time steps, $\Delta t = 0.175$, 0.150, and 0.05, are used to illustrate the accuracy. For $\Delta t = 0.175 > \Delta t_{\text{cri}}$, the solution is unstable, whereas for $\Delta t < \Delta t_{\text{cri}}$, it is stable but inaccurate. The period of the solution is given by

$$T = 2\pi/\omega = 0.27635$$

For two- and four-element meshes of Euler–Bernoulli elements, the critical time steps are (details are not presented here)

$$(\Delta t_{\text{cri}})_2 = 0.00897, \quad (\Delta t_{\text{cri}})_4 = 0.00135$$

where the subscripts refer to the number of elements in the mesh. The transverse deflection obtained with the one and two Euler–Bernoulli elements ($\Delta t = 0.005$) in half beam for a complete period (0, 0.28) are shown in Fig. 6.2.6.

The problem can also be analyzed using the Timoshenko beam element (RIE). In writing the governing equations [see (6.1.43*a*) and (6.1.43*b*)] of the TBT as they apply to the present problem, we first identify the coefficients GAK_s, EI, ρA, and ρI consistent with those in the differential equation (6.2.53*a*). We have $EI = 1.0$ and $\rho A = 1.0$. Then GAK_s can be computed as

$$GAK_s = \frac{E}{2(1+\nu)} BHK_s = \frac{EI}{2(1+\nu)} \frac{12}{H^2} \frac{5}{6} = \frac{4}{H^2} EI \tag{6.2.54}$$

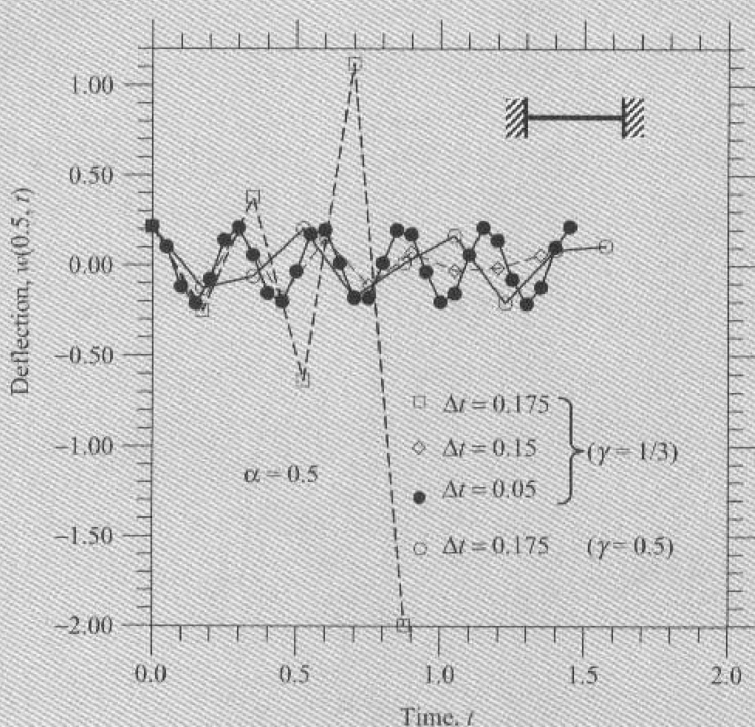

Figure 6.2.5 Center deflection $w(0.5, t)$ versus time t for a clamped beam.

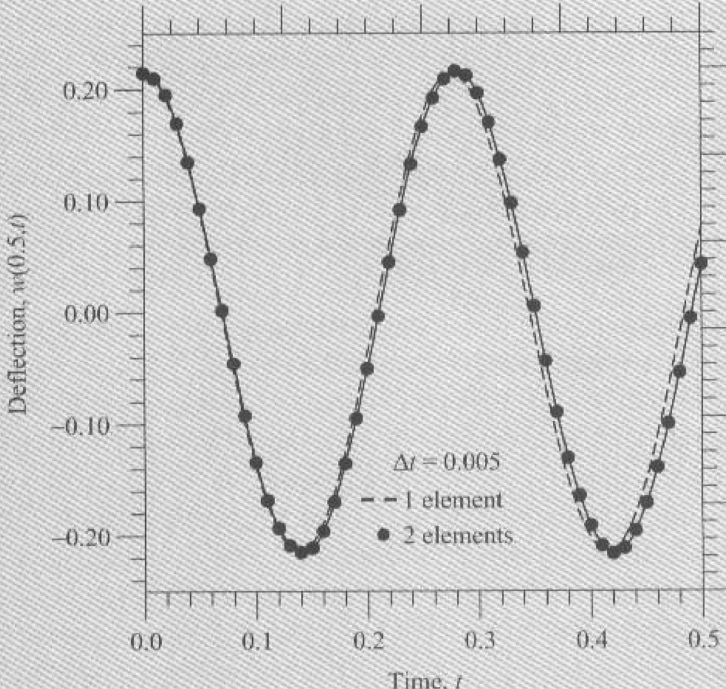

Figure 6.2.6 Center transverse deflection versus time for a clamped beam subjected to an initial transverse deflection ($\Delta t = 0.005$, $\alpha = 0.5$, and $\gamma = 0.5$).

where B is the width and H the height of the beam, and $I = \frac{1}{12}BH^3$, $\nu = 0.25$, and $K_s = \frac{5}{6}$ are used in arriving at the last expression. Similarly,

$$\rho I = \rho \frac{1}{12} BH^3 = \frac{1}{12}\rho A H^2 \tag{6.2.55}$$

Thus, the governing equations of the TBT for the problem at hand are

$$\frac{\partial^2 w}{\partial t^2} - \frac{4}{H^2}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} + \Psi\right) = 0 \tag{6.2.56a}$$

$$\frac{H^2}{12}\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial x^2} + \frac{4}{H^2}\left(\frac{\partial w}{\partial x} + \Psi\right) = 0 \tag{6.2.56b}$$

The values of $w(0.5, t)$ as obtained using the Euler–Bernoulli and Timoshenko elements (both elements include the rotary inertia) for various numbers of elements are presented in Table 6.2.2. The time step is taken to be $\Delta t = 0.005$, which is smaller than the critical time step of the two-element mesh of the Euler–Bernoulli beam element when $\gamma = \frac{1}{3}$. Plots of $w(0.5, t)$ obtained with two and four linear Timoshenko beam elements, two quadratic Timoshenko beam elements for $L/H = 100$ (since $L = 1.0$, we take $H = 0.01$; for $L/H = 100$ the shear deformation effect is negligible) along with the two-element solution of the Euler–Bernoulli beam element are shown in Fig. 6.2.7. The two linear element mesh of TBT beam elements predicts transient response that differs significantly from the EBT solution, and the TBT solution converges to the EBT solution as the number of elements or the degree of approximation is increased. Should we use conditionally stable schemes, it can be shown that the Timoshenko beam element requires larger Δt_{cri} than the Euler–Bernoulli beam element. This is because, as L/H is decreased, the ω_{max} predicted by the TBT is smaller than that predicted by the EBT.

Table 6.2.2 Effect of mesh on the transient response of a beam clamped at both ends ($\Delta t = 0.005$).

	EBT elements					TBT elements*		
	$\alpha = 0.5, \gamma = 1/3$		$\alpha = 0.5, \gamma = 0.5$			$\alpha = 0.5, \gamma = 0.5$		
Time t	1	2	1	2	4	2L	4L	2Q
0.00	0.2146	0.2146	0.2146	0.2146	0.2146	0.2146	0.2146	0.2146
0.01	0.2091	0.2098	0.2091	0.2098	0.2098	0.2124	0.2099	0.2116
0.02	0.1928	0.1951	0.1928	0.1951	0.1951	0.1938	0.1936	0.1951
0.03	0.1666	0.1696	0.1667	0.1696	0.1698	0.1550	0.1686	0.1690
0.04	0.1319	0.1346	0.1320	0.1348	0.1350	0.1145	0.1374	0.1427
0.05	0.0904	0.0930	0.0905	0.0931	0.0935	0.0695	0.0980	0.1067
0.06	0.0442	0.0481	0.0443	0.0482	0.0483	0.0093	0.0073	0.0657
0.07	−0.0043	0.0014	−0.0041	0.0014	0.0018	−0.0467	−0.0403	0.0234
0.08	−0.0525	−0.0462	−0.0523	−0.0459	−0.0455	−0.0917	−0.0844	−0.0267
0.09	−0.0980	−0.0926	−0.0978	−0.0923	−0.0916	−0.1422	−0.1254	−0.0706
0.10	−0.1385	−0.1345	−0.1383	−0.1342	−0.1336	−0.1833	−0.1588	−0.1100
0.11	−0.1719	−0.1685	−0.1717	−0.1685	−0.1682	−0.2010	−0.1875	−0.1461
0.12	−0.1964	−0.1933	−0.1963	−0.1933	−0.1932	−0.2154	−0.2069	−0.1717
0.13	−0.2108	−0.2088	−0.2108	−0.2088	−0.2087	−0.2205	−0.2063	−0.1969
0.14	−0.2144	−0.2153	−0.2144	−0.2150	−0.2148	−0.1986	−0.2138	−0.2110
0.15	−0.2070	−0.2113	−0.2071	−0.2112	−0.2111	−0.1696	−0.2134	−0.2146

* 2L = two linear elements; 4L = four linear elements; 2Q = two quadratic elements.

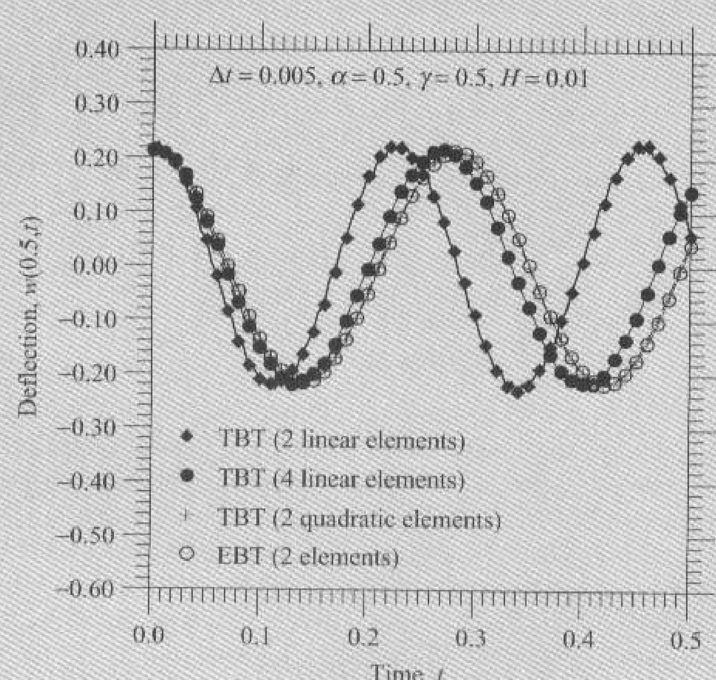

Figure 6.2.7 Transient response of a beam clamped at both ends, according to the TBT and EBT ($EI = 1$, $\rho A = 1$, $H = 0.01$, $\Delta t = 0.005$, $\alpha = 0.5$, and $\gamma = 0.5$).

6.3 SUMMARY

In this chapter, finite element formulations of eigenvalue problems and time-dependent problems are developed. One-dimensional, second- and fourth-order equations (beams) have been discussed. The eigenvalue problems studied include problems of heat transfer (and the like), bars, and beams. In the case of bars and beams, the eigenvalue problems arise in connection with natural vibrations and buckling of columns. Except for the solution procedure, the finite element formulation of eigenvalue problems is entirely analogous to boundary value problems.

Finite element models of time-dependent problems described by parabolic and hyperbolic equations have also been presented. A two-step procedure to derive finite element models from differential equations has been described. In the first step, we seek spatial approximations of the dependent variables of the problem as linear combinations of nodal values that are functions of time and interpolation functions that are functions of space. This procedure is entirely analogous to the finite element formulation presented for boundary value problems in Chapters 3–5. The end result of this step is a set of ordinary differential equations (in time) among the nodal values. In the second step, the ordinary differential equations are further approximated using the finite difference approximation of the time derivatives. The resulting algebraic equations can be solved for repeatedly, marching in time. Examples using both transient heat transfer and beam bending are presented.

PROBLEMS

Section 6.1

Most problems are formulative in nature. For eigenvalue problems, we need to write the final characteristic equations for the eigenvalues.

6.1 Determine the first two eigenvalues associated with the heat transfer problem whose governing equations and boundary conditions are given by (Fig. P6.1)

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+b\frac{\partial u}{\partial t}+cu=0 \quad \text{for } 0<x<L$$

$$u(0)=0, \quad \left(a\frac{\partial u}{\partial x}+\beta u\right)\bigg|_{x=L}=0$$

where a, b, c, and β are constants. Use (*a*) two linear finite elements and (*b*) one quadratic element in the domain to solve the problem.

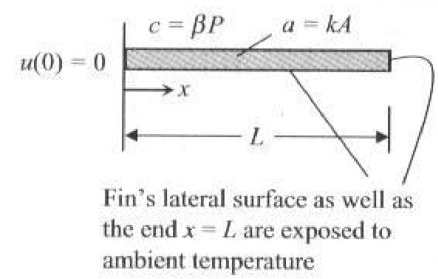

Figure P6.1

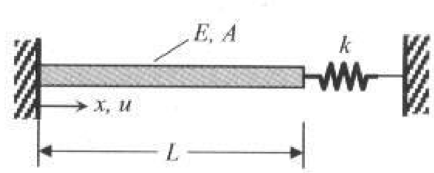

Figure P6.2

6.2 Determine the first two longitudinal frequencies of a rod (with Young's modulus E, area of cross section A, and length L) that is fixed at one end (say, at $x = 0$) and supported axially at the other end (at $x = L$) by a linear elastic spring (with spring constant k), as shown in Fig. P6.1:

$$-EA\frac{\partial^2 u}{\partial x^2} + \rho A\frac{\partial^2 u}{\partial t^2} = 0 \quad \text{for } 0 < x < L$$

$$u(0) = 0, \quad \left(EA\frac{du}{dx} + ku\right)\bigg|_{x=L} = 0$$

Use (*a*) two linear finite elements and (*b*) one quadratic element in the domain to solve the problem. *Answer:* (a) The characteristic equation is $7\lambda^2 - (10 + 4c)\lambda + (1 + 2c) = 0$, $c = kL/2EA$, $\lambda = (\rho h^2/6E)\omega^2$.

6.3 Determine the smallest natural frequency of a beam with clamped ends and of constant cross-sectional area A, moment of inertia I, and length L. Use the symmetry and two Euler–Bernoulli beam elements in the half beam.

6.4 Resolve the above problem with two RIEs in the half-beam.

6.5 Consider a beam (of Young's modulus E, shear modulus G, area of cross section A, second moment area about the axis of bending I, and length L) with its left end ($x = 0$) clamped and its right end ($x = L$) supported vertically by a linear elastic spring (see Fig. P6.5). Determine the fundamental natural frequency using (*a*) one Euler–Bernoulli beam element and (*b*) one Timoshenko beam element (IIE) (use the same mass matrix in both elements).

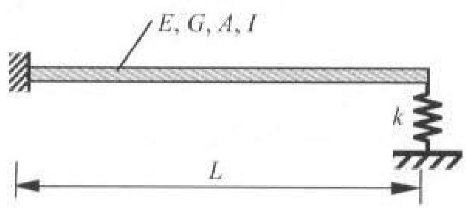

Figure P6.5

6.6 Consider a simply supported beam (of Young's modulus E, mass density ρ, area of cross section A, second moment of area about the axis of bending I, and length L) with an elastic support at the center of the beam (see Fig. P6.6). Determine the fundamental natural frequency using the minimum number of Euler–Bernoulli beam elements. *Answer:* The characteristic polynomial is $455\lambda^2 - 2(129 + c)\lambda + 3 + 2c = 0$.

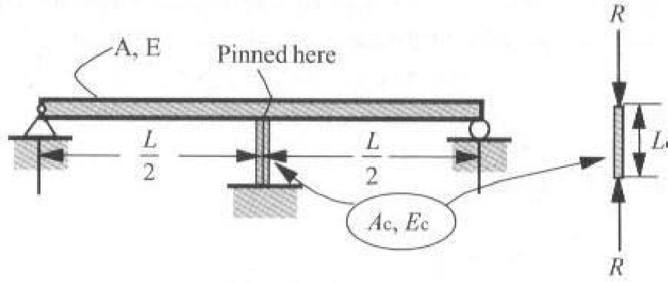

Figure P6.6

6.7 Determine the critical buckling load of a cantilever beam (A, I, L, E) using (*a*) one Euler–Bernoulli beam element and (*b*) one Timoshenko beam element.

6.8 The natural vibration of a beam under applied axial compressive load N^0 is governed by the differential equation

$$EI\frac{d^4w}{dx^4} + N^0\frac{d^2w}{dx^2} = \lambda w$$

where λ denotes nondimensional frequency of natural vibration and EI is the flexural stiffness of the beam. (*a*) Determine the fundamental (i.e., smallest) natural frequency ω of a cantilever beam (i.e., fixed at one end and free at the other end) of length L with axial compressive load N_0 using one beam element. (*b*) What is the buckling load of the beam? You are required to give the final characteristic equation in each case.

6.9 Determine the fundamental natural frequency of the truss shown in Fig. P6.9 (you are required only to formulate the problem).

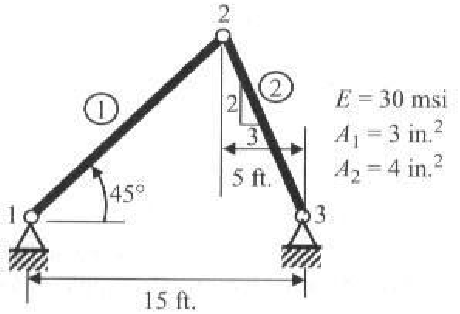

Figure P6.9

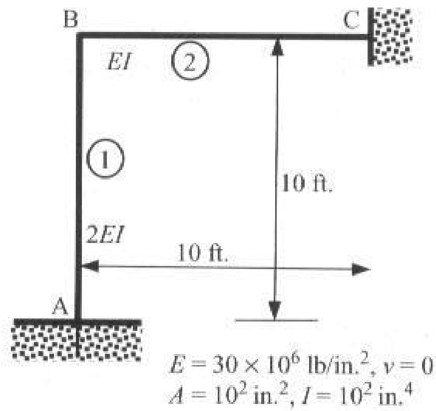

Figure P6.10

6.10 Determine the fundamental natural frequency of the truss shown in Fig. P6.10 (you are required only to formulate the problem).

6.11 Determine the first two longitudinal natural frequencies of a rod (A, E, L, m), fixed at one end and with an attached mass m_2 at the other. Use two linear elements. *Hint:* Note that the boundary conditions for the problem are $u(0) = 0$ and $(EA\ \partial u/\partial x + m_2\ \partial^2 u/\partial t^2)|_{x=L} = 0$.

6.12 The equation governing torsional vibration of a circular rod is

$$-GJ\frac{\partial^2\phi}{\partial x^2} + mJ\frac{\partial^2\phi}{\partial t^2} = 0$$

where ϕ is the angular displacement, J the moment of inertia, G the shear modulus, and m the density. Determine the fundamental torsional frequency of a rod with disk (J_1) attached at each end. Use the symmetry and (*a*) two linear elements and (*b*) one quadratic element.

6.13 The equations governing the motion of a beam according to TBT can be reduced to the single equation

$$a^2\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} - b^2\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{b^2 m}{kG}\frac{\partial^4 w}{\partial t^4} = 0$$

where $a^2 = EI/mA$ and $b^2 = I/A$. Here E is the Young's modulus, G is the shear modulus, m is the mass per unit length, A is the area of cross section, and I is the moment of inertia. Assuming that $(b^2 m/kG) << 1$ (i.e., neglect the last term in the governing equation), formulate the finite element model of the (*a*) eigenvalue problem for the determination of natural frequencies and (*b*) fully discretized problem for the determination of the transient response.

6.14 Use the finite element model of Problem 6.13 to determine the fundamental frequency of a simply supported beam.

6.15 Find the critical buckling load P_{cri} by determining the eigenvalues of the equation

$$EI\frac{d^4w}{dx^4} + P_{cri}\frac{d^2w}{dx^2} = 0 \quad \text{for } 0 < x < L$$

$$w(0) = w(L) = 0, \quad \left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x=0} = \left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x=L} = 0$$

Use one Euler–Bernoulli element in the half-beam. *Answer:* $P_{cri} = 9.9439EI/L^2$.

Section 6.2

6.16 Consider the partial differential equation arising in connection with unsteady heat transfer in an insulated rod:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) = f \quad \text{for } 0 < x < L$$

$$u(0,t) = 0, \quad u(x,0) = u_0, \quad \left[a\frac{\partial u}{\partial x} + \beta(u - u_\infty) + \hat{q}\right]\bigg|_{x=L} = 0$$

Following the procedure outlined in Section 6.2, derive the semidiscrete variational form, the semidiscrete finite element model, and the fully discretized finite element equations for a typical element.

6.17 Using a two-element (linear) model and the semidiscrete finite element equations derived in Problem 6.16, determine the nodal temperatures as functions of time for the case in which $a = 1$, $f = 0$, $u_0 = 1$, and $\hat{q} = 0$. Use the Laplace transform technique [see Reddy (2002)] to solve the ordinary differential equations in time.

6.18 Consider a uniform bar of cross-sectional area A, modulus of elasticity E, mass density m, and length L. The axial displacement under the action of time-dependent axial forces is governed by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \quad a = \left(\frac{E}{m}\right)^{1/2}$$

Determine the transient response [i.e., find $u(x,t)$] of the bar when the end $x = 0$ is fixed and the end $x = L$ is subjected to a force P_0. Assume zero initial conditions. Use one linear element to approximate the spatial variation of the solution, and solve the resulting ordinary differential equation in time exactly to obtain

$$u_2(x,t) = \frac{P_0 L}{AE}\frac{x}{L}(1 - \cos\alpha t), \quad \alpha = \sqrt{3}\frac{a}{L}$$

6.19 Resolve Problem 6.18 with a mesh of two linear elements. Use the Laplace transform method to solve the two ordinary differential equations in time.

6.20 Solve Problem 6.18 when the right end is subjected to an axial force F_0 and supported by an axial spring of stiffness k. *Answer:*

$$u_2(t) = c(1 - \cos\beta t), \quad c = \frac{3F_0}{mAL\beta^2}, \quad \beta = \sqrt{3}\frac{a}{L}\left(1 + \frac{kL}{EA}\right)^{1/2}$$

6.21 A bar of length L moving with velocity v_0 strikes a spring of stiffness k. Determine the motion $u(x, t)$ from the instant when the end $x = 0$ strikes the spring. Use one linear element.

6.22 A uniform rod of length L and mass m is fixed at $x = 0$ and loaded with a mass M at $x = L$. Determine the motion $u(x, t)$ of the system when the mass M is subjected to a force P_0. Use one linear element. *Answer:*

$$u_2(t) = c(1 - \cos\lambda t), \quad c = \frac{P_0 L}{AE}, \quad \lambda = \sqrt{3}\frac{a}{L}\left(\frac{3M}{AL} + m\right)^{-1}$$

6.23 The flow of liquid in a pipe, subjected to a surge-of-pressure wave (i.e., a water hammer), experiences a surge pressure p, which is governed by the equation

$$\frac{\partial^2 p}{\partial t^2} - c^2\frac{\partial^2 p}{\partial x^2} = 0, \quad c^2 = \frac{1}{m}\left(\frac{1}{k} + \frac{D}{bE}\right)^{-1}$$

where m is the mass density and K the bulk modulus of the fluid, D is the diameter and b the thickness of the pipe, and E is the modulus of elasticity of the pipe material. Determine the pressure $p(x, t)$ using one linear finite element for the following boundary and initial conditions:

$$p(0, t) = 0, \quad \frac{\partial p}{\partial x}(L, t) = 0, \quad p(x, 0) = p_0, \quad \dot{p}(x, 0) = 0$$

6.24 Consider the problem of determining the temperature distribution of a solid cylinder, initially at a uniform temperature T_0 and cooled in a medium of zero temperature (i.e., $T_\infty = 0$). The governing equation of the problem is

$$\rho c\frac{\partial T}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) = 0$$

The boundary conditions are

$$\frac{\partial T}{\partial r}(0, t) = 0, \quad \left(rk\frac{\partial T}{\partial r} + \beta T\right)\Big|_{r=R} = 0$$

The initial condition is $T(r, t) = T_0$. Determine the temperature distribution $T(r, t)$ using one linear element. Take $R = 2.5$ cm, $k = 215$ W/(m·°C), $\beta = 525$ W/(m·°C), $T_0 = 130$°C, $\rho = 2700$ kg/m^2, and $c = 0.9$ kJ/(kg·°C). What is the heat loss at the surface? Formulate the problem.

6.25 Determine the nondimensional temperature $\theta(r, t)$ in the region bounded by two long cylindrical surfaces of radii R_1 and R_2. The dimensionless heat conduction equation is

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right) + \frac{\partial\theta}{\partial t} = 0$$

with boundary and initial conditions

$$\frac{\partial\theta}{\partial r}(R_1, t) = 0, \quad \theta(R_2, t) = 1, \quad \theta(r, 0) = 0$$

6.26 Show that (6.2.28*a*), (6.2.28*b*), (6.2.29*a*), and (6.2.29*b*) can be reworked to match the form in Eq. (6.2.38) as

$$[H]\{\ddot{u}\}_{s+1} = \{\tilde{F}\}_{s+1}$$

and define $[H]$ and $\{\tilde{F}\}_{s+1}$.

6.27 A uniform cantilever beam of length L, moment of inertia I, modulus of elasticity E, and mass m begins to vibrate with initial displacement

$$w(x, 0) = w_0 x^2 / L^2$$

and zero initial velocity. Find its displacement at the free end at any subsequent time. Use one Euler–Bernoulli beam element to determine the solution. Solve the resulting differential equations in time using the Laplace transform method.

6.28 Resolve Problem 6.27 using one Timoshenko beam element.

REFERENCES FOR ADDITIONAL READING

1. Argyris, J. H. and Scharpf, O. W., "Finite Elements in Time and Space," *Aeronautical Journal of the Royal Society,* **73**, 1041–1044, 1969.
2. Bathe, K. J., *Finite Element Procedures in Engineering Analysis,* Prentice-Hall, Englewood Cliffs, NJ, 1982.
3. Bathe, K. J. and Wilson, E. L., "Stability and Accuracy Analysis of Direct Integration Methods," *International Journal of Earthquake Engineering and Structural Dynamics,* **1**, 283–291, 1973.
4. Belytschko, T., "An Overview of Semidiscretization and Time Integration Procedures," in Belytschko T. and Hughes T. J. R. (eds.), *Computational Methods for Transient Analysis,* North-Holland, Amsterdam, pp. 1–65, 1983.
5. Goudreau, G. L., and Taylor, R. L., "Evaluation of Numerical Integration Methods in Elastodynamics," *Journal of Computer Methods in Applied Mechanics and Engineering,* **2**(1), 69–97, 1973.
6. Hilber, H. M., "Analysis and Design of Numerical Integration Methods in Structural Dynamics," EERC Report no. 77–29, Earthquake Engineering Research Center, University of California, Berkeley, CA, November 1976.
7. Houbolt, J. C., "A Recurrence Matrix Solution for the Dynamic Response of Elastic Aircraft," *Journal of Aeronautical Science,* **17**, 540–550, 1950.
8. Newmark, N. M., "A Method of Computation for Structural Dynamics," *Journal of Engineering Mechanics Division, ASCE,* **85**, 67–94, 1959.
9. Nickell, R. E., "On the Stability of Approximation Operators in Problems of Structural Dynamics," *International Journal of Solids and Structures,* **7**, 301–319, 1971.
10. Reddy, J. N., *Theory and Analysis of Elastic Plates,* Taylor and Francis, Philadelphia, PA, 1999*a*.
11. Reddy, J. N., "On the Dynamic Behavior of the Timoshenko Beam Finite Elements," *Sadhana* (Journal of the Indian Academy of Sciences), **24**, Part 3, 175–198, 1999*b*.
12. Reddy, J. N., "On the Derivation of the Superconvergent Timoshenko Beam Finite Element," *Int. J. Comput. Civil and Struct. Engng.,* **1**(2), 71–84, 2000.
13. Reddy, J. N., *Energy Principles and Variational Methods in Applied Mechanics,* John Wiley, New York, 2002.
14. Wood, W. L., "Control of Crank–Nicolson Noise in the Numerical Solution of the Heat Conduction Equation," *International Journal for Numerical Methods in Engineering,* **11**, 1059–1065, 1977.