Chapter 4

SECOND-ORDER DIFFERENTIAL EQUATIONS IN ONE DIMENSION: APPLICATIONS

4.1 PRELIMINARY COMMENTS

In Chapter 3 we developed weak forms and finite element models of continuum problems described by a fairly general second-order differential equation. For discrete systems, such as a network of springs or electrical circuits, no differential equations exist and the weak form concept is not applicable. Therefore, an alternate approach based on the laws of physics must be used to develop finite element models (i.e., relations between the cause and effect) of such systems. Physical principles can also be used to develop finite element models of continuum problems (as discussed in Remark 7 of Chapter 3) but the approach cannot be used to derive finite element models with higher-order approximation of the field variable.

The objective of this chapter is two-fold. First, we derive finite element models of some typical discrete systems. Finite element models of discrete systems are developed using physical laws familiar to most engineering and applied science majors, and the approach requires no concept of weak form. Second, we present numerical examples of application of finite element models developed for both discrete systems and continuum systems. We will consider several examples to illustrate the steps involved in the finite element analysis of one-dimensional second-order differential equations arising in heat transfer, fluid mechanics, and solid mechanics. The examples presented here make use of the element equations already developed in Chapter 3. While the notation used for the dependent variables, independent coordinates, and data of problems from field to field is different, the reader should keep the common mathematical structure in mind and not get confused with the change of notation from problem to problem and field to field.

4.2 DISCRETE SYSTEMS

4.2.1 Linear Elastic Spring

A linear elastic spring is a discrete element (i.e., not a continuum) whose load-displacement relationship can be expressed as

$$F = k\delta \tag{4.2.1}$$

where F is the force (N) at the right end, δ is the displacement (m) of the right end of the spring relative to the left end in the direction of the force, and k is the constant known as the *spring constant* (N/m). The spring constant depends on the elastic modulus, area of cross section, and number of turns in the coil of the spring. Often a spring is used to characterize the elastic behavior of complex physical systems.

A relationship between the end forces (F_1^e, F_2^e) and end displacements (δ_1^e, δ_2^e) of a typical spring element [see Fig. 4.2.1(a)] can be developed as discussed in Remark 7 for a bar element. The force F_1^e at node 1 is equal to the spring constant multiplied by the relative displacement of node 1 with respect to node 2, $\delta_1^e - \delta_2^e$:

$$F_1^e = k_e(\delta_1^e - \delta_2^e) = k_e\delta_1^e - k_e\delta_2^e$$

Similarly, the force at node 2 is equal to

$$F_2^e = k_e(\delta_2^e - \delta_1^e) = -k_e\delta_1^e + k_e\delta_2^e$$

Note that the force equilibrium, $F_1^e + F_2^e = 0$, is automatically satisfied. The above equations can be written in matrix form as

$$k_e\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \delta_1^e \\ \delta_2^e \end{Bmatrix} = \begin{Bmatrix} F_1^e \\ F_2^e \end{Bmatrix} \tag{4.2.2}$$

Equation (4.2.2) is applicable to any spring element whose force-displacement relation is linear. Thus a typical spring in a network of springs of different spring constants obey Eq. (4.2.2).

Example 4.2.1

Consider the spring assemblage shown in Fig. 4.2.1(b). We wish to determine the displacement of the rigid block and forces in the springs. Assume that the rigid block is required to remain vertical (i.e., no tilting from its vertical position). Since the rigid block must remain vertical, all points on it will move horizontally by the same amount; hence, all global nodes on the block must have the same node number, say 2. Each spring in the assemblage has the same force-displacement relationship as in Eq. (4.2.2), except that element number will be different for different elements.

The three elements are connected at node 2 through the rigid block. Hence, the continuity and equilibrium conditions at node 2 require

$$\delta_2^{(1)} = \delta_1^{(2)} = \delta_1^{(3)} = U_2 \tag{4.2.3a}$$

$$F_2^{(1)} + F_1^{(2)} + F_1^{(3)} = F_2 \tag{4.2.3b}$$

The equilibrium condition (4.2.3*b*) suggests that we must add the second equation of element 1, the first equation of element 2, and the first equation of element 3 together to replace the sum of three unknowns forces $F_2^{(1)} + F_1^{(2)} + F_1^{(3)}$ with the known force F_2. Thus, we have four equations: the first equation of element 1, the sum of the three equations stated above, the second equation of element 2, and the second equation of element 3:

$$
\begin{aligned}
k_1U_1 - k_1U_2 &= F_1 \\
-k_1U_1 + (k_1 + k_2 + k_3)\,U_2 - k_2U_3 - k_3U_4 &= F_2 \\
-k_2U_2 + k_2U_3 &= F_3 \\
-k_3U_2 + k_3U_4 &= F_4
\end{aligned}
\tag{4.2.4a}
$$

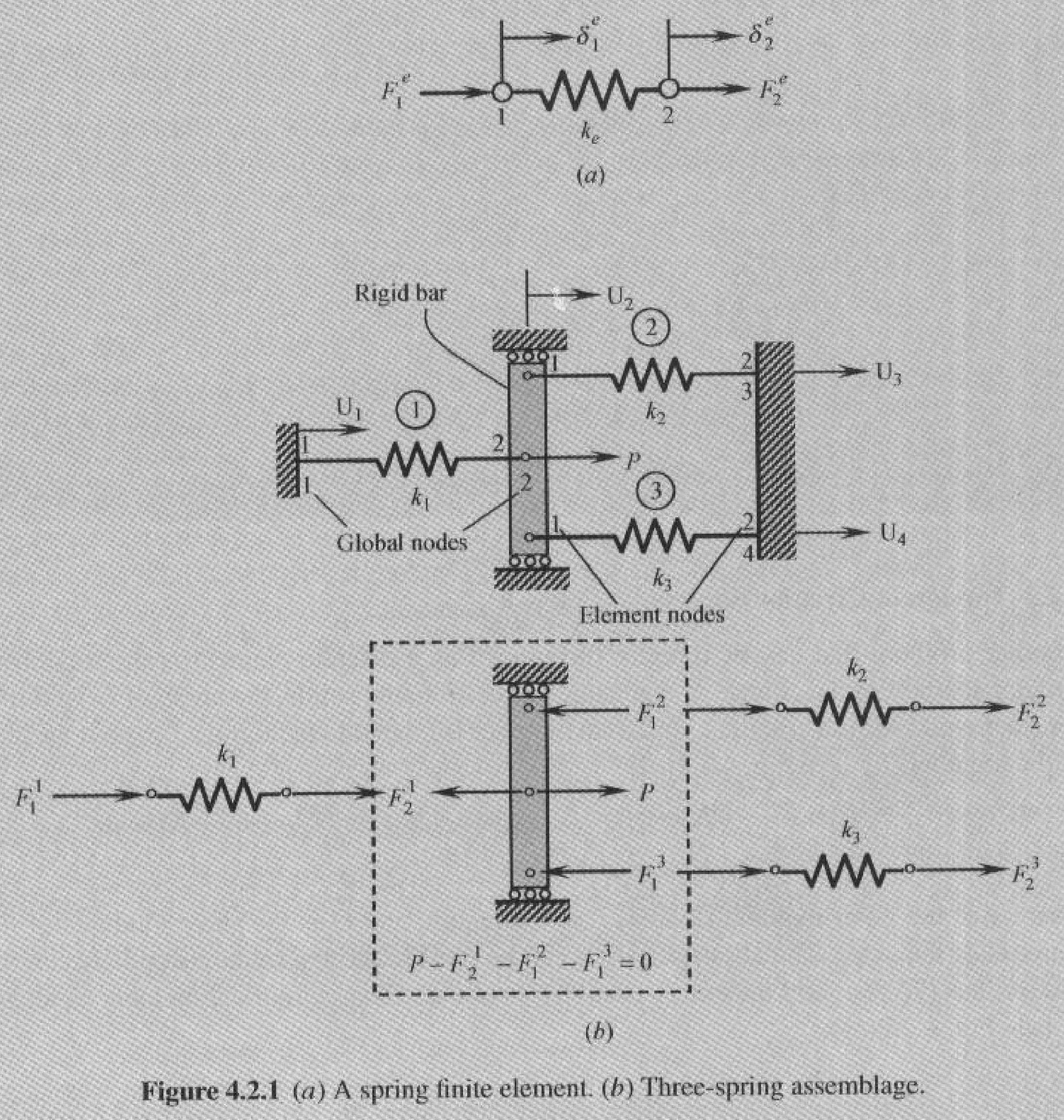


Figure 4.2.1 (*a*) A spring finite element. (*b*) Three-spring assemblage.

In matrix form, the above equations can be expressed as

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} \tag{4.2.4b}$$

Next, we identify the boundary conditions and impose them on Eq. (4.2.4b). From Fig. 4.2.1, it is clear that the displacements of nodes 1, 3, and 4 are zero, and the force at node 2 is specified to be P:

$$U_1 = U_3 = U_4 = 0, \qquad F_2 = P \tag{4.2.5}$$

Thus, there are four unknowns ($F_1 = F_1^{(1)}, U_2, F_3 = F_2^{(2)}, F_4 = F_2^{(3)}$) and four equations. Using the second equation in (4.2.4a), we determine U_2 (condensed equation for the displacement)

$$(k_1+k_2+k_3)U_2 = P \text{ or } U_2 = \frac{P}{k_1+k_2+k_3} \tag{4.2.6}$$

The forces F_1, F_3, and F_4 can be calculated using equations 1, 3, and 4 of (4.2.4a). The condensed equations for forces are

$$\begin{aligned} F_1 = F_1^{(1)} &= -k_1U_2 = -\frac{Pk_1}{k_1+k_2+k_3} \\ F_3 = F_2^{(2)} &= -k_2U_2 = -\frac{Pk_2}{k_1+k_2+k_3} \\ F_4 = F_2^{(3)} &= -k_3U_2 = -\frac{Pk_3}{k_1+k_2+k_3} \end{aligned} \tag{4.2.7}$$

4.2.2 Torsion of Circular Shafts

Another problem that can be directly formulated as a discrete element is the torsion of circular shafts. From a course on mechanics of deformable solids, the angle of twist θ of a constant cross-section circular cylindrical member is related to the torque T (about the axis of the member) by [see Fig. 4.2.2(a)]

$$T = \frac{GJ}{L}\theta \tag{4.2.8}$$

where J is the polar moment of area, L is the length, and G is the shear modulus of the material of the shaft. The above equation can be used to write a relationship between the end torques (T_1^e, T_2^e) and the end twists (θ_1^e, θ_2^e) of a circular cylindrical member of length h_e [see Fig. 4.2.2(b)]:

$$T_1^e = \frac{G_eJ_e}{h_e}\left(\theta_1^e - \theta_2^e\right), \; T_2^e = \frac{G_eJ_e}{h_e}\left(\theta_2^e - \theta_1^e\right)$$

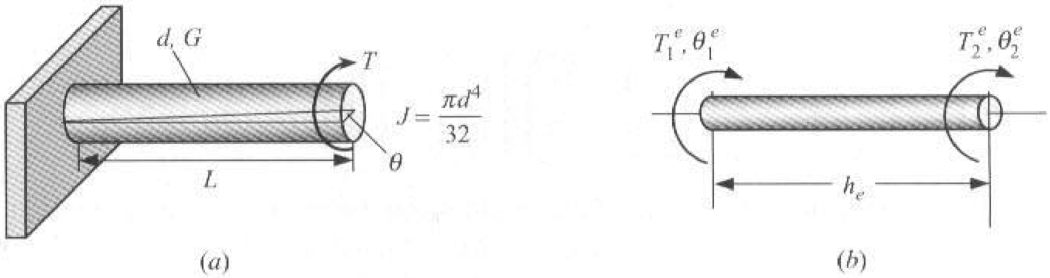


Figure 4.2.2 Torsion of a circular shaft.

or

$$\frac{G_e J_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix} \tag{4.2.9}$$

4.2.3 Electrical Resistor Circuits

There is a direct analogy between a network of mechanical springs and a direct current electric resistor network. Ohm's law provides the relationship between flow of electric current I (amperes) through an ideal resistor and voltage drop V (volts) between the ends of the resistor

$$V = IR \tag{4.2.10}$$

where R denotes the electric resistance (ohms) of the wire.

Kirchhoff's voltage rule states that the algebraic sum of the voltage changes in any loop must be equal to zero. Applied to a single resistor, the rule gives [see Fig. 4.2.3(a)]

$$I_1^e R_e + V_2^e - V_1^e = 0, \quad I_2^e R_e + V_1^e - V_2^e = 0$$

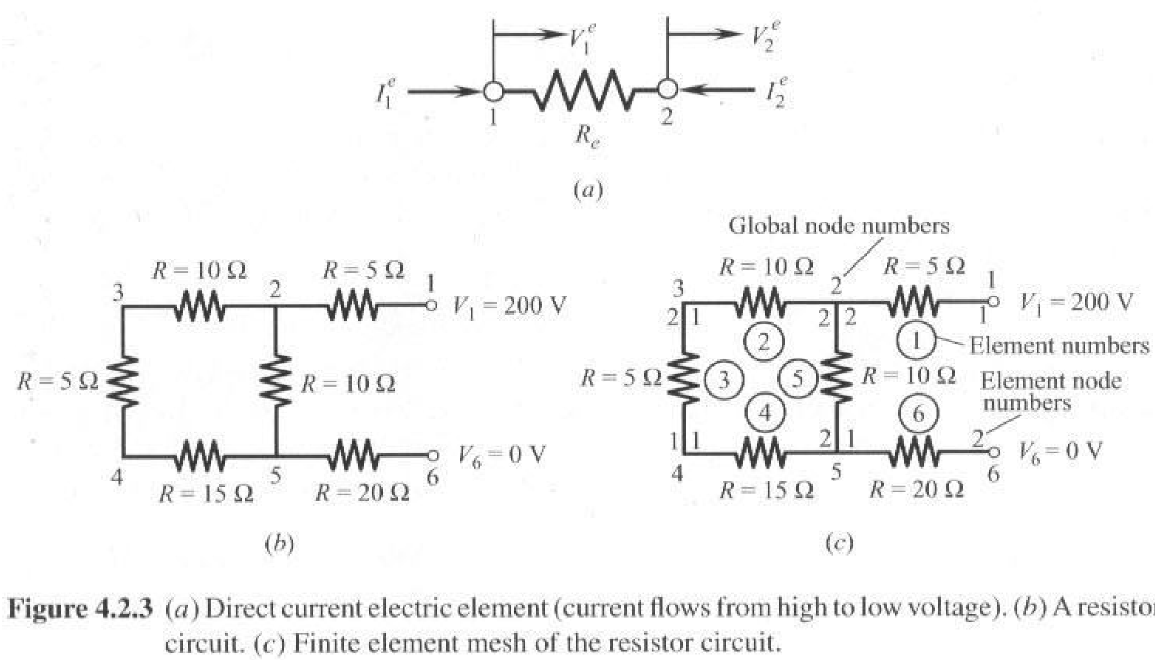


Figure 4.2.3 (a) Direct current electric element (current flows from high to low voltage). (b) A resistor circuit. (c) Finite element mesh of the resistor circuit.

or in matrix form

$$\frac{1}{R_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} V_1^e \\ V_2^e \end{Bmatrix} = \begin{Bmatrix} I_1^e \\ I_2^e \end{Bmatrix} \tag{4.2.11}$$

Thus, once again we have the same form of relationship between voltages and currents as in the case of springs. The quantity $1/R_e$ is known as the *electrical conductance*.

The assembly of resistor equations is based on the following rules:

1. Voltage is single-valued.
2. *Kirchhoff current rule:* The sum of all currents entering a node is equal to zero.

Example 4.2.2

Consider the resistor circuit shown in Fig. 4.2.3(*b*). We wish to determine the currents in the loops and voltage at the nodes. The element numbering, element node numbering, and global node numbering are indicated in Fig. 4.2.3(*c*). The element node numbering is important in assembling the element equations. From the element notation indicated in Fig. 4.2.3(*a*), it is clear that current flows from node 1 to node 2 of the element. The element node numbering in Fig. 4.2.3(*c*) indicates the assumed direction of the currents.

The assembled coefficient matrix is given by

$$[K] = \begin{array}{cc} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 & 0 & 0 \\ & K_{22}^1 + K_{22}^2 + K_{22}^5 & K_{21}^2 & 0 & K_{21}^5 & 0 \\ & & K_{11}^2 + K_{22}^3 & K_{21}^3 & 0 & 0 \\ & & & K_{11}^3 + K_{11}^4 & K_{12}^4 & 0 \\ \text{symm.} & & & & K_{22}^4 + K_{11}^5 + K_{11}^6 & K_{12}^6 \\ & & & & & K_{22}^6 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

$$= \begin{bmatrix} 0.2 & -0.2 & 0 & 0 & 0 & 0 \\ & 0.2+0.1+0.1 & -0.1 & 0 & -0.1 & 0 \\ & & 0.1+0.2 & -0.2 & 0 & 0 \\ & & & 0.2+0.0667 & -0.0667 & 0 \\ \text{symm.} & & & & 0.0667+0.1+0.05 & -0.05 \\ & & & & & 0.05 \end{bmatrix} \tag{4.2.12}$$

The boundary conditions are $V_1 = 200$ V and $V_6 = 0$ V. The condensed equations for the nodal voltages are obtained by omitting the first row and last row of the system, and then moving the terms involving V_1 and V_6 to the right. We obtain

$$\begin{bmatrix} 0.4 & -0.1 & 0.0 & -0.1 \\ -0.1 & 0.3 & -0.2 & 0.0 \\ 0.0 & -0.2 & 0.2667 & -0.0667 \\ -0.1 & 0.0 & -0.0667 & 0.2167 \end{bmatrix} \begin{Bmatrix} V_2 \\ V_3 \\ V_4 \\ V_5 \end{Bmatrix} = \begin{Bmatrix} 0.2V_1 \\ 0 \\ 0 \\ 0.05V_6 \end{Bmatrix}$$

The solution of these equations is given by (obtained with the aid of a computer)

$$V_2 = 169.23 \text{ V},\ V_3 = 153.85 \text{ V},\ V_4 = 146.15 \text{ V},\ V_5 = 123.08 \text{ V}$$

The condensed equations for the unknown currents at nodes 1 and 6 can be calculated from equations 1 and 6 of the system. We have

$$I_1^{(1)} = 0.2V_1 - 0.2V_2 = 40 - 33.846 = 6.154 \text{ A}$$

$$I_2^{(6)} = -0.05V_5 + 0.05V_6 = -6.154 \text{ A}$$

The negative sign on $I_2^{(6)}$ indicates that the current is flowing out of global node 6.

The currents through each element can be calculated using the element equations (4.2.11). For example, the nodal currents in resistor 5 are given by

$$\begin{Bmatrix} I_1^{(5)} \\ I_2^{(5)} \end{Bmatrix} = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix} \begin{Bmatrix} 123.08 \\ 169.23 \end{Bmatrix} = \begin{Bmatrix} -4.615 \\ 4.615 \end{Bmatrix}$$

which indicates that the net current flow in resistor 5 is from its node 2 to node 1 (or global node 2 to global node 5), and its value is 4.615 amps (which is the same as 6.154 − 1.539). The finite element solutions for the voltages and currents are shown in Figure 4.2.4.

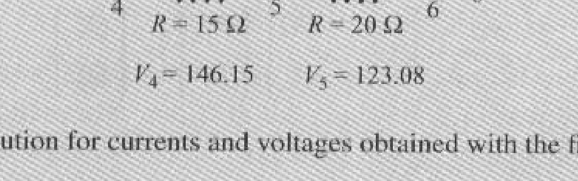


Figure 4.2.4 The solution for currents and voltages obtained with the finite element method.

4.2.4 Fluid Flow through Pipes

Another example of a discrete element is provided by steady, fully developed, flows of viscous incompressible fluids through circular pipes. The velocity of fully developed laminar

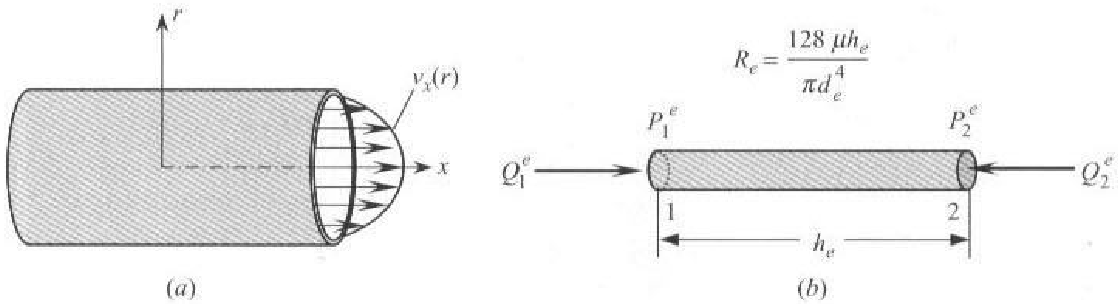


Figure 4.2.5 Flow of viscous fluids through pipes.

flow of viscous fluids through circular pipes is given by

$$v_x = -\frac{1}{4\mu}\frac{dP}{dx}\left[1 - \left(\frac{2r}{d}\right)^2\right] \tag{4.2.13a}$$

where dP/dx is the pressure gradient, d is the diameter of the pipe, and μ is the viscosity of the fluid [see Fig. 4.2.5(a)]. The volume rate of flow, Q, is obtained by integrating v_x over the pipe cross section. Thus, the relationship between Q and the pressure gradient dP/dx is given by the equation

$$Q = -\frac{\pi d^4}{128\mu}\frac{dP}{dx} \tag{4.2.13b}$$

The negative sign indicates that the flow is in the direction of negative pressure gradient.

Equation (4.2.13b) can be used to develop a relationship between the nodal values of the volume rate of flow, (Q_1^e, Q_2^e) and the pressure, (P_1^e, P_2^e), of a pipe element of length h_e and diameter d_e. The volume rate of flow entering node 1 is given by [see Fig. 4.2.5(b)]

$$Q_1^e = -\frac{\pi d_e^4}{128\mu h_e}(P_2^e - P_1^e)$$

Similarly, the volume rate of flow entering node 2 is

$$Q_2^e = -\frac{\pi d^4}{128\mu h_e}(P_1^e - P_2^e)$$

Thus, we have

$$\frac{\pi d^4}{128\mu h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} P_1^e \\ P_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{4.2.14}$$

The constant, $R_e = 128\mu h_e/\pi d_e^4$ is called the *pipe resistance*, in analogy with the electrical resistance [see Eq. (4.2.11)].

4.3 HEAT TRANSFER

4.3.1 Governing Equations

The equations governing conduction heat transfer were discussed in Example 1.2.2. Here we briefly-review the pertinent equations for our use.

The Fourier heat conduction law for one-dimensional systems states that the heat flow $q(x)$ is relate the temperature gradient $\partial T/\partial x$ by the relation (with heat flow in the positive direction of x)

$$q = -kA\frac{\partial T}{\partial x} \tag{4.3.1}$$

where k is the thermal conductivity of the material, A the cross-sectional area, and T the temperature. The negative sign in (4.3.1) indicates that heat flows downhill on the temperature scale. The balance of energy requires that

$$\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + Ag = \rho cA\frac{\partial T}{\partial t} \tag{4.3.2}$$

where g is the heat energy generated per unit volume, ρ is the density, c is the specific heat of the material, and t is time. Equation (4.3.2) governs the transient heat conduction in a slab or fin (i.e., a one-dimensional system) when the heat flow in the normal to the x-direction is zero. For a plane wall, we take $A = 1$.

In the case of radially symmetric problems with cylindrical geometries, (4.3.2) takes a different form. Consider a long cylinder of inner radius R_{i}, outer radius R_{o}, and length L. When L is very large compared with the diameter, it is assumed that heat flows in the radial direction r. The transient radially symmetric heat flow in a cylinder is governed by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + g = \rho c\frac{\partial T}{\partial t} \tag{4.3.3}$$

A cylindrical fuel element of a nuclear reactor, a current-carrying electrical wire, and a thick-walled circular tube provide examples of one-dimensional radial systems.

The boundary conditions for heat conduction involve specifying either the temperature T or the heat flow Q at a point:

$$T = T_0 \quad \text{or} \quad Q \equiv -kA\frac{\partial T}{\partial x} = Q_0 \tag{4.3.4}$$

It is known that when a heated surface is exposed to a cooling medium, such as air or liquid, the surface will cool faster. We say that the heat is convected away. The *convection heat transfer* between the surface and the medium in contact is given by *Newton's law of cooling*:

$$Q = \beta A(T_s - T_\infty) \tag{4.3.5}$$

where T_s is the surface temperature, T_∞ is the temperature of the surrounding medium, called the *ambient temperature*; and β is the *convection heat transfer coefficient* or *film conductance* (or film coefficient). The heat flow due to conduction and convection at a boundary point must be in balance with the applied flow Q_0:

$$\pm kA\frac{\partial T}{\partial x} + \beta A(T - T_\infty) + Q_0 = 0 \tag{4.3.6}$$

The sign of the first term in (4.3.6) is negative when the heat flow is from the fluid at T_∞ to the surface at the left end of the element, and it is positive when the heat flow is from the fluid at T_∞ to the surface at the right end.

Convection of heat from a surface to the surrounding fluid can be increased by attaching thin strips of conducting metal to the surface. The metal strips are called *fins*. For a fin with heat flow along its length, heat can convect across the lateral surface of the fin [see Fig. 4.3.1(a)]. To account for the convection of heat through the surface, we must add the rate of heat loss by convection to the right-hand side of (4.3.2):

$$\frac{\partial}{\partial x}\left(Ak\frac{\partial T}{\partial x}\right) + Aq = \rho c A\frac{\partial T}{\partial t} + P\beta(T - T_\infty) \tag{4.3.7a}$$

where P is the perimeter and β is the film coefficient. Equation (4.3.7a) can be expressed in the alternative form

$$\rho c A\frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + P\beta T = Ag + P\beta T_\infty \tag{4.3.7b}$$

The units of various quantities (in metric system) are as follows:

T	°C (celsius)	k	$W/(m \cdot {}^\circ C)$
g	W/m^3	ρ	kg/m^3
c	J/(kg · °C)	β	W/(m^2 · °C)

For a steady state, we set the time derivatives in (4.3.2), (4.3.3), (4.3.7a), and (4.3.7b) equal to zero. The steady-state equations for various one-dimensional systems are summarized below [see Fig. 4.3.1(b) and (c); see Eqs. (1.2.14) and (1.2.17)].

Plane Wall $[Q = k(dT/dx)]$

$$-\frac{d}{dx}\left(k\frac{dT}{dx}\right) = Ag \tag{4.3.8}$$

Fin $[Q = kA(dT/dx)]$

$$-\frac{d}{dx}\left(kA\frac{dT}{dx}\right) + cT = Ag + cT_\infty, \quad c = P\beta \tag{4.3.9}$$

Cylindrical System $[Q = k(dT/dr)]$

$$-\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = g(r) \tag{4.3.10}$$

The essential and natural boundary conditions associated with these equations are of the form

$$T = T_0, \quad Q + \beta A(T - T_\infty) + Q_0 = 0$$

Equations (4.3.8)–(4.3.10) are a special case of the model equation (3.2.1) discussed in Section 3.2, with $a = kA$, $c = P\beta$, and $f \rightarrow Ag + P\beta T_\infty$. We immediately have the finite element model of Eqs. (4.3.8) and (4.3.9) from (3.2.31a) and (3.2.31b):

$$[K^e]\{T^e\} = \{f^e\} + \{Q^e\} \tag{4.3.11a}$$

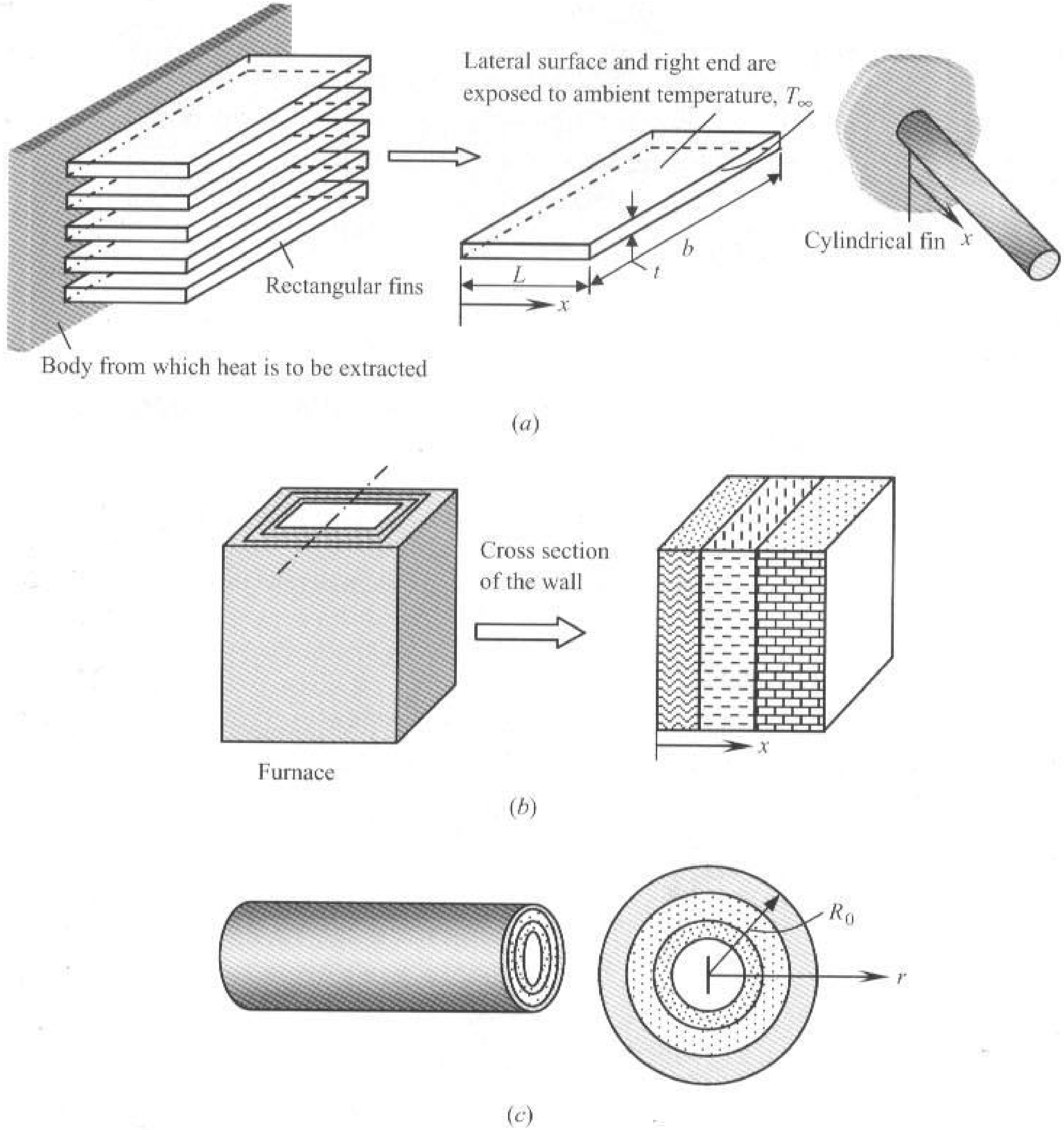


Figure 4.3.1 Heat transfer in (*a*) fins, (*b*) plane wall, and (*c*) radially symmetric system.

Here

$$K_{ij}^e = \int_{x_a}^{x_b} \left(kA \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + P\beta \psi_i^e \psi_j^e \right) dx, \quad f_i^e = \int_{x_a}^{x_b} \psi_i^e (Ag + P\beta T_\infty)\, dx$$

$$Q_1^e = \left(-kA \frac{dT}{dx} \right)\bigg|_{x_a}, \quad Q_2^e = \left(kA \frac{dT}{dx} \right)\bigg|_{x_b} \tag{4.3.11b}$$

where Q_1^e and Q_2^e denote heat flow *into* the element at the nodes.

Equation (4.3.10) is also a special case of the model boundary value problem. However, in developing the weak forms of (4.3.10), integration must be carried over a typical volume element of each system, as discussed in Section 3.4 [see Eq. (3.4.2)].

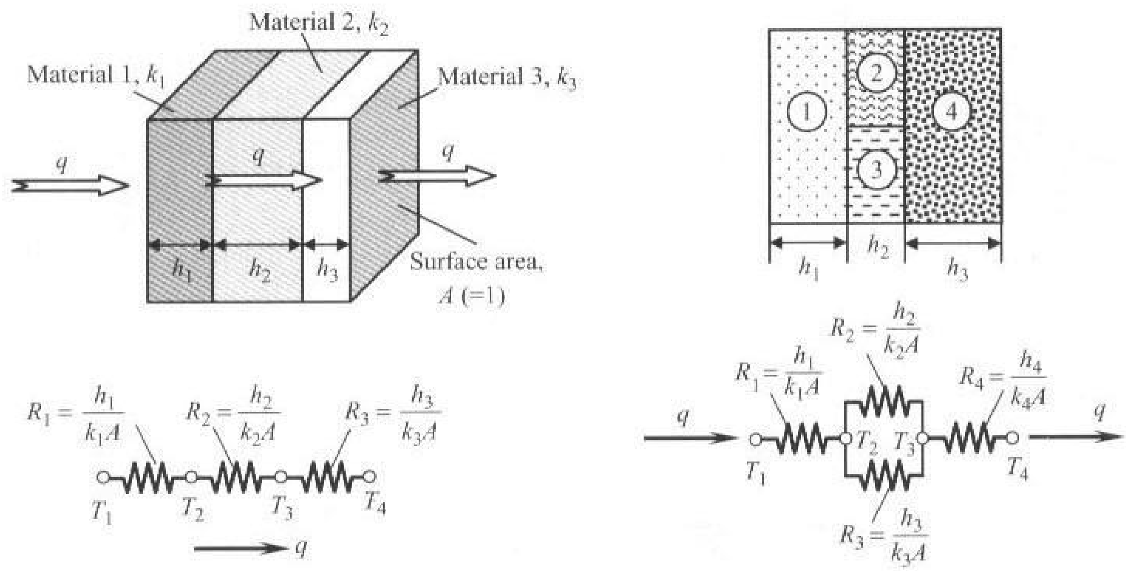


Figure 4.3.2 One-dimensional heat transfer through composite walls and their thermal circuits.

4.3.2 Finite Element Models

It is interesting to note the analogy between Eq. (4.2.11) of an electric resistor and Eq. (3.3.5*b*) of one-dimensional heat transfer (see Remark 7 of Section 3.3):

$$\frac{A_e k_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{4.3.12}$$

If we identify *thermal resistance* R_{th}^e by

$$R_{th}^e = \frac{h_e}{k_e A_e} \tag{4.3.13}$$

Equations (4.2.11) and (4.3.12) are the same with the following correspondence:

$$R_e \sim R_{th}^e, \quad I_i^e \sim Q_i^e, \quad V_i^e \sim T_i^e \tag{4.3.14}$$

This allows us to model complicated problems involving both series and parallel thermal resistances. Typical problems and their electrical analogies are shown in Figure 4.3.2.

4.3.3 Numerical Examples

Example 4.3.1

A composite wall consists of three materials, as shown in Fig. 4.3.3. The inside wall temperature is 200°C and the outside air temperature is 50°C with a convection coefficient of $\beta =$ 10 W (m^2·K). We wish to determine the temperature along the composite wall.

First, we note that the problem is governed by the equation

$$-kA\frac{d^2T}{dx^2} = 0, \quad 0 < x < L \tag{4.3.15}$$

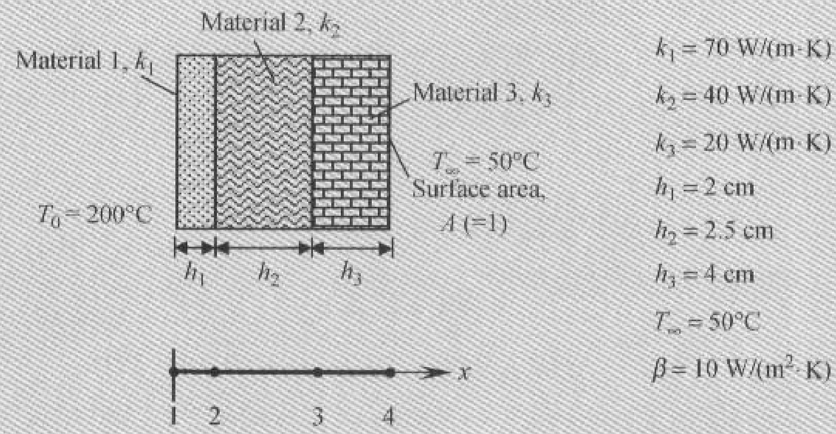


Figure 4.3.3 Heat transfer problem discussed in Example 4.3.1.

and is subjected to the boundary conditions

$$T(0) = T_0, \quad \left[kA\frac{dT}{dx} + \beta A(T - T_\infty)\right]_{x=L} = 0 \tag{4.3.16}$$

where A is the area of cross section (can be taken to be equal to 1) and k is the conductivity.

The data is discontinuous (i.e., different values of $a_e = k_e A_e$ and h_e for different elements). The source term $f = 0$ and therefore $\{f\}^{(e)} = \{0\}$ for $e = 1, 2, 3$. For a mesh of three linear elements (minimum needed), we have

$$[\mathbf{K}]^{(1)} = \frac{k_1 A}{h_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{70 \times 1}{0.02}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3500 & -3500 \\ -3500 & 3500 \end{bmatrix}$$

$$[\mathbf{K}]^{(2)} = \frac{k_2 A}{h_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{40 \times 1}{0.025}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1600 & -1600 \\ -1600 & 1600 \end{bmatrix}$$

$$[\mathbf{K}]^{(3)} = \frac{k_3 A}{h_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{20 \times 1}{0.04}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}$$

The assembled equations are

$$\begin{bmatrix} 3500 & -3500 & 0 & 0 \\ -3500 & 3500 + 1600 & -1600 & 0 \\ 0 & -1600 & 1600 + 500 & -500 \\ 0 & 0 & -500 & 500 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 \end{Bmatrix}$$

The boundary and balance conditions are

$$U_1 = 200,\ Q_2^1 + Q_1^2 = 0,\ Q_2^2 + Q_1^3 = 0,\ Q_2^3 = -\beta A(U_4 - T_\infty) = -10 \times 1(U_4 - 50)$$

Hence, the condensed equations (obtained by omitting first row and column and modifying the right-hand side) for the unknown temperatures are

$$\begin{bmatrix} 3500 + 1600 & -1600 & 0 \\ -1600 & 1600 + 500 & -500 \\ 0 & -500 & 500 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 3500 \times 200 \\ 0 \\ -10U_4 + 500 \end{Bmatrix}$$

or

$$\begin{bmatrix} 5100 & -1600 & 0 \\ -1600 & 2100 & -500 \\ 0 & -500 & 510 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 7 \times 10^5 \\ 0 \\ 500 \end{Bmatrix}$$

The solution is given by

$$U_1 = 200^\circ\text{C},\ U_2 = 199.58^\circ\text{C},\ U_3 = 198.67^\circ\text{C},\ U_4 = 195.76^\circ\text{C}$$

The condensed equations for the unknown heats (per unit area) are

$$Q_1^{(1)} = 3500U_1 - 3500U_2 = 3500 \times 200 - 3500 \times 199.58354 = 1457.6\ \text{W/m}^2 \text{(heat in)}$$

$$Q_2^{(3)} = -10 \times 1\,(U_4 - 50) = -10(195.76 - 50) = -1457.6\ \text{W/m}^2\ \text{(heat out)}$$

It can be shown that for this problem the finite element solution for the nodal temperatures and heats coincides with the exact solution (see Remark 6 of Chapter 3), which is given by

$$T_{\text{exact}}(x) = \begin{cases} A_1 + A_2 x, & 0 < x < h_1 \\ B_1 + B_2 x, & h_1 < x < h_1 + h_2 \\ C_1 + C_2 x, & h_1 + h_2 < x < h_1 + h_2 + h_3 = L \end{cases} \tag{4.3.17a}$$

where

$$A_1 = T_0,\ A_2 = \frac{T_\infty - T_0}{\Delta},\ B_1 = T_0 + h_1\left(1 - \frac{k_1}{k_2}\right)A_2$$

$$B_2 = \frac{k_1}{k_2}A_2,\ C_1 = T_\infty - k_1\left(\frac{1}{\beta} + \frac{L}{k_3}\right)A_2,\ C_2 = \frac{k_2}{k_3}B_2$$

$$\Delta = k_1\left(\frac{h_1}{k_1} + \frac{h_2}{k_2} + \frac{h_3}{k_3} + \frac{1}{\beta}\right) \tag{4.3.17b}$$

Example 4.3.2

Rectangular fins are used to remove heat from a heated surface [see Fig. 4.3.1(a)]. The fins are exposed to ambient air at T_∞. The heat transfer coefficient associated with fin material and the air is β. Assuming that heat is conducted along the length of the fin and uniform along the width and thickness directions, we wish to determine the temperature distribution along the fin and heat loss per fin for two different sets of boundary conditions.

The governing differential equation and boundary conditions of the problem are

$$-kA\frac{d^2T}{dx^2} + \beta P\,(T - T_\infty) = 0, \quad 0 < x < L \tag{4.3.18}$$

$$\textbf{Set 1:}\quad T(0) = T_0,\quad \left[kA\frac{dT}{dx} + \beta A\,(T - T_\infty)\right]_{x=L} = 0 \tag{4.3.19a}$$

$$\textbf{Set 2:}\quad T(0) = T_0,\quad T(L) = T_L \tag{4.3.19b}$$

where T is the temperature, k the conductivity, β is the heat transfer coefficient, P is the perimeter, and A is the area of cross section. Compared to the model equation (3.2.1), we have the following data:

$$a = kA, \; c = P\beta, \; f = P\beta T_\infty$$

For this case, $c \neq 0$, and the finite element solution at the nodes does not coincide with the exact solution.

We consider a uniform mesh (i.e., $h_1 = h_2 = h_3 = h_4 = L/4$) of four linear elements (see Fig. 4.3.4). The element matrices are

$$[K]^{(e)} = \frac{kA}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{P\beta h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \; \{f\}^{(e)} = \frac{P\beta T_\infty}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

The assembled system of equations is given by ($h_1 = h_2 = h_3 = h_4 = h = L/4$)

$$\left(\frac{kA}{h}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} + \frac{\beta P h}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix} + \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ f_2^3 + f_1^4 \\ f_2^4 \end{Bmatrix}$$

Set 1 Boundary Conditions

The boundary and balance conditions are

$$U_1 = T_0, \; Q_2^1 + Q_1^2 = 0, \; Q_2^2 + Q_1^3 = 0, \; Q_2^3 + Q_1^4 = 0, \; Q_2^4 = -\beta A U_5 + \beta A T_\infty$$

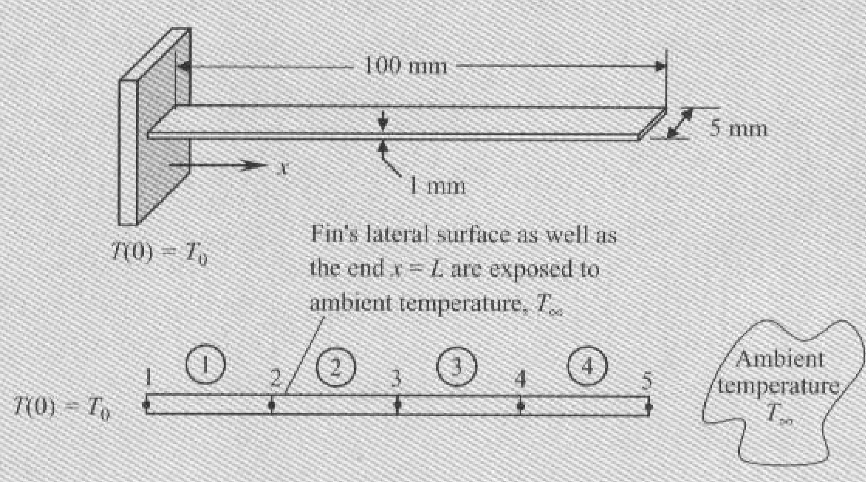


Figure 4.3.4 Finite element mesh of a rectangular fin.

Hence, the condensed equations are

$$\left(\frac{kA}{h}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} + \frac{\beta Ph}{6}\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2+\alpha \end{bmatrix}\right)\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} \left(\frac{kA}{h} - \frac{\beta Ph}{6}\right) T_0 \\ 0 \\ 0 \\ \beta A T_\infty \end{Bmatrix} + \frac{P\beta T_\infty h}{2}\begin{Bmatrix} 1+1 \\ 1+1 \\ 1+1 \\ 1 \end{Bmatrix}$$

$$Q_1^1 = -f_1^1 + \left(\frac{kA}{h} + \frac{\beta Ph}{3}\right) T_0 + \left(-\frac{kA}{h} + \frac{\beta Ph}{6}\right) U_2$$

$$Q_2^4 = -\beta A U_5 + \beta A T_\infty$$

where $\alpha = \beta A/(\beta Ph/6) = 6A/Ph$.

For the choice of the following data (material of the fin is copper),

$$k = 385\ \mathrm{W(m \cdot {}^\circ C)},\ \beta = 25\ \mathrm{W(m^2 \cdot {}^\circ C)},\ T_0 = 100^\circ\mathrm{C},\ T_\infty = 20^\circ\mathrm{C}$$

$$L = 100\ \mathrm{mm},\ t = 1\ \mathrm{mm},\ b = 5\ \mathrm{mm}$$

we have

$$\frac{kA}{h} = \frac{385 \times 5 \times 10^{-6}}{25 \times 10^{-3}} = 0.077,\quad \frac{\beta Ph}{6} = \frac{25 \times 12 \times 10^{-3} \times 25 \times 10^{-3}}{6} = 0.00125$$

$$\beta PT_\infty h = 25 \times 12 \times 10^{-3} \times 20 \times 25 \times 10^{-3} = 0.15,\ \beta AT_\infty = 25 \times 5 \times 10^{-6} \times 20 = 0.0025$$

$$\alpha = 6\frac{A}{Ph} = \frac{6 \times 5}{12 \times 25} = 0.1$$

The condensed equations for the unknown nodal temperatures become

$$\left(\begin{bmatrix} 0.154 & -0.077 & 0 & 0 \\ -0.077 & 0.154 & -0.077 & 0 \\ 0 & -0.077 & 0.154 & -0.077 \\ 0 & 0 & -0.077 & 0.077 \end{bmatrix} + \begin{bmatrix} 0.005 & 0.00125 & 0 & 0 \\ 0.00125 & 0.005 & 0.00125 & 0 \\ 0 & 0.00125 & 0.005 & 0.00125 \\ 0 & 0 & 0.00125 & 0.002625 \end{bmatrix}\right)$$

$$\times \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} (0.077 - 0.0025)\,100 \\ 0 \\ 0 \\ 25 \times 5 \times 10^{-6} \times 20 \end{Bmatrix} + \begin{Bmatrix} 0.15 \\ 0.15 \\ 0.15 \\ 0.075 \end{Bmatrix}$$

The solution of these equations is

$$U_1 = 100.0^\circ\mathrm{C},\ U_2 = 82.283^\circ\mathrm{C},\ U_3 = 70.732^\circ\mathrm{C},\ U_4 = 64.204^\circ\mathrm{C},\ U_5 = 62.053^\circ\mathrm{C}$$

The heat input at node 1 is

$$Q_1^1 = -0.075 + (0.077 + 0.0025)\,100 + (-0.077 + 0.00125)\,82.283 = 1.642 \text{ W}$$

The total heat loss from the surface of the fin can be calculated using

$$\begin{aligned}
Q &= \sum_{e=1}^{4} Q^e + \beta A(T(L) - T_\infty) = \sum_{e=1}^{4} \int_{x_a}^{x_b} P\beta(T - T_\infty)dx + kA(T(L) - T_\infty) \\
&= \sum_{e=1}^{4} \int_{x_a}^{x_b} P\beta\left(T_1^e \psi_1^e + T_2^e \psi_2^e - T_\infty\right)dx + \beta A(U_5 - T_\infty) \\
&= \sum_{e=1}^{4} \beta P h_e \left(\frac{T_1^e + T_2^e}{2} - T_\infty\right) + \beta A(U_5 - T_\infty) \\
&= 0.0075\,(0.5U_1 + U_2 + U_3 + U_4 + 0.5U_5 - 4T_\infty) + 25 \times 5 \times 10^{-6}(62.053 - 20) \\
&= 1.6368 + 0.00526 = 1.642 \text{ W}
\end{aligned}$$

The exact solution of Eqs. (4.3.18) and (4.3.19*a*) is

$$\frac{T(x) - T_\infty}{T_0 - T_\infty} = \left[\frac{\cosh m(L - x) + (\beta/mk)\sinh m(L - x)}{\cosh mL + (\beta/mk)\sinh mL}\right],\ m^2 = \frac{\beta P}{Ak} \tag{4.3.20a}$$

$$Q(0) = -kA\frac{dT}{dx} = (T_0 - T_\infty)M\left[\frac{\sinh mL + (\beta/mk)\cosh mL}{\cosh mL + (\beta/mk)\sinh mL}\right],\ M^2 = \beta PAk \tag{4.3.20b}$$

Evaluating the exact solution at the nodes, we obtain

$$T(0.025) = 82.414^\circ\text{C},\ T(0.05) = 70.958^\circ\text{C},\ T(0.075) = 64.505^\circ\text{C},\ T(0.1) = 62.422^\circ\text{C}$$

and $Q_1^1 = 1.62995$ W.

Next, we consider a mesh of two quadratic elements to analyze the problem described by Eqs. (4.3.18) and (4.3.19*a*). The assembled equations of two quadratic elements ($h_1 = h_2 = L/2$) is

$$\left(\frac{kA}{3h}\begin{bmatrix} 7 & -8 & 1 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 \\ 1 & -8 & 7+7 & -8 & 1 \\ 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & 1 & -8 & 7 \end{bmatrix} + \frac{\beta P h}{30}\begin{bmatrix} 4 & 2 & -1 & 0 & 0 \\ 2 & 16 & 2 & 0 & 0 \\ -1 & 2 & 4+4 & 2 & -1 \\ 0 & 0 & 2 & 16 & 2 \\ 0 & 0 & -1 & 2 & 4 \end{bmatrix}\right)\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_2^2 \\ Q_3^2 \end{Bmatrix} + \frac{\beta P T_\infty h}{6}\begin{Bmatrix} 1 \\ 4 \\ 1+1 \\ 4 \\ 1 \end{Bmatrix}$$

The boundary and balance conditions are [for boundary conditions in Eq. (4.3.19a)]

$$U_1 = T_0,\ Q_2^1 = 0,\ Q_3^1 + Q_1^2 = 0,\ Q_2^2 = 0,\ Q_3^2 = -\beta A U_5 + \beta A T_\infty$$

Hence, the condensed equations are

$$\left(\frac{kA}{3h} \begin{bmatrix} 16 & -8 & 0 & 0 \\ -8 & 14 & -8 & 1 \\ 0 & -8 & 16 & -8 \\ 0 & 1 & -8 & 7 \end{bmatrix} + \frac{\beta P h}{30} \begin{bmatrix} 16 & 2 & 0 & 0 \\ 2 & 8 & 2 & -1 \\ 0 & 2 & 16 & 2 \\ 0 & -1 & 2 & 4+\alpha \end{bmatrix} \right) \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} \left(\frac{7kA}{3h} - \frac{4\beta P h}{30}\right) T_0 \\ 0 \\ 0 \\ \beta A T_\infty \end{Bmatrix} + \frac{\beta P T_\infty h}{6} \begin{Bmatrix} 4 \\ 2 \\ 4 \\ 1 \end{Bmatrix}$$

where $\alpha = \beta A/(\beta P h/30) = 30A/Ph$.

The solution of the condensed equations for the unknown temperatures is

$$U_1 = 100.0^\circ\text{C},\ U_2 = 82.374^\circ\text{C},\ U_3 = 70.884^\circ\text{C},\ U_4 = 64.380^\circ\text{C},\ U_5 = 62.240^\circ\text{C}$$

The heat input at node 1 is

$$Q_1^1 = -0.05 + 0.091833 \times 100 - 0.10167 \times 82.374 + 0.012333 \times 70.884 = 1.633\ \text{W}$$

The total heat loss from the surface of the fin can be calculated using

$$Q = \sum_{e=1}^{4} \int_{x_a}^{x_b} P\beta \left(T_1^e \psi_1^e + T_2^e \psi_2^e + T_3^e \psi_3^e - T_\infty\right) dx + \beta A (U_5 - T_\infty)$$

$$= \sum_{e=1}^{4} \beta P h_e \left(\frac{T_1^e + 4T_2^e + T_3^e}{6} - T_\infty \right) + \beta A (U_5 - T_\infty)$$

$$= \frac{\beta P h}{6} (U_1 + 4U_2 + 2U_3 + 4U_4 + U_5 - 12 T_\infty) = 1.62756 + 0.00528 = 1.633\ \text{W}$$

Set 2 Boundary Conditions

For Set 2 boundary conditions (4.3.19b), the finite element boundary and balance conditions for the mesh of four linear elements are

$$U_1 = T_0,\ Q_2^1 + Q_1^2 = 0,\ Q_2^2 + Q_1^3 = 0,\ Q_2^3 + Q_1^4 = 0,\ U_5 = T_L = 20^\circ\text{C}$$

and the finite element solution is

$$U_1 = 100.0^\circ\text{C},\ U_2 = 73.955^\circ\text{C},\ U_3 = 53.252^\circ\text{C},\ U_4 = 35.842^\circ\text{C},\ U_5 = 20^\circ\text{C}$$

The heats at nodes 1 and 5 are

$$Q_1^1 = -0.075 + (0.077 + 0.0025)\,100 + (-0.077 + 0.00125)\,73.955 = 2.273\ \text{W}$$

$$Q_2^4 = -0.075 + (-0.077 + 0.00125)\,35.842 + (0.077 + 0.0025)\,20 = -1.2\ \text{W}$$

Thus, the heat loss from the end of the fin is overestimated by assuming that the end $x = L$ is at the ambient temperature. The total heat loss from the lateral surface of the fin is given by

$$Q = \sum_{e=1}^{4} Q^e = 0.0075\,(0.5U_1 + U_2 + U_3 + U_4 + 0.5U_5 - 4T_\infty) = 1.073 \text{ W}$$

The exact solution of Eqs. (4.3.18) and (4.3.19*b*) is ($\theta_L = T_L - T_\infty$ and $\theta_0 = T_0 - T_\infty$)

$$T(x) - T_\infty = \left[\frac{\theta_L \sinh mx + \theta_0 \sinh m(L-x)}{\sinh mL}\right], \; m^2 = \frac{\beta P}{Ak} \tag{4.3.21a}$$

$$Q(0) = -kA\tfrac{dT}{dx} = M\left[\frac{\theta_0 \cosh mL - \theta_L}{\sinh mL}\right], \; M^2 = \beta PAk \tag{4.3.21b}$$

Evaluating the exact solution at the nodes, we obtain

$$T(0.025) = 74^\circ\text{C},\; T(0.05) = 53.3^\circ\text{C},\; T(0.075) = 35.87^\circ\text{C},\; T(0.1) = 20^\circ\text{C}$$

and $Q_1^1 = 2.268$ W.

The next example is concerned with heat transfer in a rod and comparison of finite difference and finite element analysis steps.

Example 4.3.3

A steel rod of diameter $D = 0.02$ m, length $L = 0.05$m, and thermal conductivity $k = 50$ W/(m·°C) is exposed to ambient air at $T_\infty = 20$°C with a heat transfer coefficient $\beta = 100$ W/(m²·°C). The left end of the rod is maintained at temperature $T_0 = 320$°C and the other end is insulated. We wish to determine the temperature distribution and the heat input at the left end of the rod [see Fig. 4.3.5(*a*)].

The governing equation of the problem is the same as in Eq. (4.3.18). We rewrite Eq. (4.3.18) in the nondimensional form

$$-\frac{d^2\theta}{dx^2} + m^2\theta = 0 \quad \text{for} \quad 0 < x < L \tag{4.3.22}$$

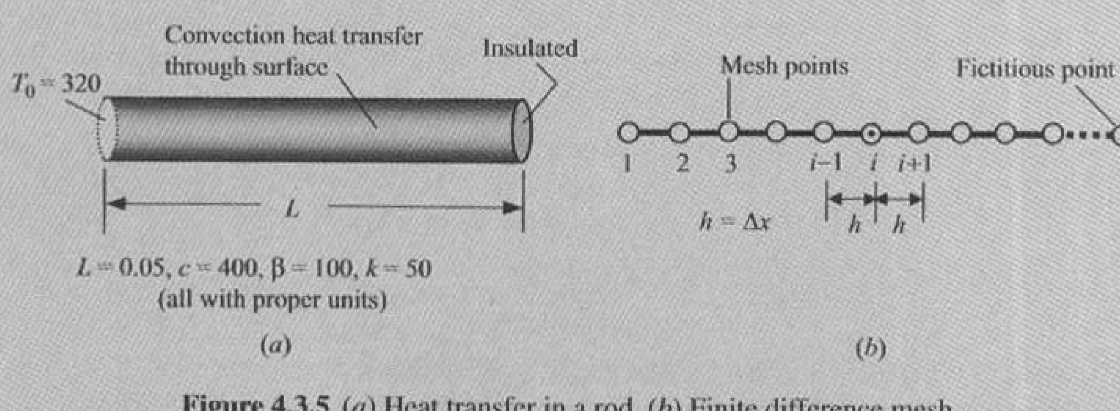


Figure 4.3.5 (*a*) Heat transfer in a rod. (*b*) Finite difference mesh.

where $\theta = T - T_\infty$, T being the temperature, and m^2 is given by

$$m^2 = \frac{\beta P}{Ak} = \frac{\beta \pi D}{\frac{1}{4}\pi D^2 k} = \frac{4\beta}{kD} \tag{4.3.23}$$

The boundary conditions of the problem become

$$\theta(0) = T(0) - T_\infty = 300^\circ\text{C}, \quad \left(\frac{d\theta}{dx}\right)\bigg|_{x=L} = 0 \tag{4.3.24}$$

The exact solution is given by [see Eqs. (4.3.20a) and (4.3.20b)]

$$\theta(x) = \theta(0)\frac{\cosh m(L-x)}{\cosh mL}, \quad Q(0) = \theta(0)\sqrt{\beta P A k}\frac{\sinh mL}{\cosh mL} \tag{4.3.25}$$

For the sake of comparison, we will consider both finite difference and finite element solutions of the problem.

Finite Difference Solutions

The second-derivative may be approximated with the centered finite difference formula (see Example 1.3.2)

$$\frac{d^2\theta}{dx^2} \approx \frac{1}{h^2}(\theta_{i+1} - 2\theta_i + \theta_{i-1}) \tag{4.3.26}$$

Substituting the above formula for the second derivative into Eq. (4.3.22), we arrive at

$$-\theta_{i+1} + (2 + m^2h^2)\theta_i - \theta_{i-1} = 0 \tag{4.3.27}$$

which is valid for any point where θ is not specified.

We choose a mesh of three points ($h = 0.025$), two end points and one in the middle [note that there is no concept of elements in the finite difference method; instead, we identify the number of mesh points (or nodes) in the domain at which we apply the formula (4.3.27), i.e., we arrive at the global equations directly]. Applying the formula (4.3.27) at nodes 2 and 3, we obtain ($m^2 = 400$)

$$(2 + 400h^2)\theta_2 - \theta_3 = \theta_1, \quad -\theta_2 + (2 + 400h^2)\theta_3 - \theta_4 = 0 \tag{4.3.28}$$

where $\theta_1 = \theta(0) = 300^\circ$C. Note that θ_4 is the value of θ at the fictitious node 4, which is considered to be a mirror image because of the boundary condition $d\theta/dx = 0$ [see Fig. 4.3.5(b)]. To eliminate θ_4, we can use one of the following formulas:

$$\left(\frac{d\theta}{dx}\right)_{x=0} = \frac{\theta_4 - \theta_3}{h} = 0 \quad \text{(forward)} \tag{4.3.29a}$$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = \frac{\theta_4 - \theta_2}{2h} = 0 \quad \text{(centered)} \tag{4.3.29b}$$

The latter is of order $O(h^2)$, consistent with the centered difference formula (4.3.26). Using (4.3.29b), we set $\theta_4 = \theta_2$ in Eq. (4.3.28). Equations (4.3.28) can be written in matrix form as

(the coefficient matrix is not symmetric)

$$\begin{bmatrix} 2.25 & -1 \\ -2 & 2.25 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 300 \\ 0 \end{Bmatrix}$$

The solution of these equations is

$$\theta_2 = 220.41^\circ\text{C}, \quad \theta_3 = 195.92^\circ\text{C}$$

The heat at node 1 ($x = 0$) can be computed using the definition

$$Q(0) = kA\left(-\frac{d\theta}{dx}\right)_{x=0} = kA\frac{\theta_1 - \theta_2}{h} = 50.01 \text{ W}$$

Next, we use a mesh of five points. Applying Eq. (4.3.27) at nodes 2, 3, 4, and 5, and using $\theta_6 = \theta_4$, we obtain ($h = 0.0125$)

$$\begin{bmatrix} 2.0625 & -1 & 0 & 0 \\ -1 & 2.0625 & -1 & 0 \\ 0 & -1 & 2.0625 & -1 \\ 0 & 0 & -2 & 2.0625 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{Bmatrix} = \begin{Bmatrix} 300 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The solution of these equations is

$$\theta_2 = 251.89^\circ\text{C}, \quad \theta_3 = 219.53^\circ\text{C}, \quad \theta_4 = 200.89^\circ\text{C}, \quad \theta_5 = 194.80^\circ\text{C}$$

The heat at node 1 ($x = 0$) is

$$Q(0) = kA\left(-\frac{d\theta}{dx}\right)_{x=0} = kA\frac{\theta_1 - \theta_2}{h} = 60.46 \text{ W}$$

Finite Element Solutions

For the choice of two linear elements, the assembled system of equations is (U_I denotes the temperature $\theta(x)$ at the Ith global node)

$$\begin{bmatrix} 43.333 & -38.333 & 0 \\ -38.333 & 86.667 & -38.333 \\ 0 & -38.333 & 43.333 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} + Q_1^{(2)} \\ Q_2^{(2)} \end{Bmatrix} \tag{4.3.30}$$

The boundary conditions are

$$U_1 = 300^\circ\text{C}, \quad Q_2^{(1)} + Q_1^{(2)} = 0, \quad Q_2^{(2)} = 0$$

Hence, we have

$$\begin{bmatrix} 86.667 & -38.333 \\ -38.333 & 43.333 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 38.333 \times 300 \\ 0 \end{Bmatrix}$$

The solution of these equations is

$$U_2 = 217.98^\circ\text{C}, \quad U_3 = 192.83^\circ\text{C}$$

The heat at node 1 ($x = 0$) can be computed using the first equation of the first element

$$Q(0) = kAQ_1^{(1)} = kA(43.333U_1 - 38.333U_2) = 72.95 \text{ W}$$

Once we have the nodal values U_1, U_2, and U_3, values at other points (intermediate to the nodes) can be computed using the interpolation

$$\theta^e(\bar{x}) = \sum_{j=1}^{2} \theta_j^e \psi_j^e(\bar{x}), \quad 0 \le \bar{x} \le h_e$$

where $\theta_1^1 = U_1$, $\theta_2^1 = U_2 = \theta_1^2$, and $\theta_2^2 = U_3$.

For the choice of four linear elements, the assembled system of equations is

$$\begin{bmatrix} 81.667 & -79.167 & 0 & 0 & 0 \\ -79.167 & 163.333 & -79.167 & 0 & 0 \\ 0 & -79.167 & 163.333 & -79.167 & 0 \\ 0 & 0 & -79.167 & 163.333 & -79.167 \\ 0 & 0 & 0 & -79.167 & 81.667 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} + Q_1^{(2)} \\ Q_2^{(2)} + Q_1^{(3)} \\ Q_2^{(3)} + Q_1^{(4)} \\ Q_2^{(4)} \end{Bmatrix} \tag{4.3.31}$$

The boundary conditions are ($e = 1, 2, 3$):

$$U_1 = 300^\circ\text{C}, \quad Q_2^{(e)} + Q_1^{(e+1)} = 0, \quad Q_2^{(4)} = 0$$

Hence, we have

$$\begin{bmatrix} 163.333 & -79.167 & 0 & 0 \\ -79.167 & 81.667 & -79.167 & 0 \\ 0 & -79.167 & 81.667 & -79.167 \\ 0 & 0 & -79.167 & 81.667 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 79.167 \times 300 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The solution of these equations is

$$U_2 = 251.52^\circ\text{C}, \quad U_3 = 218.92^\circ\text{C}, \quad U_4 = 200.16^\circ\text{C}, \quad U_5 = 194.03^\circ\text{C}$$

The heat at node 1 ($x = 0$) is

$$Q(0) = kAQ_1^{(1)} = kA(81.667U_1 - 79.167U_2) = 72.07 \text{ W}$$

A comparison of the numerical results with the exact solution is presented in Table 4.3.1. The finite element solution is the best solution in the sence that it gives the minimum value of the integral of the square of $|u - u_h|$ over the domain. The exact value of $Q(0)$ is 71.78 W.

Table 4.3.1 Comparison of finite difference and finite element solutions with the exact solution of $-\dfrac{d^2\theta}{dx^2}+400\theta=0,\ \theta(0)=300,\ \dfrac{d\theta}{dx}(0.05)=0.$

x	Exact solution	FEM[†] solution		FDM solution	
		$N=2$	$N=4$	$N=2$	$N=4$
0.0000	300.00	300.00	300.00	300.00	300.00
0.0125	251.71	258.99*	251.52	260.21*	251.89
0.0250	219.23	217.98	218.92	220.41	219.53
0.0375	200.52	205.41*	200.16	208.17*	200.89
0.0500	194.42	192.83	194.03	195.92	194.80

*Values computed by interpolation.
† FEM, finite element method; FDM, finite difference method.

The last example of heat transfer deals with radially symmetric heat transfer in a cylinder.

Example 4.3.4

Consider a *long*, homogeneous, solid cylinder of radius R_0 [see Fig. 4.3.1(*c*)] in which energy is generated at a constant rate g_0 (W/m³). The boundary surface at $r=R_0$ is maintained at a constant temperature T_0. We wish to calculate the temperature distribution $T(r)$ and heat flux $q(r)=-kdT/dr$ (or heat $Q=AkdT/dr$).

The governing equation for this problem is given by (4.3.10) with $g=g_0$. The boundary conditions are

$$T(R_0)=T_0,\quad \left(2\pi kr\frac{dT}{dr}\right)\bigg|_{r=0}=0 \tag{4.3.32}$$

The zero-flux boundary condition at $r=0$ is a result of the radial symmetry at $r=0$. If the cylinder is hollow with inner radius R_i then the boundary condition at $r=R_i$ can be a specified temperature, specified heat flux, or convection boundary condition, depending on the situation. The finite element solution will not be exact at nodes because a is not constant.

The finite element model of the governing equation is [see Eqs. (3.4.5*a*) and (3.4.5*b*)]

$$[K^e]\{T^e\}=\{f^e\}+\{Q^e\} \tag{4.3.33}$$

where

$$K_{ij}^e=2\pi\int_{r_a}^{r_b}kr\frac{d\psi_i^e}{dr}\frac{d\psi_j^e}{dr}\,dr,\quad f_i^e=2\pi\int_{r_a}^{r_b}\psi_i^e g_0 r\,dr \tag{4.3.34a}$$

$$Q_1^e=-2\pi k\left(r\frac{dT}{dr}\right)\bigg|_{r_a},\quad Q_2^e=2\pi k\left(r\frac{dT}{dr}\right)\bigg|_{r_b} \tag{4.3.34b}$$

and (r_a, r_b) are the global coordinates of typical element $\Omega_e=(r_a, r_b)$.

For linear interpolation of $T(r)$, the element equations for a typical element are given by Eq. (3.4.7). The element equations for individual elements are obtained from these by giving the element length h_e and the global coordinates of the element nodes, $r_e = r_a$ and $r_{e+1} = r_b$.

For the mesh of one linear element, we have $r_a = r_1 = 0$ and $r_2 = h_e = R_0$, and

$$\pi k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \frac{\pi g_0 R_0}{3} \begin{Bmatrix} R_0 \\ 2R_0 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}$$

The boundary conditions imply $U_2 = T_0$ and $Q_1^1 = 0$. Hence the temperature at node 1 is

$$U_1 = \frac{g_0 R_0^2}{3k} + T_0$$

and the heat at $r = R_0$ is

$$Q_2^1 = \pi k(U_2 - U_1) - \frac{2}{3}\pi g_0 R_0^2 = -\pi g_0 R_0^2$$

The negative sign indicates that heat is removed from the body (because $dT/dr < 0$). The one-element solution as a function of the radial coordinate r is

$$T_h(r) = U_1 \psi_1^1(r) + U_2 \psi_2^1(r) = \frac{g_0 R_0^2}{3k}\left(1 - \frac{r}{R_0}\right) + T_0$$

and the heat flux is

$$q(r) \equiv -k\frac{dT_h}{dr} = \frac{1}{3} g_0 R_0$$

For a mesh of two linear elements, we take $h_1 = h_2 = \frac{1}{2}R_0$, $r_1 = 0$, $r_2 = h_1 = \frac{1}{2}R_0$, and $r_3 = h_1 + h_2 = R_0$. The two-element assembly gives

$$\pi k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+3 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \frac{\pi g_0 R_0^2}{6} \begin{Bmatrix} \frac{1}{2} \\ 1+2 \\ \frac{5}{2}+2 \end{Bmatrix} \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

Imposing the boundary conditions $U_3 = T_0$, $Q_2^1 + Q_1^2 = 0$, and $Q_1^1 = 0$, the condensed equations for the unknown temperatures are

$$\pi k \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \frac{\pi g_0 R_0^2}{12} \begin{Bmatrix} 1 \\ 6 \end{Bmatrix} + \pi k \begin{Bmatrix} 0 \\ 3T_0 \end{Bmatrix}$$

The nodal values are

$$U_1 = \frac{5}{18}\frac{g_0 R_0^2}{k} + T_0, \quad U_2 = \frac{7}{36}\frac{g_0 R_0^2}{k} + T_0$$

From equilibrium, Q_2^2 is computed as

$$Q_2^2 = -\frac{5}{12}\pi g_0 R_0^2 + 3\pi k(U_3 - U_2) = -\pi g_0 R_0^2$$

The finite element solution becomes

$$T_h(r) = \begin{cases} U_1\psi_1^1(r) + U_2\psi_2^1(r) = \left(\dfrac{5}{18}\dfrac{g_0R_0^2}{k} + T_0\right)\left(1 - 2\dfrac{r}{R_0}\right) + 2\left(\dfrac{7}{36}\dfrac{g_0R_0^2}{k} + T_0\right)\dfrac{r}{R_0} \\ U_2\psi_1^2 + U_3\psi_2^2 = 2\left(\dfrac{7}{36}\dfrac{g_0R_0^2}{k} + T_0\right)\left(1 - \dfrac{r}{R_0}\right) + T_0\left(2\dfrac{r}{R_0} - 1\right) \end{cases}$$

$$= \begin{cases} \dfrac{1}{18}\dfrac{g_0R_0^2}{k}\left(5 - 3\dfrac{r}{R_0}\right) + T_0, & \text{for } 0 \le r \le \dfrac{1}{2}R_0 \\ \dfrac{7}{18}\dfrac{g_0R_0^2}{k}\left(1 - \dfrac{r}{R_0}\right) + T_0, & \text{for } \dfrac{1}{2}R_0 \le r \le R_0 \end{cases}$$

The exact solution of the problem is

$$T(r) = \frac{g_0R_0^2}{4k}\left[1 - \left(\frac{r}{R_0}\right)^2\right] + T_0(^\circ\text{C}) \tag{4.3.35}$$

$$q(r) = \frac{1}{2}g_0 r \ (\text{W/m}^2), \quad Q(R_0) = \left(2\pi kr\frac{dT}{dr}\right)\bigg|_{R_0} = -\pi g_0 R_0^2 \ (\text{W}) \tag{4.3.36}$$

The temperature at the center of the cylinder according to the exact solution is $T(0) = g_0R_0^2/4k + T_0$, whereas it is $g_0R_0^2/3k + T_0$ and $5g_0R_0^2/18k + T_0$ according to the one- and two-element models, respectively.

The finite element solutions obtained using one-, two-, four-, and eight-element meshes of linear elements are compared with the exact solution in Table 4.3.2. Convergence of the finite element solutions, $\bar{T} = (T - T_0)k/g_0R_0^2$, to the exact solution with an increasing number of elements is clear (see Fig. 4.3.6). Figure 4.3.7 shows plots of $\bar{Q}(r) = Q(r)/2\pi R_0 g_0$ and $Q(r) = 2\pi kr\,dT/dr$, versus $\bar{r} = r/R_0$, as computed in the finite element analysis and the exact solution.

Table 4.3.2 Comparison of the finite element and exact solutions for heat transfer in a radially symmetric cylinder $R_0 = 0.01$ m, $g_0 = 2 \times 10^8$ W/m^3, $k = 20$ W/(m·°C), $T_0 = 100$°C.

	Temperature $T(r)$[†]				
r/R_0	One element	Two elements	Four elements	Eight elements	Exact
0.000	<u>433.33</u>	<u>377.78</u>	<u>358.73</u>	<u>352.63</u>	350.00
0.125	391.67	356.24	348.31	<u>347.42</u>	346.09
0.250	350.00	335.11	<u>337.90</u>	<u>335.27</u>	334.38
0.375	308.33	315.28	313.59	<u>315.48</u>	314.84
0.500	266.67	<u>294.44</u>	<u>289.29</u>	<u>287.95</u>	287.50
0.625	225.00	245.83	249.70	<u>252.65</u>	252.34
0.750	183.33	197.22	<u>210.12</u>	<u>209.56</u>	209.38
0.875	141.67	148.61	155.06	<u>158.68</u>	158.59
1.000	<u>100.00</u>	<u>100.00</u>	<u>100.00</u>	<u>100.00</u>	100.00

† The underlined terms are nodal values and others are interpolated values.

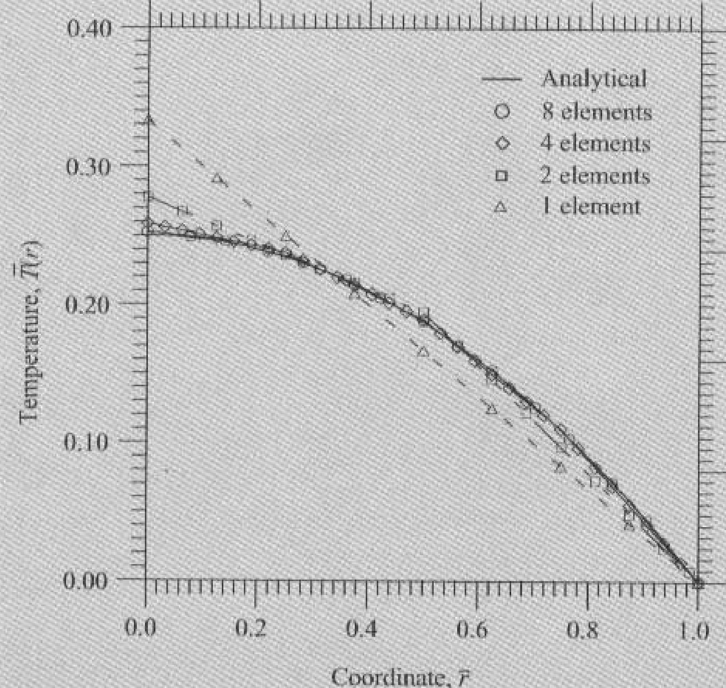


Figure 4.3.6 Comparison of the finite element solutions with the exact solution for heat transfer in a radially symmetric problem with cylindrical geometry.

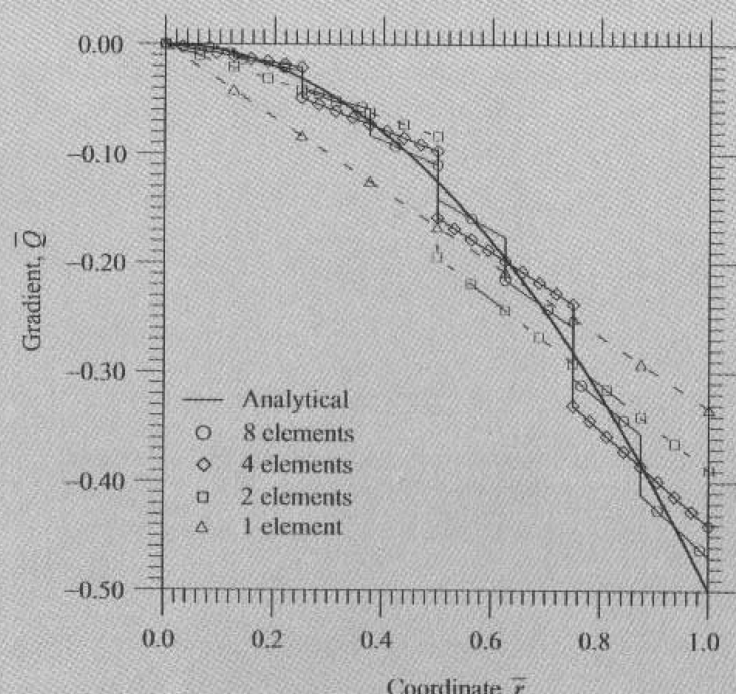


Figure 4.3.7 Comparison of the finite element solution with the exact solution for the temperature gradient in a radially symmetric problem with cylindrical geometry.

4.4 FLUID MECHANICS

4.4.1 Governing Equations

All bulk matter in nature exists in one of two forms: solid or fluid. A solid body is characterized by the relative immobility of its molecules whereas a fluid state is characterized by relative mobility of its molecules. Fluids can exist either as gases or liquids. The field of fluid mechanics is concerned with the motion of fluids and the conditions affecting the motion (see Reddy and Gartling, 2001).

The basic equations of fluid mechanics are derived from the global laws of conservation of mass, momentum, and energy. Conservation of mass gives the continuity equation, while the conservation of momentum results in the equations of motion. The conservation of energy, considered in the last section, is the first law of thermodynamics, and it results in Eqs. (4.3.8)–(4.3.10) for one-dimensional systems when thermal-fluid coupling is omitted. For additional details, see Schlichting (1979), Bird et al. (1960), and Reddy and Gartling (2001). More details are provided in Chapter 10, which is dedicated to finite element models of two-dimensional flows of viscous incompressible fluids.

Here, we consider so-called parallel flow, where only one velocity component is different from zero resulting in all the fluid particles moving in one direction, i.e., $u = u(x, y, z)$, where u is the velocity component along the x coordinate. We assume that there are no body forces. The z-momentum equation requires that $u = u(x, y)$. The conservation of mass in this case reduces to

$$\frac{\partial u}{\partial x} = 0$$

which implies that $u = u(y)$. The y-momentum equation simplifies to

$$\frac{\partial P}{\partial y} = 0$$

which implies that $P = P(x)$, where P is the pressure. The x-momentum equation simplifies to

$$\mu \frac{d^2 u}{dy^2} = \frac{dP}{dx} \tag{4.4.1}$$

The energy equation for this problem reduces to

$$\rho c u \frac{\partial T}{\partial x} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \mu \left(\frac{du}{dy}\right)^2 \tag{4.4.2}$$

Here we are primarily interested in the finite element analysis of Eq. (4.4.1).

4.4.2 Finite Element Model

Equation (4.4.1) is a special case of the model equation (3.2.1) with the following correspondence:

$$f = -\frac{dP}{dx}, \quad a = \mu = \text{constant}, \quad c = 0, \quad x = y \tag{4.4.3}$$

Therefore, the finite element model in (3.2.31a) and (3.2.31b) is valid for this problem:

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\} \quad \text{or} \quad K^e u^e = f^e + Q^e \tag{4.4.4a}$$

where

$$K_{ij}^e = \int_{y_a}^{y_b} \mu \frac{d\psi_i^e}{dy}\frac{d\psi_j^e}{dy}\,dy, \quad f_i^e = \int_{y_a}^{y_b} \left(-\frac{dP}{dx}\right)\psi_i^e\,dy$$
$$Q_1^e = -\left(\mu\frac{du}{dy}\right)\bigg|_a, \quad Q_2^e = -\left(\mu\frac{du}{dy}\right)\bigg|_b \tag{4.4.4b}$$

Next, we consider an example.

Example 4.4.1

Consider parallel flow between two long flat walls separated by a distance $2L$ [see Fig. 4.4.1(a)]. We wish to determine the velocity distribution $u(y)$, $-L < y < L$, for a given pressure gradient $-dP/dx$, using the finite element method.

For a two-element mesh of linear elements ($h = L$), we have

$$\frac{\mu}{h}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \frac{f_0 h}{2}\begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

We consider two sets of boundary conditions [see Fig. 4.4.1(b)]

Set 1: $u(-L) = 0, \ u(L) = 0$ (two stationary walls)

Set 2: $u(-L) = 0, \ u(L) = U_0$ (bottom wall stationary and top wall moving) (4.4.5)

For the first case, we may use symmetry and model domain $0 < x < L$. Here we consider the full domain for the two sets of boundary conditions.

For Set 1, we have $U_1 = U_3 = 0$. The finite element solution is given by

$$U_2 = \frac{f_0 L^2}{2\mu}, \quad u_h(y) = \frac{f_0 L^2}{2\mu}\frac{y}{L} \tag{4.4.6}$$

For set 2, we have $U_1 = 0$ and $U_3 = U_0$. The finite element solution is

$$U_2 = \frac{f_0 L^2}{2\mu} + \frac{1}{2}U_0, \quad u_h(y) = \left(\frac{f_0 L^2}{2\mu} + \frac{1}{2}U_0\right)\frac{y}{L} \tag{4.4.7}$$

For a one-element mesh of the quadratic element ($h = 2L$), we have

$$\frac{\mu}{6L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \frac{f_0 L}{3}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ 0 \\ Q_3^1 \end{Bmatrix} \tag{4.4.8}$$

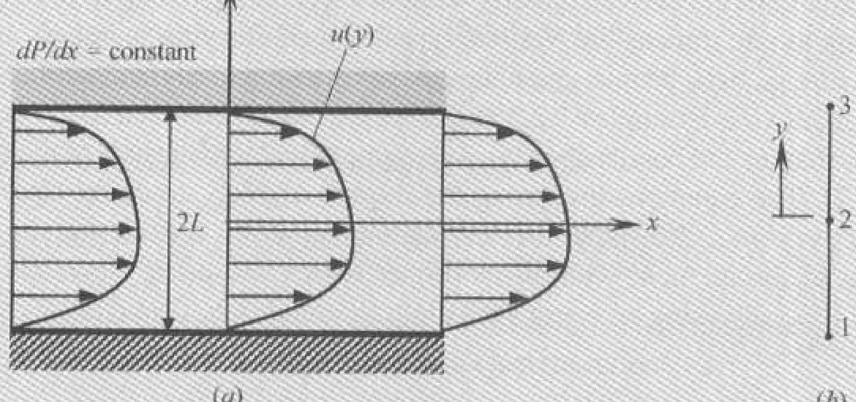


Figure 4.4.1 (*a*) Flow between parallel plates. (*b*) Finite element mesh; $u(L) = 0$ for set 1 and $u(L) = U_0$ for set 2.

and the finite element solutions for the two sets of boundary conditions are

$$U_2 = \frac{f_0 L^2}{2\mu} \quad \text{for Set 1}$$
$$U_2 = \frac{f_0 L^2}{2\mu} + \frac{1}{2}U_0 \quad \text{for Set 2} \tag{4.4.9}$$

Although the nodal values predicted in the linear- and quadratic-element meshes are the same, they vary linearly and quadratically between nodes of linear and quadratic elements, respectively.

The exact solutions for the two sets of boundary conditions in (4.4.5) are $(-L \le y \le L)$

$$u(y) = \frac{f_0 L^2}{2\mu}\left(1 - \frac{y^2}{L^2}\right) \quad \text{for Set 1}$$
$$u(y) = U_0 \frac{1}{2}\left(1 + \frac{y}{L}\right) + \frac{f_0 L^2}{2\mu}\left(1 - \frac{y^2}{L^2}\right) \quad \text{for Set 2} \tag{4.4.10}$$

Note that the finite element solutions at the nodes are exact, as expected. The quadratic-element solution agrees with the exact solutions (4.4.10) for every value of y.

4.5 SOLID AND STRUCTURAL MECHANICS

4.5.1 Preliminary Comments

Solid mechanics is that branch of mechanics dealing with the motion and deformation of solids. The Lagrangian description of motion is used to express the global conservation laws. The conservation of mass for solid bodies is trivially satisfied because of the fixed material viewpoint used in the Lagrangian description. The conservation of momentum is nothing but Newton's second law of motion. Under isothermal conditions, the energy equation uncouples from the momentum equations, and we need only consider the equations of motion or equilibrium (see Example 2.3.3).

Unlike in fluid mechanics, the equations governing solid bodies undergoing different forms of deformations are derived directly without specializing the three-dimensional elasticity equations to one dimension. Various types of load-carrying members are called by different names, e.g., bars, beams, and plates. A *bar* is a structural member that is subjected to only axial loads (see Examples 1.2.3 and 2.3.2), while a *beam* is a member that is subjected to loads that tend to bend it about an axis perpendicular to the axis of the member (see Example 2.4.2). The equations governing the motion of such structural elements are not deduced directly from (2.3.52), but they are derived either by considering the equilibrium of an element of the member with all its proper forces and using Newton's second law (Example 1.2.3), or by using an energy principle (Examples 2.3.2 and 2.4.2).

4.5.2 Finite Element Model of Bars and Cables

The equation of motion governing axial deformation of a bar is (see Example 1.2.3)

$$\rho A \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = f(x,t) \tag{4.5.1}$$

For static problems, Eq. (4.5.1) reduces to

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f(x) \tag{4.5.2}$$

It should be recalled that Eq. (4.5.2) is derived under the assumption that all material points on the line x = constant (i.e., all points on any cross section) move by the same amount $u(x)$. This is equivalent to the assumption that the stress on any cross section is uniform. Equation (4.5.2) is the same as the model equation (3.2.1), with $a = EA$ and $c = 0$. Hence, the finite element model in (3.2.31a) and (3.2.31b) is valid for bars.

The average transverse deflection $u(x)$ of a cable made of elastic material is also governed by an equation of the form:

$$-\frac{d}{dx}\left(T\frac{du}{dx}\right) = f(x) \tag{4.5.3}$$

where T is the uniform tension in the cable and f is the distributed transverse force. Again, Eq. (4.5.3) is a special case of Eq. (3.2.1), with $a = T$ and $c = 0$.

In structural mechanics problems, the quadratic functional of (3.2.10) takes the special meaning of total potential energy, Π^e, which can be expressed in the form

$$\Pi^e = \frac{1}{2}\int_{x_a}^{x_b} \varepsilon^{\mathrm{T}}\sigma A_e \; dx - \int_{x_a}^{x_b} uf\, dx - \sum_i u_i^e Q_i^e$$

where u is the displacement, ε the strain and σ the stress. The finite element approximation (3.2.24) of u can be written as

$$u_h = \{\psi_1^e \; \psi_2^e \; \ldots \; \psi_n^e\} \begin{Bmatrix} u_1^e \\ u_2^e \\ \vdots \\ u_n^e \end{Bmatrix} = \mathbf{N}^e\mathbf{u}^e$$

and the strains and stresses take the form

$$\varepsilon = \frac{du}{dx} = \frac{d}{dx}(\mathbf{N}^e\mathbf{u}^e) \equiv \mathbf{B}^e\mathbf{u}^e$$
$$\sigma = E\varepsilon = E_e\mathbf{B}^e\mathbf{u}^e$$

and the expression for the total potential energy becomes

$$\Pi^e = \frac{1}{2}\int_{x_a}^{x_b} A_e E_e \mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{u}\,dx - \int_{x_a}^{x_b} \mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} f\,dx - \mathbf{u}^{\mathrm{T}}\mathbf{Q}$$

where the element label on **B** and **u** is omitted for brevity. Then the principle of minimum total potential energy, $\delta\Pi^e = 0$, yields the finite element model

$$\delta\Pi^e = \delta\mathbf{u}^{\mathrm{T}}\left[\left(\int_{x_a}^{x_b} A_e E_e \mathbf{B}^{\mathrm{T}}\mathbf{B}\,dx\right)\mathbf{u} - \int_{x_a}^{x_b} \mathbf{B}^{\mathrm{T}} f\,dx - \mathbf{Q}\right] = 0$$

or

$$\mathbf{K}^e\mathbf{u}^e = \mathbf{f}^e + \mathbf{Q}^e \tag{4.5.4a}$$

where

$$\mathbf{K}^e = \int_{x_a}^{x_b} A_e E_e \mathbf{B}^{\mathrm{T}}\mathbf{B}\,dx, \quad \mathbf{f}^e = \int_{x_a}^{x_b} \mathbf{B}^{\mathrm{T}} f\,dx, \quad \mathbf{B} = \frac{d\mathbf{N}}{dx} \tag{4.5.4b}$$

These are just matrix form of the equations already presented in (3.2.31*a*) and (3.2.31*b*).

4.5.3 Numerical Examples

In this section we consider a number of examples of finite element analysis of bars.

Example 4.5.1

A bridge is supported by several concrete piers, and the geometry and loads of a typical pier are shown in Fig. 4.5.1. The load 20 $\mathrm{kN/m^2}$ represents the weight of the bridge and an assumed distribution of the traffic on the bridge. The concrete weighs approximately 25 $\mathrm{kN/m^3}$ and its modulus is $E = 28 \times 10^6$ $\mathrm{kN/m^2}$. We wish to analyze the pier for displacements and stresses using the finite element method. The pier is indeed a three-dimensional structure. However, we wish to approximate the deformation and stress fields in the pier as one-dimensional.

We represent the distributed force at the top of the pier as a point force

$$F = (0.5 \times 0.5)20 = 5 \text{ kN}$$

The weight of the concrete is represented as the body force per unit length. The total force at any distance x is equal to the weight of the concrete above that point. The weight at a distance x is equal to the product of the volume of the body above x and the specific weight of the concrete:

$$W(x) = 0.5\frac{0.5 + (0.5 + 0.5x)}{2}x \times 25.0 = 6.25(1 + 0.5x)x$$

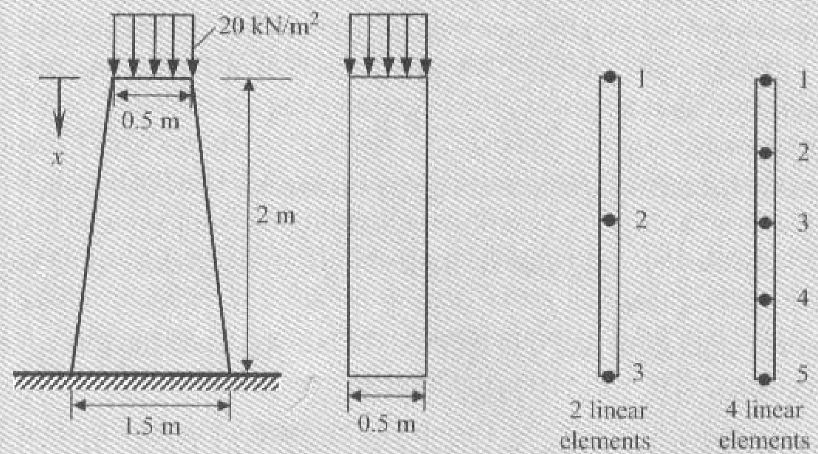


Figure 4.5.1 The geometry and loading in the concrete pier problem of Example 4.5.1.

The body force per unit length is computed from

$$f = \frac{dW}{dx} = 6.25(1 + x)$$

This completes the load representation of the problem.

The governing differential equation for the problem is given by (4.5.2), with $E = 28 \times 10^6$ kN/m^2 and cross-sectional area $A(x)$:

$$A(x) = (0.5 + 0.5x)0.5 = \frac{1}{4}(1 + x)$$

Thus, the concrete pier problem is idealized as a one-dimensional problem whose axial displacement u is governed by the equation

$$-\frac{d}{dx}\left[\frac{1}{4}E(1 + x)\frac{du}{dx}\right] = 6.25(1 + x) \tag{4.5.5a}$$

subject to the boundary conditions

$$\left[\frac{1}{4}E(1 + x)\frac{du}{dx}\right]\bigg|_{x=0} = -5, \quad u(2) = 0 \tag{4.5.5b}$$

Equation (4.5.5a) is a special case of the model equation (3.2.1) with the following correspondance:

$$a(x) = 0.25E(1 + x), \quad f(x) = 6.25(1 + x) \tag{4.5.6}$$

For a typical linear element, the stiffness matrix and force vector can be computed using Eq. (3.2.31b). We have

$$\begin{aligned} [K^e] &= \frac{E}{4h_e}\left[1 + \frac{1}{2}(x_e + x_{e+1})\right]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ \{f^e\} &= 6.25\frac{h_e}{2}\left(\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \frac{1}{3}\begin{Bmatrix} x_{e+1} + 2x_e \\ 2x_{e+1} + x_e \end{Bmatrix}\right) \end{aligned} \tag{4.5.7}$$

Let us consider a two-element mesh with $h_1 = h_2 = 1$ m. We have

$$[K^1] = \frac{E}{4}\begin{bmatrix} 1.5 & -1.5 \\ -1.5 & 1.5 \end{bmatrix}, \quad \{f^1\} = \frac{6.25}{6}\begin{Bmatrix} 3+1 \\ 3+2 \end{Bmatrix} = \begin{Bmatrix} 4.167 \\ 5.208 \end{Bmatrix}$$

$$[K^2] = \frac{E}{4}\begin{bmatrix} 2.5 & -2.5 \\ -2.5 & 2.5 \end{bmatrix}, \quad \{f^2\} = \frac{6.25}{6}\begin{Bmatrix} 3+4 \\ 3+5 \end{Bmatrix} = \begin{Bmatrix} 7.292 \\ 8.333 \end{Bmatrix}$$

The assembled equations are

$$E\begin{bmatrix} 0.375 & -0.375 & 0.000 \\ -0.375 & 1.000 & -0.625 \\ 0.000 & -0.625 & 0.625 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}\begin{Bmatrix} 4.167 \\ 12.500 \\ 8.333 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

The boundary and equilibrium conditions require

$$U_3 = 0, \quad Q_2^1 + Q_1^2 = 0, \quad Q_1^1 = 5\text{ kN}$$

The condensed equations for the unknown displacements and forces are

$$E\begin{bmatrix} 0.375 & -0.375 \\ -0.375 & 1.000 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 9.167 \\ 12.500 \end{Bmatrix}, \quad Q_2^2 = -0.6255U_2 - 8.333$$

The solution is given by (positive displacements, because of the coordinate system used, indicate that the pier is in compression)

$$U_1 = 2.111 \times 10^{-6}\text{m}, \quad U_2 = 1.238 \times 10^{-6}\text{m}, \quad Q_2^2 = -30\text{ kN}$$

Hence, the stress at the fixed end is (compressive)

$$\sigma_x = Q_2^2/A = -30/0.75 = -40\text{ kN/m}^2$$

The four-element model gives 2.008×10^{-6} m and 1.228×10^{-6}m, respectively.

The exact solution of Eqs. (4.5.5a) and (4.5.5b) is

$$u(x) = \frac{1}{E}\left[56.25 - 6.25(1+x)^2 - 7.5\ln\left(\frac{1+x}{3}\right)\right] \tag{4.5.8}$$

The exact values of u at nodes 1 and 2 are

$$u(0) = 2.08 \times 10^{-6}\text{m}, \quad u(1) = 1.225 \times 10^{-6}\text{m}$$

The finite element solution at the nodes is not exact because $a = EA$ is not a constant in this problem.

Example 4.5.2

Consider the composite bar consisting of a tapered steel bar fastened to an aluminum rod of uniform cross section and subjected to loads as shown in Figure 4.5.2. We wish to determine the displacement field in the bar using the finite element method.

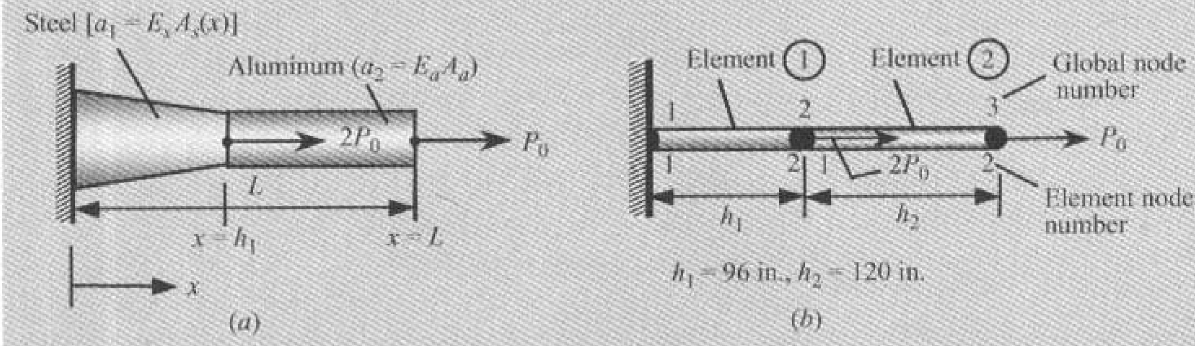


Figure 4.5.2 Axial deformation of a composite member (*a*) Geometry and loading. (*b*) Finite element representation.

The governing equations are given by

$$
\begin{aligned}
-\frac{d}{dx}\left(E_s A_s \frac{du}{dx}\right) &= 0, \quad 0 < x < h_1 \\
-\frac{d}{dx}\left(E_a A_a \frac{du}{dx}\right) &= 0, \quad h_1 < x < h_1 + h_2 = L
\end{aligned}
\tag{4.5.9}
$$

where the subscript "s" refers to steel and "a" to aluminum. The boundary conditions are obvious. We consider the following data:

$$
\begin{aligned}
&E_s = 30 \times 10^6 \text{ psi}, \; A_s = (c_1 + c_2 x)^2, \; E_a = 10^7 \text{ psi} \\
&A_a = 1 \text{ in.}^2, \; h_1 = 96 \text{ in.}, \; L = 216 \text{ in.}, \; P_0 = 10,000 \text{ lb}
\end{aligned}
\tag{4.5.10}
$$

The finite element equations for a uniform bar element with constant $E_e A_e$ and $f(x) = 0$ are given by [see Eq. (3.3.5*a*)]

$$
\frac{E_e A_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}
\tag{4.5.11a}
$$

where Q_i^e are the end forces

$$
Q_1^e = \left[-EA\frac{du}{dx}\right]_{x_a}, \quad Q_2^e = \left[EA\frac{du}{dx}\right]_{x_b}
\tag{4.5.11b}
$$

For the present problem, A_e is not constant, but Eq. (4.6.11a) is still valid with

$$
A_e = (c_1^e)^2 + \frac{1}{3}(c_2^e)^2(x_b^2 + x_a^2 + x_a x_b) + c_1^e c_2^e (x_b + x_a)
$$

To see this, we calculate K_{ij}^e for the problem using the local coordinate $\bar{x}$ ($x = \bar{x} + x_a$). We have

$$
K_{ij}^e = \int_{x_a}^{x_b} E_e (c_1^e + c_2^e x)^2 \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\,dx = \int_0^{h_e} E_e \left[c_1^e + c_2^e(\bar{x} + x_a)\right]^2 \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\,d\bar{x}
$$

$$
K_{11}^e = \frac{E_e}{h_e^2}\int_{x_a}^{x_b} (c_1^e + c_2^e x)^2\,dx
$$

$$= \frac{E_e}{h_e^2}\left[(c_1^e)^2 h_e + \frac{1}{3}(c_2^e)^2(x_b^3 - x_a^3) + c_1^e c_2^e (x_b^2 - x_a^2)\right]$$

$$= \frac{E_e}{h_e}\left[(c_1^e)^2 + \frac{1}{3}(c_2^e)^2(x_b^2 + x_a^2 + x_a x_b) + c_1^e c_2^e (x_b + x_a)\right] = \frac{E_e A_e}{h_e}$$

$K_{12}^e = K_{21}^e = -K_{11}^e$ and $K_{22}^e = K_{11}^e$, where we have used the algebraic equalities, $a^2 - b^2 = (a-b)(a+b)$ and $a^3 - b^3 = (a-b)(a^2+b^2+ab)$ to simplify the expressions.

For two linear elements of lengths $h_1 = 96$ in. and $h_2 = 120$ in., we have $c_1^1 = 1.5$, $c_2^1 = -0.5/96$, $c_1^2 = 1$, $c_2^2 = 0$, and the element stiffness matrices become

$$[K^1] = \frac{4.75}{96} \times 10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^4 \begin{bmatrix} 49.479 & -49.479 \\ -49.479 & 49.479 \end{bmatrix}$$

$$[K^2] = \frac{1}{120} \times 10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^4 \begin{bmatrix} 8.333 & -8.333 \\ -8.333 & 8.333 \end{bmatrix}$$

The assembled equations are

$$10^4 \begin{bmatrix} 49.479 & -49.479 & 0.000 \\ -49.479 & 57.812 & -8.333 \\ 0.000 & -8.333 & 8.333 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

The boundary conditions imply

$$U_1 = 0, \quad Q_2^1 + Q_1^2 = 2P_0, \quad Q_2^2 = P_0$$

Hence, the condensed equations are

$$10^4 \begin{bmatrix} 57.812 & -8.333 \\ -8.333 & 8.333 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 2P_0 \\ P_0 \end{Bmatrix}, \quad Q_1^1 = -10^4 \times 49.479\, U_2$$

whose solution is

$$U_2 = 0.06063 \text{ in.}, \quad U_3 = 0.18063 \text{ in.}, \quad Q_1^1 = -30,000 \text{ lb}$$

The negative sign indicates that the reaction is acting away from the end (i.e., tensile force). The magnitude of Q_1^1 is consistent with the static equilibrium of the forces:

$$Q_1^1 + 2P_0 + P_0 = 0 \text{ or } Q_1^1 = -3P_0 = -30,000 \text{ lb}$$

The axial displacement at any point x along the bar is given by

$$u_h(x) = \begin{cases} u_1^{(1)}\psi_1^{(1)} + u_2^{(1)}\psi_2^{(1)} = 0.06063x/96, & 0 \le x \le 96 \\ u_1^{(2)}\psi_1^{(2)} + u_2^{(2)}\psi_2^{(2)} = -0.03537 + 0.001x, & 96 \le x \le 216 \end{cases}$$

and its first derivative is given by

$$\frac{du_h}{dx} = \begin{cases} \dfrac{0.06063}{96}, & 0 \le x \le 96 \\ 0.0001, & 96 \le x \le 216 \end{cases}$$

The exact solution of Eqs. (4.5.9) subject to the boundary conditions

$$u(0) = 0, \left[\left(a\frac{du}{dx}\right)_{x=96^+} - \left(a\frac{du}{dx}\right)_{x=96^-}\right] = 2P_0, \left(a\frac{du}{dx}\right)_{x=216} = P_0$$

is given by

$$u(x) = \begin{cases} 0.128[x/(288-x)], & 0 \le x \le 96 \\ 0.001(x-32), & 96 \le x \le 216 \end{cases}$$

$$\frac{du}{dx} = \begin{cases} 36.864/(288-x)^2, & 0 \le x \le 96 \\ 0.001, & 96 \le x \le 216 \end{cases}$$

In particular, the exact solution at nodes 2 and 3 is given by

$$u(96) = 0.064 \text{ in.}, \quad u(216) = 0.1840 \text{ in.}$$

Thus, the two-element solution is about 1.8 percent off from the maximum displacement.

Next consider a two-element mesh of quadratic elements. The element matrix and force vector for element 1 are

$$[K^1] = 10^4 \begin{bmatrix} 142.19 & -159.37 & 17.18 \\ -159.37 & 266.67 & -107.29 \\ 17.18 & -107.29 & 90.10 \end{bmatrix}, \quad \{F^1\} = \begin{Bmatrix} Q_1^1 \\ 0 \\ Q_3^1 \end{Bmatrix}$$

The assembled stiffness matrix is of order 5×5, and it is of the form

$$[K] = \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 & 0 & 0 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 + K_{11}^2 & K_{12}^2 & K_{13}^2 \\ 0 & 0 & K_{21}^2 & K_{22}^2 & K_{23}^2 \\ 0 & 0 & K_{31}^2 & K_{32}^2 & K_{33}^2 \end{bmatrix}$$

After imposing boundary conditions ($U_1 = 0$, $Q_3^1 + Q_1^2 = 20,000$, $Q_3^2 = 10,000$), and solving the resulting 4×4 equations, we obtain

$$U_2 = 0.02572 \text{ in.}, U_3 = 0.06392 \text{ in.}, U_4 = 0.12392 \text{ in.}, U_5 = 0.18392 \text{ in.}$$

The two-element solution obtained using the quadratic element is very accurate, as can be seen from a comparison of the finite element solution with the exact solution presented in Table 4.5.1.

Table 4.5.1 Comparison of the finite element solutions with the exact solution of the bar problem in Example 4.5.2.

		Linear					Quadratic	
x (in.)	Exact	(1, 1)†	(2, 1)	(3, 2)	(4, 2)	(6, 2)	(1, 1)	(2, 1)
16	0.00753	—	—	—	—	0.00752	—	—
24	0.01600	—	—	—	0.01161	—	—	0.01164
32	0.01600	—	—	0.01593	—	0.01598	—	—
48	0.02560	—	0.02532	—	0.02553	0.02557	0.02572	0.02560
64	0.03657	—	—	0.03638	—	0.03652	—	—
72	0.04267	—	—	—	0.04253	—	—	0.04268
80	0.04923	—	—	—	—	0.04916	—	—
96	0.06400	0.06063	0.06309	0.06359	0.06377	0.06390	0.06392	0.06399
156	0.12392	—	—	—	0.12377	0.12390	0.12392	0.12399
216	0.18400	0.18063	0.18309	0.18359	0.18377	0.18390	0.18392	0.18399

† (m, n) means m elements in the interval (0,96) and n elements in the interval (96,216); all elements in each interval are of the same size.

Example 4.5.3

Consider a hollow circular cylinder with inner radius a, outer radius b, and length L. The outer surface of the hollow cylinder is assumed to be fixed and its inner surface ideally bonded to a rigid circular cylindrical core of radius a and length L, as shown in Fig. 4.5.3. Suppose that an axial force P is applied to the rigid core along its centroidal axis. We wish to find the axial displacement δ of the rigid core by assuming that the displacement field in the hollow cylinder is of the form

$$u_r = u_\theta = 0, \quad u_z = U(r) \tag{4.5.12}$$

where (u_r, u_θ, u_z) are the displacements along (r, θ, z) coordinates.

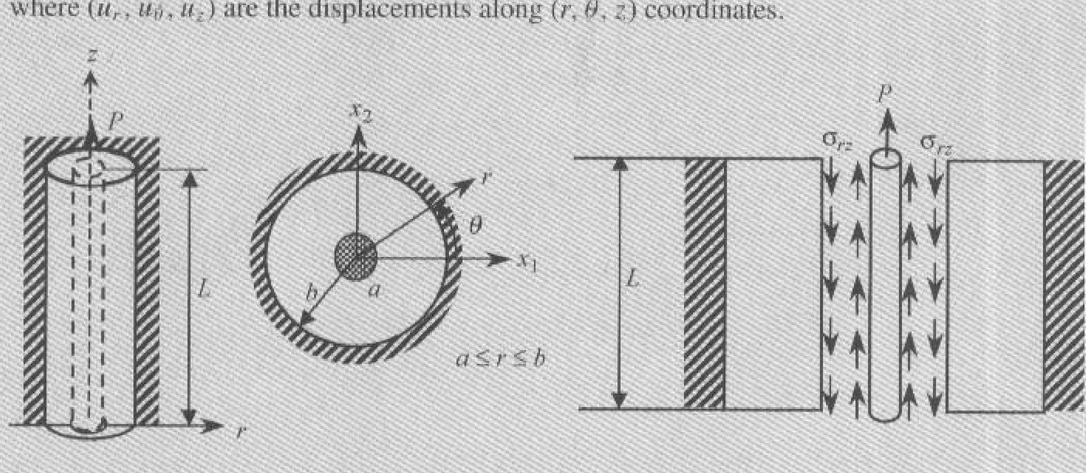


Figure 4.5.3 Axisymmetric deformation of a hollow cylinder fixed at the outer surface and pulled by a rigid core at the inner surface.

The equation governing $U(r)$ can be determined as follows. First, we note that

$$\varepsilon_{rr} = \varepsilon_{\theta\theta} = \varepsilon_{zz} = \varepsilon_{r\theta} = \varepsilon_{\theta z} = 0, \quad 2\varepsilon_{rz} = \frac{dU}{dr} \tag{4.5.13a}$$

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{\theta r} = \sigma_{\theta z} = 0; \quad \sigma_{rz} = G\frac{dU}{dr} \tag{4.5.13b}$$

where G is the shear modulus. Of the three stress-equilibrium equations in the cylindrical coordinates, the two equilibrium equations associated with r and θ directions are trivially satisfield (in the absence of body forces) by the stress field. The third equilibrium equation

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{\partial \sigma_{\theta z}}{\partial \theta} + r\frac{\partial \sigma_{zz}}{\partial z}\right] = 0$$

yields the equation

$$\frac{1}{r}\frac{d}{dr}\left(rG\frac{dU}{dr}\right) = 0 \tag{4.5.14}$$

The boundary conditions on $U(r)$ are

$$u_z(b) = 0 \rightarrow U(b) = 0, \quad -2\pi L(r\sigma_{rz})|_{r=a} = P \rightarrow -\left(rG\frac{dU}{dr}\right)_{r=a} = \frac{P}{2\pi L} \tag{4.5.15}$$

This completes the theoretical formulation of the problem. We wish to analyze the problem using the finite element method.

The finite element model of Eq. (4.5.14) [cf. Eq. (3.4.1)] is given by Eqs. (3.4.5a) and (3.4.5b) with $a(r) = rG$ and $f = 0$. One linear element in the domain yields ($r_a = a$ and $h = b - a$)

$$\frac{2\pi G}{h}\left(r_a + \frac{1}{2}h\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}$$

and with $U_2 = 0$ and $Q_1^1 = P/L$, we obtain

$$U_1 = \frac{P}{\pi LG}\frac{(b-a)}{(b+a)} \tag{4.5.16}$$

The exact solution of Eqs. (4.5.14) and (4.5.15) is given by

$$U(r) = -\frac{P}{2\pi LG}\log(r/b), \quad U(a) \equiv \delta = \frac{P}{2\pi LG}\log(b/a) \tag{4.5.17}$$

The one element solution in Eq. (4.5.16) corresponds to the first term in the logarithmic series of δ [i.e., $\log(a/b)$].

Use of one quadratic element gives ($h = b - a$)

$$\frac{2\pi G}{6h}\begin{bmatrix} 3h+14a & -(4h+16a) & h+2a \\ -(4h+16a) & 16h+32a & -(12h+16a) \\ h+2a & -(12h+16a) & 11h+14a \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ 0 \\ Q_3^1 \end{Bmatrix} \tag{4.5.18}$$

Using the boundary conditions $U_3 = 0$ and $Q_1^1 = P/L$, we obtain the following condensed equations in U_1 and U_2:

$$\frac{2\pi G}{6h}\begin{bmatrix} 3h+14a & -(4h+16a) \\ -(4h+16a) & 16h+32a \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} P/L \\ 0 \end{Bmatrix} \tag{4.5.19}$$

whose solution is

$$U_1 = \frac{3hP}{\pi GL}\frac{h+2a}{3h^2+20ah+28a^2}$$
$$U_2 = \frac{3hP}{\pi GL}\frac{h+4a}{12h^2+80ah+112a^2} \tag{4.5.20}$$

The finite element solution becomes

$$U_h(r) = U_1(1-r/h)(1-2r/h) + 4U_2(r/h)(1-r/h)$$

A comparison of the finite element solutions obtained with linear and quadratic elements with the analytical solution is shown in Fig. 4.5.4. The four-element mesh of quadratic elements virtually gives the exact solution.

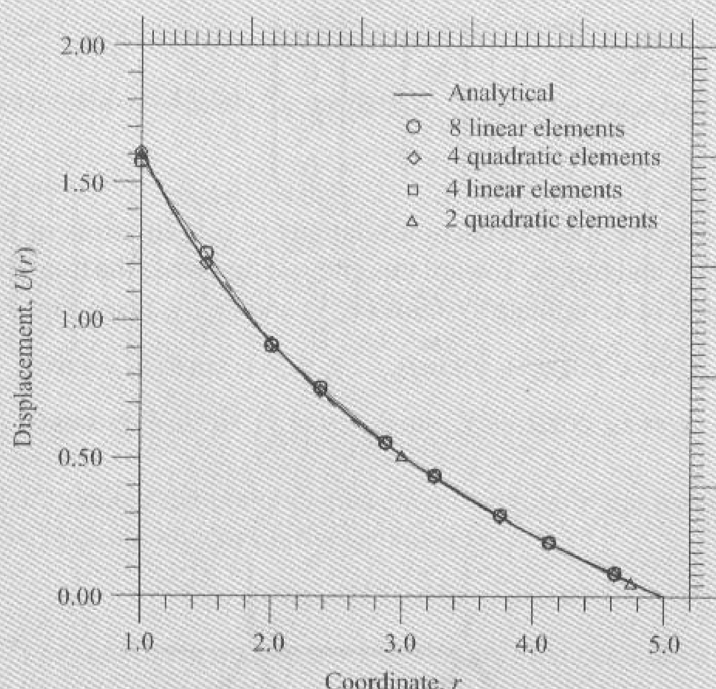


Figure 4.5.4 Comparison of finite element solutions with the analytical solution for axisymmetric deformation of a hollow cylinder fixed at the outer surface and pulled by a rigid core at the inner surface.

4.6 PLANE TRUSSES

4.6.1 Introduction

Consider a structure consisting of several bar elements connected to each other by pins, as shown in Figure 4.6.1. The members may rotate freely about the axis of the pin. Consequently, each member carries only axial forces. The planar structure with pin-connected members (i.e., all members lie in the same plane) is called a *plane truss*. Since each member is oriented differently with respect to a global coordinate system (x, y), it is necessary to transform the force displacement relations that were derived in element coordinate system $(\bar{x}, \bar{y})$ to the global coordinate system (x, y) so that the structure stiffness can be assembled from the element stiffness referred to the same global coordinate system.

4.6.2 Basic Truss Element

First, we consider a uniform bar element with constant EA and oriented at an angle θ_e, measured counterclockwise, from the positive x-axis. If the member coordinate system $(\bar{x}_e, \bar{y}_e)$ is taken as shown in Figure 4.6.2(a), and $(\bar{u}_i^e, \bar{v}_i^e)$ and $(\bar{F}_i^e, 0)$ denote the displacements and forces at node i with respect to the member coordinate system $(\bar{x}_e, \bar{y}_e)$, respectively, the element equations (4.5.11a) can be expressed as (we now use the notation $\bar{Q}_i^e + \bar{f}_i^e = \bar{F}_i^e$)

$$\frac{E_e A_e}{h_e}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{Bmatrix} \bar{F}_1^e \\ 0 \\ \bar{F}_2^e \\ 0 \end{Bmatrix} \quad \text{or} \quad [\bar{K}^e]\{\bar{\Delta}^e\} = \{\bar{F}^e\} \tag{4.6.1}$$

We wish to write the force-deflection relations (4.6.1) in terms of the corresponding global displacements and forces. Toward this end, we first write the transformation relations between the two sets of coordinate systems (x, y) and $(\bar{x}_e, \bar{y}_e)$ (see Fig. 4.6.2)

$$\bar{x}_e = x \cos\theta_e + y \sin\theta_e, \quad \bar{y}_e = -x \sin\theta_e + y \cos\theta_e$$

$$x = \bar{x}_e \cos\theta_e - \bar{y}_e \sin\theta_e, \quad y = \bar{x}_e \sin\theta_e + \bar{y}_e \cos\theta_e$$

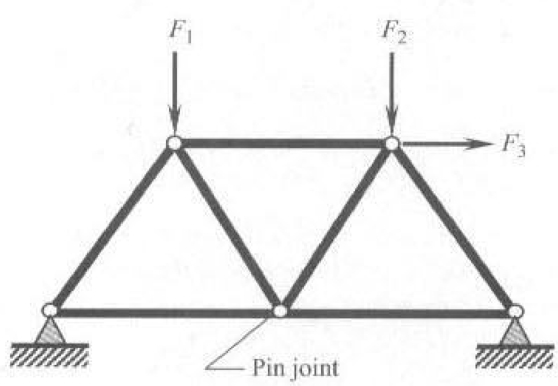


Figure 4.6.1 A plane truss structure.

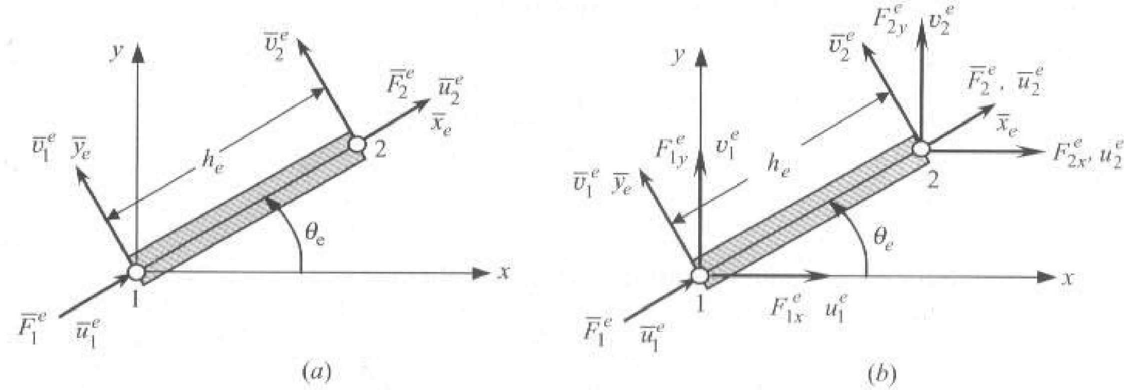


Figure 4.6.2 A bar element oriented at an angle with respect to the global coordinate system (x, y). (*a*) Forces and displacements in the element coordinates. (*b*) Forces and displacements in the global coordinates.

or, in matrix form, we have

$$\begin{Bmatrix} \bar{x}_e \\ \bar{y}_e \end{Bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e \\ -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}, \quad \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos\theta_e & -\sin\theta_e \\ \sin\theta_e & \cos\theta_e \end{bmatrix} \begin{Bmatrix} \bar{x}_e \\ \bar{y}_e \end{Bmatrix} \tag{4.6.2}$$

where θ_e is the angle between the positive x-axis and positive $\bar{x}_e$-axis, measured in the counterclockwise direction. Note that all quantities with a bar over them refer to the member (or local) coordinate system $(\bar{x}_e, \bar{y}_e)$, while the quantities without a bar refer to the global coordinate system (x, y).

The above relationship also holds for the displacements and forces of the two coordinate systems. We have

$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 & 0 \\ -\sin\theta_e & \cos\theta_e & 0 & 0 \\ 0 & 0 & \cos\theta_e & \sin\theta_e \\ 0 & 0 & -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{Bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \end{Bmatrix} \tag{4.6.3a}$$

or

$$\{\bar{\Delta}^e\} = [T^e]\{\Delta^e\} \tag{4.6.3b}$$

where $\{\bar{\Delta}^e\}$ and $\{\Delta^e\}$ denote the nodal displacement vectors in the member and structure coordinate systems, respectively. Similarly, we have

$$\{\bar{F}^e\} = [T^e]\{F^e\} \tag{4.6.4}$$

Here $\{\bar{F}^e\}$ and $\{F^e\}$ denote the nodal force vectors in the member and structure coordinate systems, respectively (see Figures 4.6.2).

4.6.3 General Truss Element

Next, we derive the relationship between the global displacements and global forces. Using Eqs. (4.6.3*b*) and (4.6.4) in Eq. (4.6.1), we obtain

$$[\bar{K}^e][T^e]\{\Delta^e\} = [T^e]\{F^e\} \tag{4.6.5}$$

Premultiplying both sides of the above equation with $[T^e]^{\mathrm{T}}$ and noting that $[T^e]^{-1} = [T^e]^{\mathrm{T}}$, we obtain

$$[T^e]^{\mathrm{T}}[\bar{K}^e][T^e]\{\Delta^e\} = \{F^e\} \quad \text{or} \quad [K^e]\{\Delta^e\} = \{F^e\} \tag{4.6.6}$$

where

$$[K^e] = [T^e]^{\mathrm{T}}[\bar{K}^e][T^e], \qquad \{F^e\} = [T^e]^{\mathrm{T}}\{\bar{F}^e\} \tag{4.6.7}$$

Carrying out the indicated matrix multiplications, we obtain

$$[K^e] = \frac{E_e A_e}{h_e}\begin{bmatrix} \cos^2\theta_e & \frac{1}{2}\sin 2\theta_e & -\cos^2\theta_e & -\frac{1}{2}\sin 2\theta_e \\ \frac{1}{2}\sin 2\theta_e & \sin^2\theta_e & -\frac{1}{2}\sin 2\theta_e & -\sin^2\theta_e \\ -\cos^2\theta_e & -\frac{1}{2}\sin 2\theta_e & \cos^2\theta_e & \frac{1}{2}\sin 2\theta_e \\ -\frac{1}{2}\sin 2\theta_e & -\sin^2\theta_e & \frac{1}{2}\sin 2\theta_e & \sin^2\theta_e \end{bmatrix} \tag{4.6.8}$$

$$\{F^e\} = \begin{Bmatrix} F_1^e \\ F_2^e \\ F_3^e \\ F_4^e \end{Bmatrix} = \begin{Bmatrix} \bar{F}_1^e\cos\theta_e \\ \bar{F}_1^e\sin\theta_e \\ \bar{F}_2^e\cos\theta_e \\ \bar{F}_2^e\sin\theta_e \end{Bmatrix} + \begin{Bmatrix} \bar{f}_1^e\cos\theta_e \\ \bar{f}_1^e\sin\theta_e \\ \bar{f}_2^e\cos\theta_e \\ \bar{f}_2^e\sin\theta_e \end{Bmatrix} \tag{4.6.9}$$

where $\bar{f}_i^e$ are computed using Eq. (3.2.31*b*) [also see Eq. (3.2.34)]

$$\bar{f}_i^e = \int_0^{h_e} f(\bar{x})\psi_i^e(\bar{x})\,d\bar{x} \tag{4.6.10}$$

Equations (4.6.8) and (4.6.9) provide the means to compute the element stiffness matrix $[K^e]$ and force vector $\{F^e\}$, respectively, both referred to the global coordinate system, of a bar element oriented at an angle θ_e. The assembly of elements with their stiffness matrix and force vector in the global coordinates follows the same ideas as discussed before except that we must note that each node now has two displacement degrees of freedom. These ideas are illustrated in the following example.

Example 4.6.1

Consider a three-member truss shown in Figure 4.6.3(*a*). All members of the truss have identical areas of cross section A and modulus E. The hinged supports at joints A, B, and C allow free rotation of the members about the z-axis (taken positive out of the plane of the paper). We wish to determine the horizontal and vertical displacements at the joint C and forces in each member of the structure.

Finite Element Mesh

We use three finite elements to model the structure. Any further subdivision of the members does not add to the accuracy because the finite element solutions for displacements and forces at the nodes are exact for all truss problems. The global node numbers and element numbers are shown in Figure 4.6.3(*b*). There are two displacement degrees of freedom, horizontal and

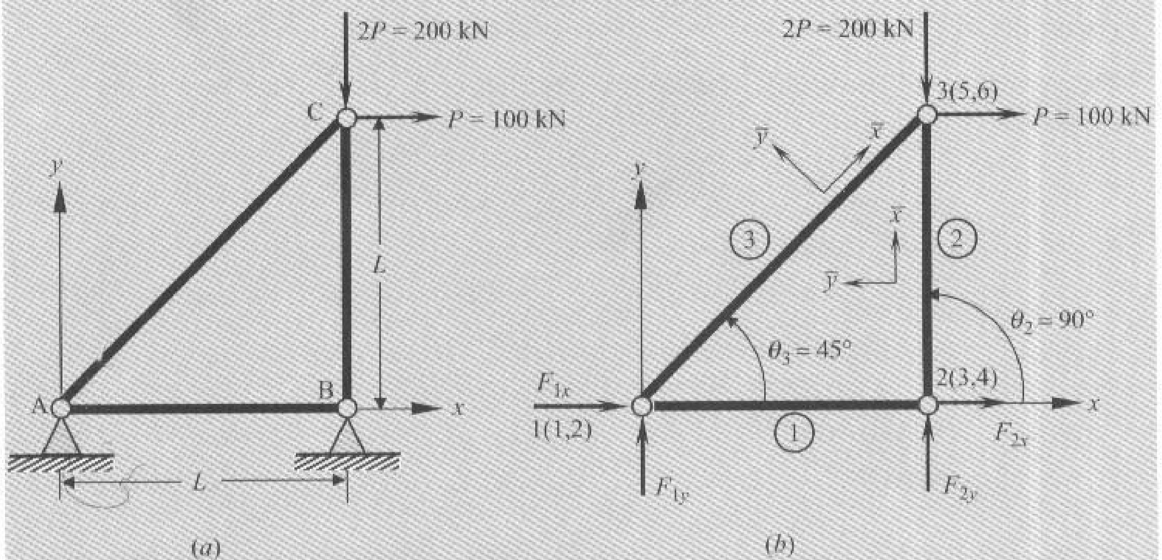


Figure 4.6.3 Geometry and finite element representation of a plane truss. (*a*) Geometry and applied loads. (*b*) Element numbering and reaction forces.

vertical displacements, at each node of the element. The element stiffness matrix in the local coordinate system is given by Eq. (4.6.1) while the stiffness matrix and force vector (with $\bar{f}_i^e = 0$) in the global coordinate system is given by Eqs. (4.6.8) and (4.6.9), respectively. The element data and connectivity is given in the following table.

Element number	Global nodes	Geometric properties	Material property	Orientation
1	1 2	$A,\ h_1 = L$	E	$\theta_1 = 0°$
2	2 3	$A,\ h_2 = L$	E	$\theta_2 = 90°$
3	1 3	$A,\ h_3 = \sqrt{2}L$	E	$\theta_3 = 45°$

Element Matrices

The element stiffness matrices are given by $[1/(2\sqrt{2}) = 0.3536]$

$$[K^1] = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad [K^2] = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[K^3] = \frac{EA}{L}\begin{bmatrix} 0.3536 & 0.3536 & -0.3536 & -0.3536 \\ 0.3536 & 0.3536 & -0.3536 & -0.3536 \\ -0.3536 & -0.3536 & 0.3536 & 0.3536 \\ -0.3536 & -0.3536 & 0.3536 & 0.3536 \end{bmatrix} \tag{4.6.11}$$

Assembly of Elements

The assembled stiffness matrix is given by (subscripts of K_{ij} refer to the local degrees of freedom)

$$
[K] = \begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
K_{11}^1 + K_{11}^3 & K_{12}^1 + K_{12}^3 & K_{13}^1 & K_{14}^1 & K_{13}^3 & K_{14}^3 \\
 & K_{22}^1 + K_{22}^3 & K_{23}^1 & K_{24}^1 & K_{23}^3 & K_{24}^3 \\
 & & K_{33}^1 + K_{11}^2 & K_{34}^1 + K_{12}^2 & K_{13}^2 & K_{14}^2 \\
 & \text{symm.} & & K_{44}^1 + K_{22}^2 & K_{23}^2 & K_{24}^2 \\
 & & & & K_{33}^2 + K_{33}^3 & K_{34}^2 + K_{34}^3 \\
 & & & & & K_{44}^2 + K_{44}^3
\end{bmatrix}
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\tag{4.6.12}
$$

The element stiffness matrices of (4.6.11) can be used in Eq. (4.6.12) to obtain the assembled global stiffness matrix

$$
[K] = \frac{EA}{L} \left[\begin{array}{cccc|cc}
1.3536 & 0.3536 & -1.0 & 0.0 & -0.3536 & -0.3536 \\
 & 0.3536 & 0.0 & 0.0 & -0.3536 & -0.3536 \\
 & & 1.0 & 0.0 & 0.0 & 0.0 \\
 & & & 1.0 & 0.0 & -1.0 \\
\hline
\text{symm.} & & & & 0.3536 & 0.3536 \\
 & & & & & 1.3536
\end{array}\right]
\tag{4.6.13}
$$

The displacement continuity conditions are

$$
\begin{aligned}
u_1^1 &= u_1^3 = U_1, \quad v_1^1 = v_1^3 = V_1 \\
u_2^1 &= u_1^2 = U_2, \quad v_2^1 = v_1^2 = V_2 \\
u_2^2 &= u_2^3 = U_3, \quad v_2^2 = v_2^3 = V_3
\end{aligned}
\tag{4.6.14}
$$

and the force equilibrium conditions are

$$
\begin{aligned}
F_1^1 + F_1^3 &= F_{1x}, \quad F_2^1 + F_2^3 = F_{1y} \\
F_3^1 + F_1^2 &= F_{2x}, \quad F_4^1 + F_2^2 = F_{2y} \\
F_3^2 + F_3^3 &= F_{3x}, \quad F_4^2 + F_4^3 = F_{3y}
\end{aligned}
\tag{4.6.15}
$$

which can be used to write the global displacement and force vectors as

$$
\{\Delta\} = \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix}, \quad
\{F\} = \begin{Bmatrix} F_1^1 + F_1^3 \\ F_2^1 + F_2^3 \\ F_3^1 + F_1^2 \\ F_4^1 + F_2^2 \\ F_3^2 + F_3^3 \\ F_4^2 + F_4^3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix}
\tag{4.6.16}
$$

where (U_I, V_I) and (F_{Ix}, F_{Iy}) denote the x and y components of the displacement and external forces, respectively, at the global node I.

Boundary Conditions

The specified displacement and force degrees of freedom are

$$U_1 = V_1 = U_2 = V_2 = 0, \quad F_{3x} = P, \quad F_{3y} = -2P \tag{4.6.17}$$

The first two boundary conditions correspond to the horizontal and vertical displacements at node 1, the next two correspond to the horizontal and vertical displacements at node 2, and the last correspond to the force boundary conditions at node 3. The unknowns are: the displacements (U_3, V_3) of node 3 and forces (F_{1x}, F_{1y}) at node 1 and forces (F_{2x}, F_{2y}) at node 2.

Condensed Equations

The condensed equations for the unknown displacements (U_3, V_3) are obtained from the last two equations of the system, as indicated by the dotted lines in (4.6.13)

$$\frac{EA}{L}\begin{bmatrix} 0.3536 & 0.3536 \\ 0.3536 & 1.3536 \end{bmatrix}\begin{Bmatrix} U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \end{Bmatrix} \tag{4.6.18}$$

and the condensed equations for the unknown reactions are (from the first four equations of the system)

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} -0.3536 & -0.3536 \\ -0.3536 & -0.3536 \\ 0.0 & 0.0 \\ 0.0 & -1.0 \end{bmatrix}\begin{Bmatrix} U_3 \\ V_3 \end{Bmatrix} \tag{4.6.19}$$

Solution of the Finite Element Equations

Solving Eqs. (4.6.18) for U_5 and U_6, we obtain

$$U_3 = (3 + 2\sqrt{2})\frac{PL}{EA} = 5.828\,\frac{PL}{EA}, \quad V_3 = -\frac{3PL}{EA} \tag{4.6.20}$$

and the reaction forces are computed using Eq. (4.6.19)

$$F_{1x} = -P, \; F_{1y} = -P, \; F_{2x} = 0.0, \; F_{2y} = 3P \tag{4.6.21}$$

Postcomputation

The stress in each member can be computed from the relation

$$\sigma^e = -\frac{\bar{P}_1^e}{A_e} = \frac{\bar{P}_2^e}{A_e} \tag{4.6.22}$$

where $\bar{P}_1^e$ and $\bar{P}_2^e$ can be determined from the element equations

$$\begin{Bmatrix} \bar{P}_1^e \\ \bar{P}_2^e \end{Bmatrix} = \frac{A_e E_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \end{Bmatrix} \tag{4.6.23}$$

and the element displacements $(\bar{u}_1^e, \bar{u}_2^e)$ are determined from the transformation relation (4.6.3a)

$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 & 0 \\ -\sin\theta_e & \cos\theta_e & 0 & 0 \\ 0 & 0 & \cos\theta_e & \sin\theta_e \\ 0 & 0 & -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{Bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \end{Bmatrix} \tag{4.6.24}$$

By definition [see Eqs. (4.6.14)], we have

$$\begin{aligned} &u_1^1 = v_1^1 = u_2^1 = v_2^1 = 0, \quad u_1^2 = v_1^2 = u_1^3 = v_1^3 = 0 \\ &u_2^2 = u_2^3 = U_3 = (3 + 2\sqrt{2})\frac{PL}{EA}, \; v_2^2 = v_2^3 = V_3 = -\frac{3PL}{EA} \end{aligned} \tag{4.6.25}$$

Hence, Eqs. (4.6.22)–(4.6.25) give

$$\sigma^e = \frac{E_e}{h_e}(\bar{u}_2^e - \bar{u}_1^e) \tag{4.6.26}$$

$$\begin{aligned} \bar{u}_1^1 &= u_1^1 \cos\theta_1 + v_1^1 \sin\theta_1 = 0 \\ \bar{u}_2^1 &= u_2^1 \cos\theta_1 + v_2^1 \sin\theta_1 = 0 \\ \bar{u}_1^2 &= u_1^2 \cos\theta_2 + v_1^2 \sin\theta_2 = 0 \\ \bar{u}_2^2 &= U_3 \cos\theta_2 + V_3 \sin\theta_2 = V_3 = -\frac{3PL}{EA} \\ \bar{u}_1^3 &= u_1^3 \cos\theta_3 + v_1^3 \sin\theta_3 = 0 \\ \bar{u}_2^3 &= U_3 \cos\theta_3 + V_3 \sin\theta_3 = \frac{1}{\sqrt{2}}(U_3 + V_3) = \frac{2PL}{EA} \end{aligned} \tag{4.6.27}$$

Thus, the member forces are

$$\bar{P}_1^1 = -\bar{P}_2^1 = 0, \quad \bar{P}_1^2 = -\bar{P}_2^2 = 3P, \quad \bar{P}_1^3 = -\bar{P}_2^3 = -\sqrt{2}P \tag{4.6.28}$$

The axial stresses in the members are

$$\sigma^{(1)} = 0, \quad \sigma^{(2)} = -\frac{3P}{A}, \quad \sigma^{(3)} = \sqrt{2}\,\frac{P}{A} \tag{4.6.29}$$

Interpretation and Verification of the Results

An examination of the structure and the sense of loads applied indicate that the displacements (U_3, V_3) are qualitatively correct (positive U_3 and negative V_3). Also, the geometry of the structure indicates that it has relatively more stiffness in the vertical direction (member 2 takes much of the load directly) compared to the horizontal direction, which explains the relatively large displacement in the horizontal direction.

The forces in Eq. (4.6.21) can be verified by applying the method of sections to the free-body-diagram in Figure 4.6.3(b). Sections AA, BB, and CC (see Figure 4.6.4) yield

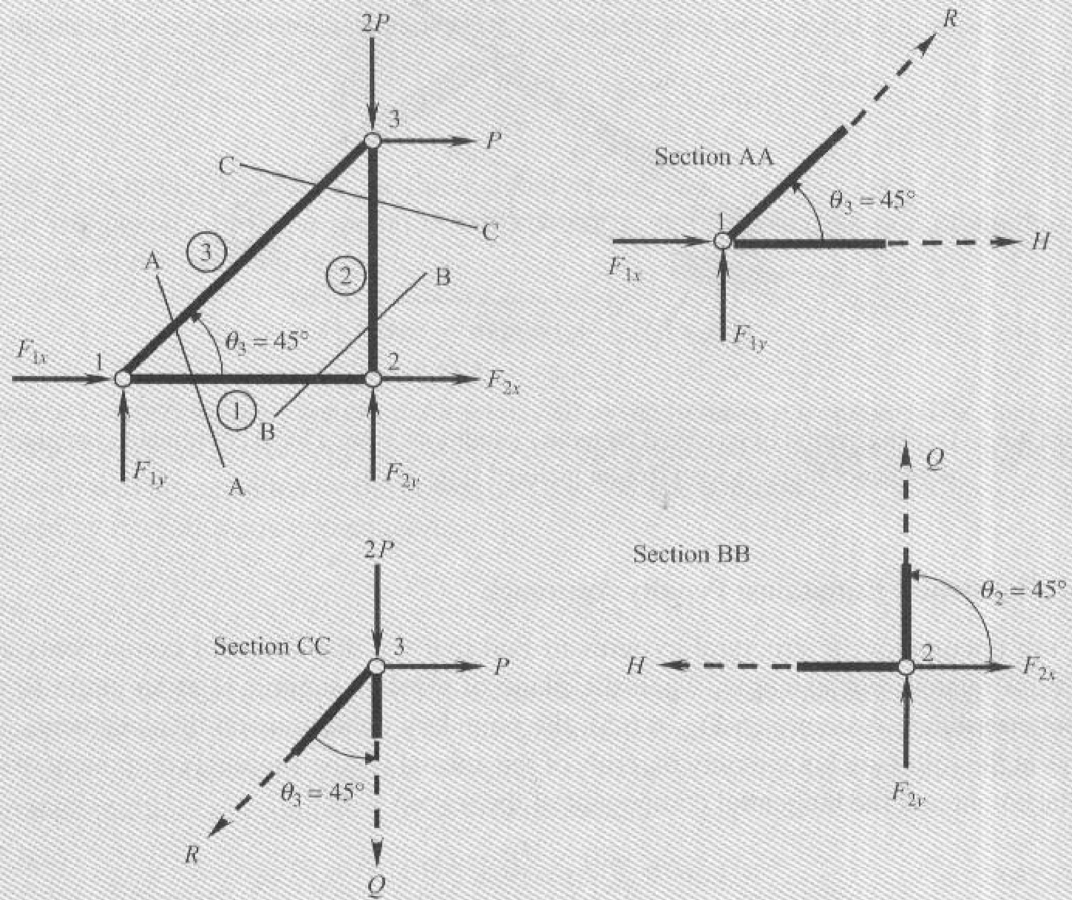


Figure 4.6.4 Method of sections to determine the member forces.

the relations ($H = 0$)

$$\frac{1}{\sqrt{2}}R + F_y^1 = 0, \quad F_x^1 + H + \frac{1}{\sqrt{2}}R = 0 \tag{4.6.30a}$$

$$F_x^2 - H = 0, \quad F_y^2 + Q = 0 \tag{4.6.30b}$$

$$-\frac{1}{\sqrt{2}}R + P = 0, \quad \frac{1}{\sqrt{2}}R + Q + 2P = 0 \tag{4.6.30c}$$

which yield

$$Q = -3P, \quad R = \sqrt{2}P, \quad F_y^2 = 3P, \quad F_y^1 = -P, \quad F_x^1 = -P, \quad H = F_x^2 = 0 \tag{4.6.30d}$$

Note that the member forces computed using the method of sections agree with those computed in the finite element method ($\bar{P}_1^1 = H$, $\bar{P}_1^2 = -Q$, and $\bar{P}_1^3 = -R$). It should be noted that this exercise is only for checking purposes, and the finite element method can be used to determine the member forces Q, R, and H, as discussed in the postcomputation. This completes the example.

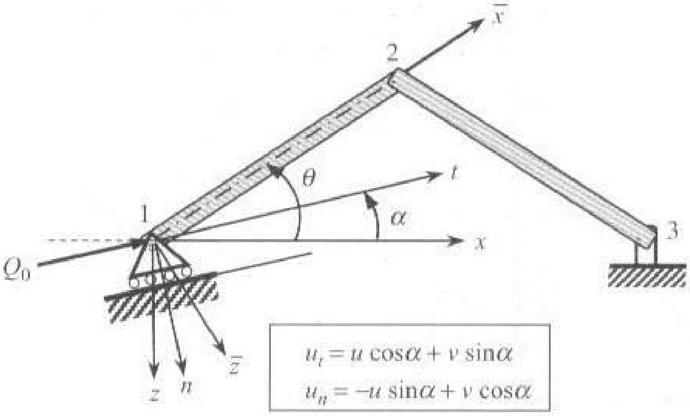


Figure 4.6.5 Transformation of specified boundary conditions from a local coordinate system to the global coordinate system (for an inclined support).

4.6.4 Constraint Equations: Penalty Approach

It is not uncommon in structural systems to find that the displacement components at a point are related. For example, when the plane of a roller support is at an angle to the global coordinate system (see Fig. 4.6.5), the boundary conditions on displacements and forces at the roller are known only in terms of the normal (to the support) component of the displacement and the tangential component of the force

$$u_n^e = 0, \quad Q_t^e = Q_0 \tag{4.6.31}$$

where u_n^e is the normal component of displacement and Q_t^e is the tangential component of the force at node 1 of the element Ω^e; Q_0 is any specified tangential force. These conditions, when expressed in terms of the global components of displacements and forces by means of the transformation of the form (4.6.3*b*) and (4.6.4), become

$$u_n^e = -u_1^e \sin\alpha + u_2^e \cos\alpha = 0 \tag{4.6.32a}$$

$$Q_t^e = Q_1^e \cos\alpha + Q_2^e \sin\alpha = Q_0 \tag{4.6.32b}$$

where (u_1^e, u_2^e) and (Q_1^e, Q_2^e) are the x and y components of the displacements and forces, respectively, at the support. Equations (4.6.32*a*) can be viewed as constraint equations among the global displacements, which have a companion relation among the associated forces, namely Eq. (4.6.32*b*). Here, we present the penalty function method through which constraint equations of the type in (4.6.32*a*) and (4.6.32*b*) can be included in the finite element equations.

The penalty function method allows us to reformulate a problem with constraints as one without constraints. The basic idea of the method can be described by considering an algebraic constrained problem:

minimize the function $f(x, y)$ *subject to the constraint* $G(x, y) = 0$

In the Lagrange multiplier method the problem is reformulated as one of determining the stationary (or critical) points of the modified function $F_L(x, y)$,

$$F_L(x, y) = f(x, y) + \lambda G(x, y) \tag{4.6.33}$$

subject to no constraints. Here λ denotes the Lagrange multiplier. The solution to the problem is obtained by setting partial derivatives of F_L with respect to x, y and λ to zero:

$$\frac{\partial F_L}{\partial x}=0, \quad \frac{\partial F_L}{\partial y}=0, \quad \frac{\partial F_L}{\partial \lambda}=0 \tag{4.6.34}$$

which gives three equations in the three unknowns (x, y, λ).

In the penalty function method, the problem is reformulated as one of finding the minimum of the modified function F_P,

$$F_P(x, y)=f(x, y)+\frac{\gamma}{2}[G(x, y)]^2 \tag{4.6.35}$$

where γ is a preassigned weight parameter, called the *penalty parameter*. The factor $\frac{1}{2}$ in Eq. (4.6.35) is used for convenience: When F_p is differentiated with respect to its arguments, the factor will be cancelled by the power on $G(x, y)$. The solution to the modified problem is given by the following two equations:

$$\frac{\partial F_P}{\partial x}=0\,, \quad \frac{\partial F_P}{\partial y}=0 \tag{4.6.36a}$$

The solution of Eqs. (4.6.36a) will be a function of the penalty parameter, (x_γ, y_γ). The larger the value of γ, the more exactly the constraint is satisfied (in a least-squares sense), and (x_γ, y_γ) approaches the actual solution (x, y) as $\gamma \to \infty$. An approximation to the Lagrange multiplier is computed from the equation,

$$\lambda_\gamma=\gamma G(x_\gamma, y_\gamma) \tag{4.6.36b}$$

We consider a specific example to illustrate the ideas presented above.

Example 4.6.2

Minimize the quadratic function

$$f(x, y)=4x^2-3y^2+2xy+6x-3y+5$$

subject to the constraint

$$G(x, y)\equiv 2x+3y=0$$

Geometrically, we seek the inflection point of the surface $f(x, y)$ that is on the line $2x+3y=0$. We solve the problem using the Lagrange multiplier method and the penalty function method.

Lagrange Multiplier Method. The modified functional is

$$F_L(x, y)=f(x, y)+\lambda(2x+3y)$$

where λ is the Lagrange multiplier to be determined. We have

$$\frac{\partial F_L}{\partial x} = 8x + 2y + 6 + 2\lambda = 0$$
$$\frac{\partial F_L}{\partial y} = -6y + 2x - 3 + 3\lambda = 0$$
$$\frac{\partial F_L}{\partial \lambda} = 2x + 3y = 0$$

Solving the three algebraic equations, we obtain

$$x = -3,\ y = 2,\ \lambda = 7$$

Penalty Function Method. The modified functional is

$$F_P(x, y) = f(x, y) + \frac{\gamma}{2}(2x + 3y)^2$$

and we have

$$\frac{\partial F_P}{\partial x} = 8x + 2y + 6 + 2\gamma(2x + 3y) = 0$$
$$\frac{\partial F_P}{\partial y} = -6y + 2x - 3 + 3\gamma(2x + 3y) = 0$$

The solution of these equations is

$$x_\gamma = \frac{15 - 36\gamma}{-26 + 12\gamma},\ y_\gamma = \frac{18 + 24\gamma}{-26 + 12\gamma}$$

The Lagrange multiplier is given by

$$\lambda_\gamma = \gamma G(x_\gamma, y_\gamma) = \frac{84\gamma}{-26 + 12\gamma}$$

Clearly, in the limit $\gamma \to \infty$, the penalty function solution approaches the exact solution:

$$\lim_{\gamma\to\infty} x_\gamma = -3,\ \lim_{\gamma\to\infty} y_\gamma = 2,\ \lim_{\gamma\to\infty} \lambda_\gamma = 7$$

An approximate solution to the problem can be obtained, within a desired accuracy, by selecting a finite value of the penalty parameter (see Table 4.6.1). This completes the example.

Table 4.6.1 Convergence of penalty function solution with increasing penalty parameter.

γ	x_γ	y_γ	λ_γ	$E_\gamma = 2x_\gamma + 3y_\gamma$
0	−0.5769	−0.6923	0.0000	−3.2308
1	1.5000	−3.0000	−6.0000	−6.0000
10	−3.6702	2.7447	8.9362	0.8936
100	−3.0537	2.0596	7.1550	0.0716
1,000	−3.0053	2.0058	7.0152	0.0068
10,000	−3.0005	2.0006	7.0015	0.0008
∞	−3.0000	2.0000	7.0000	0.0000

Now we turn our attention to constraint equations of the form used with bar elements

$$\beta_m u_m + \beta_n u_n = \beta_{mn} \tag{4.6.37}$$

where β_m, β_n and β_{mn} are known constants, and u_m and u_n are the mth and nth displacement degrees of freedom in the mesh, respectively. The functional that must be minimized subject to the constraint in (4.6.37) in this case is the total potential energy of the system (see Section 4.5.2)

$$\Pi = \frac{1}{2}\int_\Omega AE\mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{u}\, dx - \int_\Omega \mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} f\, dx - \mathbf{u}^{\mathrm{T}}\mathbf{Q} \tag{4.6.38}$$

The penalty functional is given by

$$\begin{aligned}\Pi_p = {} & \frac{1}{2}\int_\Omega AE\mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{u}\, dx - \int_\Omega \mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} f\, dx - \mathbf{u}^{\mathrm{T}}\mathbf{Q} \\ & + \frac{\gamma}{2}\left(\beta_m u_m + \beta_n u_n - \beta_{mn}\right)^2\end{aligned} \tag{4.6.39}$$

The functional Π_p attains a minimum only when $\beta_m u_m + \beta_n u_n - \beta_{mn}$ is very small, i.e., approximately satisfying the constraint (4.6.37). Setting $\delta\Pi_p = 0$ yields

$$\left(\mathbf{K} + \mathbf{K}_p\right)\mathbf{u} = \mathbf{f} + \mathbf{Q} + \mathbf{Q}_p \tag{4.6.40a}$$

where [see Eq. (4.5.4a)]

$$\mathbf{K} = \int_\Omega AE\mathbf{B}^{\mathrm{T}}\mathbf{B}\, dx, \quad \mathbf{K}_p = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \gamma\beta_m^2 & \cdots & \gamma\beta_m\beta_n & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \gamma\beta_m\beta_n & \cdots & \gamma\beta_n^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$\mathbf{f} = \int_\Omega \mathbf{B}^{\mathrm{T}} f\, dx, \quad \mathbf{Q}_p = \begin{Bmatrix} \cdots \\ \gamma\beta_{mn}\beta_m \\ \cdots \\ \gamma\beta_{mn}\beta_n \\ \cdots \end{Bmatrix} \tag{4.6.40b}$$

Thus, a modification of the stiffness and force coefficients associated with the constrained degrees of freedom will provide the desired solution to the constrained problem. As illustrated in Example 4.6.2, the value of the penalty parameter γ dictates the degree to which the constraint condition (4.6.37) is met. An analysis of the discrete problem shows that the following value of γ may be used:

$$\gamma = \max|K_{ij}| \times 10^4, \quad 1 \le i, j \le N \tag{4.6.41}$$

where N is the order of the global coefficient matrix. The reaction forces associated with the constrained displacement degrees of freedom are obtained from

$$F_{mp} = -\gamma\beta_m\left(\beta_m u_m + \beta_n u_n - \beta_{mn}\right), \quad F_{np} = -\gamma\beta_n\left(\beta_m u_m + \beta_n u_n - \beta_{mn}\right) \tag{4.6.42}$$

Because of the large magnitudes of the penalty terms, it is necessary to carry out computations in double precision (hand calculations do not give accurate results).

Example 4.6.3

Consider the structure shown in Fig. 4.6.6(a). The rigid bar ABE is supported by deformable bars AC and BD. Bar AC is made of aluminum ($E_a = 70$ GPa) and has cross-sectional area of $A_a = 500\ \text{mm}^2$; bar BD is made of steel ($E_s = 200$ GPa) and has a cross-sectional area of $A_s = 600\ \text{mm}^2$. The rigid bar carries loads of $F_1 = 10$ kN and $F_2 = 30$ kN at points O and E, respectively. We wish to determine the displacements of points A, B, and E, and the stresses in the aluminum and steel bars.

One may note that this is a statically determinate problem, i.e., the forces at points A and B can be readily determined from statics. Using the free-body diagram of the rigid bar ABE [see Fig. 4.6.6(b)], we obtain

$$F_{AC} + F_{BD} = F_1 + F_2, \quad 0.3F_{AC} + 0.5F_{BD} - 0.9F_2 = 0$$

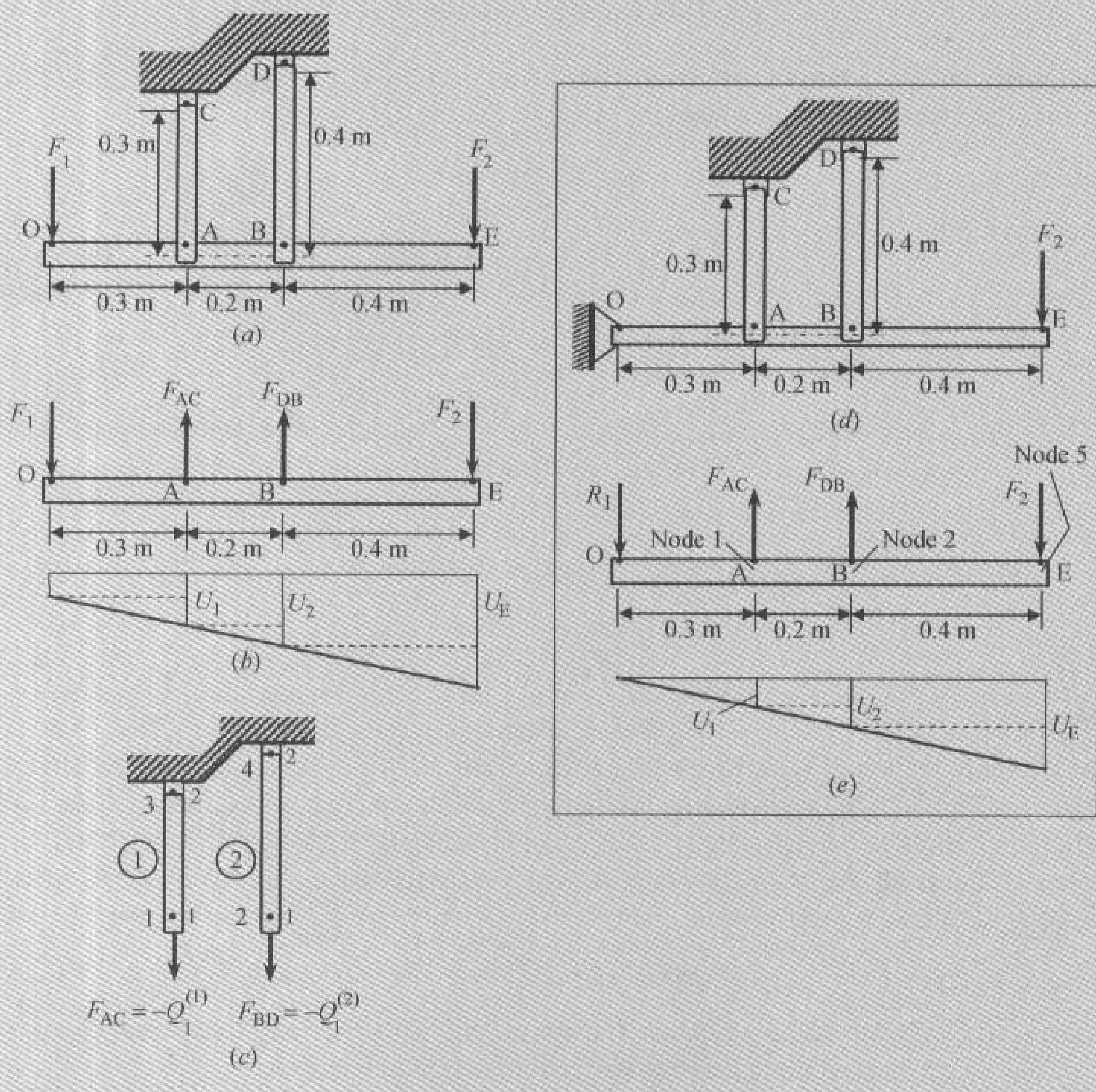


Figure 4.6.6 (a) Given structure. (b) Free-body diagram. (c) Finite element mesh. (d) Modified structure. (e) Free-body diagram of the modified structure.

which yield the values

$$F_{AC} = 2.5F_1 - 2F_2 = -35\text{kN}, \quad F_{BD} = -1.5F_1 + 3F_2 = 75\text{kN}$$

If we use two linear finite elements to represent the bars AC and BD, the assembled matrix of the structure is given by [see Fig. 4.6.6(c)]

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_2 & 0 & -k_2 \\ -k_1 & 0 & k_1 & 0 \\ 0 & -k_2 & 0 & k_2 \end{bmatrix} \end{array} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^1 \\ Q_2^2 \end{Bmatrix} \tag{4.6.43}$$

where

$$k_1 = \frac{E_a A_a}{h_1} = 116.6667 \times 10^6 \text{ N/m}, \quad k_2 = \frac{E_s A_s}{h_2} = 300 \times 10^6 \text{ N/m}$$

The boundary conditions of the problem are

$$U_3 = U_4 = 0; \quad Q_1^1 = -F_{AC} = 35 \text{ kN}, \quad Q_1^2 = -F_{BD} = -75 \text{ kN}$$

Hence, the condensed equations are given by

$$10^6 \begin{bmatrix} 116.6667 & 0 \\ 0 & 300 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 10^3 \begin{Bmatrix} 35 \\ -75 \end{Bmatrix}$$

whose solution is

$$U_1 = 0.30 \times 10^{-3} \text{ m} = 0.30 \text{ mm}, \quad U_2 = -0.25 \times 10^{-3} \text{ m} = -0.25 \text{ mm}$$

Using the principle of similar triangles, we can determine the displacement of point E. We have

$$\frac{U_E - U_1}{0.6} = \frac{U_2 - U_1}{0.2} \rightarrow U_E = 3U_2 - 2U_1 = -1.35 \text{ mm}$$

Thus, end A of the bar AC moves up by 0.3 mm, end B of BD moves down by 0.25 mm, and point E moves down by 1.35 mm. The stresses in bars AC and BD are

$$\sigma_{AC} = \frac{F_{AC}}{A_a} = -\frac{35}{500 \times 10^{-6}} = -70 \text{ MPa}, \quad \sigma_{BD} = \frac{F_{BD}}{A_s} = \frac{75}{600 \times 10^{-6}} = 125 \text{ MPa}$$

Next, consider the case in which point O is pin connected to a fixed, immovable part, as shown in Fig. 4.6.6(d). Then the problem becomes a statically indeterminate one. Of course, the finite element method can still be used to solve the problem. The assembled equations (4.6.43) are still valid for this case. However, forces Q_1^1 and Q_1^2 are not known (because we cannot solve for F_{AC} and F_{BD}). In addition, points A and B are constrained to move as the rigid member ABE is rotated about point O. This geometric constraint is equivalent to the following conditions among the displacements U_1, U_2, and U_5:

$$\frac{U_1}{0.3} = \frac{U_5}{0.9} \rightarrow 3U_1 - U_5 = 0, \quad \frac{U_2}{0.5} = \frac{U_5}{0.9} \rightarrow 1.8U_2 - U_5 = 0 \tag{4.6.44}$$

These constraints bring in an additional degree of freedom, namely U_5, into the equations. Hence, the assembled equations before including the constraint conditions are

$$10^6\begin{bmatrix} 116.67 & 0 & -116.67 & 0 & 0 \\ 0 & 300 & 0 & -300 & 0 \\ -116.67 & 0 & 116.67 & 0 & 0 \\ 0 & -300 & 0 & 300 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^1 \\ Q_2^2 \\ F_2 \end{Bmatrix} \tag{4.6.45}$$

The last row and column of the above equation are added to facilitate the addition of penalty terms. Without the addition of the penalty contributions the last equation is nonsensical.

Using the procedure developed in this section, we can include the constraints, $3U_1 - U_5 = 0$ and $1.8U_2 - U_5 = 0$ into assembled equations (4.6.43). The value of the penalty parameter is selected to be $\gamma = (300 \times 10^6)10^4$. The stiffness additions due to the two constraints are ($\beta_1 = 3$, $\beta_2 = 1.8$, $\beta_5 = -1$, and $\beta_{15} = \beta_{25} = 0$)

$$\begin{array}{c} \\ 1 \\ 5 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 5 \end{array} \\ \begin{bmatrix} (3)^2\gamma & 3(-1)\gamma \\ (-1)3\gamma & (-1)^2\gamma \end{bmatrix} \end{array} = 10^{10}\begin{bmatrix} 2700.00 & -900.00 \\ -900.00 & 300.00 \end{bmatrix}$$

$$\begin{array}{c} \\ 2 \\ 5 \end{array}\begin{array}{c} \begin{array}{cc} 2 & 5 \end{array} \\ \begin{bmatrix} (1.8)^2\gamma & (-1)1.8\gamma \\ (-1)1.8\gamma & (-1)^2\gamma \end{bmatrix} \end{array} = 10^{10}\begin{bmatrix} 972.00 & -540.00 \\ -540.00 & 300.00 \end{bmatrix} \tag{4.6.46}$$

The force additions are zero on account of $\beta_{15} = \beta_{25} = 0$. Hence, the modified finite element equations become

$$10^6\begin{bmatrix} 27{,}000{,}117 & 0 & -117 & 0 & -9{,}000{,}000 \\ 0 & 9{,}720{,}300 & 0 & -300 & -5{,}400{,}000 \\ -116.67 & 0 & 117 & 0 & 0 \\ 0 & -300 & 0 & 300 & 0 \\ -9{,}000{,}000 & -5{,}400{,}000 & 0 & 0 & 6{,}000{,}000 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^1 \\ Q_2^2 \\ F_2 \end{Bmatrix} \tag{4.6.47}$$

The condensed equations are

$$10^6\begin{bmatrix} 27{,}000{,}117 & 0 & -9{,}000{,}000 \\ 0 & 9{,}720{,}300 & -5{,}400{,}000 \\ -9{,}000{,}000 & -5{,}400{,}000 & 6{,}000{,}000 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 30 \times 10^3 \end{Bmatrix} \tag{4.6.48}$$

whose solution (using an equation solver) is (deflections are in the direction of the force)

$$U_1 = 0.09474 \text{ (mm)}, \quad U_2 = 0.15789 \text{ (mm)}, \quad U_5 = 0.28422 \text{ (mm)} \tag{4.6.49a}$$

The forces in the bars AC and BD can be calculated using Eq. (4.6.45)

$$\begin{Bmatrix} Q_1^1 \\ Q_1^2 \end{Bmatrix} = 10^6 \begin{bmatrix} 116.6667 & 0 \\ 0 & 300 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 10^3 \begin{Bmatrix} 11.053 \\ 47.367 \end{Bmatrix} \text{N}$$

The stresses are $\sigma_{AC} = 22.11$ MPa and $\sigma_{BD} = 79$ MPa. Alternatively, from Eq. (4.6.42) we have (one must include significant number of decimal points in the computation)

$$\begin{aligned} (Q_1^1)_p &= -900 \times 10^7 \left(3 \times 0.094739 \times 10^{-3} - 0.28422 \times 10^{-3}\right) = 11.053 \text{ kN} \\ (Q_1^2)_p &= -540 \times 10^7 \left(1.8 \times 0.15789 \times 10^{-3} - 0.28422 \times 10^{-3}\right) = 47.368 \text{ kN} \end{aligned} \tag{4.6.49b}$$

Next, we consider a plane truss with an inclined support. The penalty approach is used to include the constraint condition among the displacement components at the support.

Example 4.6.4

Consider the truss shown in Fig. 4.6.7(a). We wish to determine the unknown displacements of nodes 2 and 3 and the reactions associated with these displacements.

The element stiffness matrices are

$$[K^1] = 10^9 \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.126 & 0.000 & -0.126 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & -0.126 & 0.000 & 0.126 \end{bmatrix}$$

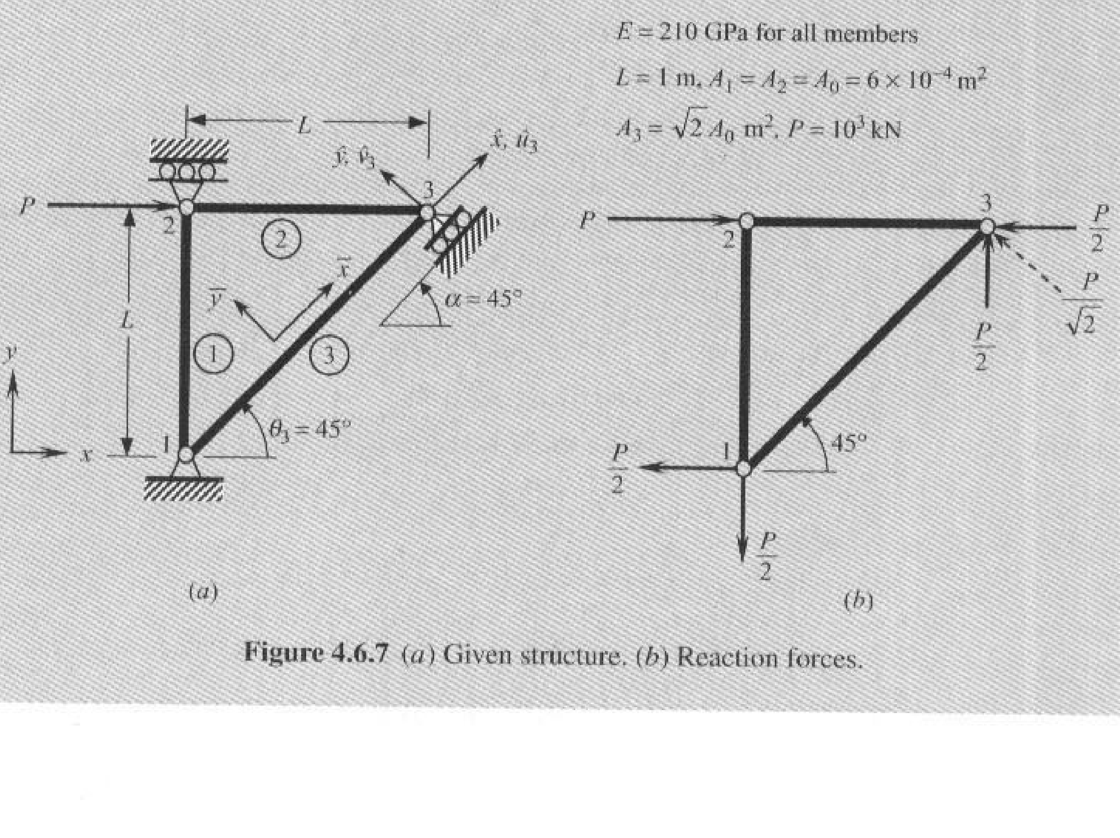


Figure 4.6.7 (a) Given structure. (b) Reaction forces.

$$[K^2] = 10^9 \begin{bmatrix} 0.126 & 0.000 & -0.126 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ -0.126 & 0.000 & 0.126 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$

$$[K^3] = 0.63 \times 10^8 \begin{bmatrix} 1.0 & 1.0 & -1.0 & -1.0 \\ 1.0 & 1.0 & -1.0 & -1.0 \\ -1.0 & -1.0 & 1.0 & 1.0 \\ -1.0 & -1.0 & 1.0 & 1.0 \end{bmatrix}$$

The assembled equations before including the constraint conditions are

$$10^8 \begin{bmatrix} 0.63 & 0.63 & 0.00 & 0.00 & -0.63 & -0.63 \\ 0.63 & 1.89 & 0.00 & -1.26 & -0.63 & -0.63 \\ 0.00 & 0.00 & 1.26 & 0.00 & -1.26 & 0.00 \\ 0.00 & -1.26 & 0.00 & 1.26 & 0.00 & 0.00 \\ 0.00 & -0.63 & -1.26 & 0.00 & 1.89 & 0.63 \\ -0.63 & -0.63 & 0.00 & 0.00 & 0.63 & 0.63 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 + Q_1^3 \\ Q_2^1 + Q_2^3 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 + Q_3^3 \\ Q_4^2 + Q_4^3 \end{Bmatrix} \tag{4.6.50}$$

The constraint condition at node 3 is

$$u_n \equiv -u \sin\alpha + v \cos\alpha = 0 \rightarrow -0.7071u + 0.7071v = 0$$

Comparing this constraint equation to the general constraint equation (4.6.37), we find that $\beta_1 = -0.7071$, $\beta_2 = 0.7071$, and $\beta_{12} = 0$.

The value of the penalty parameter is selected to be $\gamma = (1.89 \times 10^8)10^4$. The stiffness additions due to the constraint are

$$\begin{array}{cc} & \begin{array}{cc} \quad 5 & \qquad\qquad 6 \end{array} \\ \begin{array}{c} 5 \\ 6 \end{array} & \gamma \begin{bmatrix} (-0.7071)^2 & -(0.7071)^2 \\ -(0.7071)^2 & (0.7071)^2 \end{bmatrix} \end{array} = 1.89 \times 10^{12} \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

The assembled equations after including the constraint conditions are (the numbers shown are truncated but more accurate numbers are used in actual computation in a computer)

$$10^8 \begin{bmatrix} 0.63 & 0.63 & 0.00 & 0.00 & -0.63 & -0.63 \\ 0.63 & 1.89 & 0.00 & -1.26 & -0.63 & -0.63 \\ 0.00 & 0.00 & 1.26 & 0.00 & -1.26 & 0.00 \\ 0.00 & -1.26 & 0.00 & 1.26 & 0.00 & 0.00 \\ 0.00 & -0.63 & -1.26 & 0.00 & 6301.8 & 6299.2 \\ -0.63 & -0.63 & 0.00 & 0.00 & 6299.2 & 6300.5 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 + Q_1^3 \\ Q_2^1 + Q_2^3 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 + Q_3^3 \\ Q_4^2 + Q_4^3 \end{Bmatrix}$$

Imposing the boundary and force equilibrium conditions $U_1 = U_2 = U_4 = 0$ and $Q_3^1 + Q_1^2 = P$, we obtain the condensed equations

$$10^8 \begin{bmatrix} 1.26 & -1.26 & 0.00 \\ -1.26 & 6301.8 & 6299.2 \\ 0.00 & 6299.2 & 6300.5 \end{bmatrix} \begin{Bmatrix} U_3 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \end{Bmatrix}$$

The solution to these equations (as computed in double precision in a computer) is

$$U_3 = 11.905 \times 10^{-3}\ \text{m},\quad U_5 = 3.9688 \times 10^{-3}\ \text{m},\quad U_6 = 3.9680 \times 10^{-3}\ \text{m}$$

and the reactions at node 3, as computed using (4.6.50) are

$$F_{3x} = -500\ \text{kN},\quad F_{3y} = 500\ \text{kN}$$

The reactions of the whole structure are shown in Fig. 4.6.7(*b*).

4.6.5 Constraint Equations: A Direct Approach

Here, we present an exact method by which constraint equations of the type in (4.6.32*a*) and (4.6.32*b*) can be included in the assembled equations for the unknowns. The method involves expressing the global displacement degrees of freedom at the node with a constraint in terms of the local displacement degrees of freedom so that the boundary conditions can be readily imposed.

Recall from Eqs. (4.6.2) and (4.6.3*a*) that the displacements $(\hat{u}, \hat{v})$, referred to the local coordinate system $(\hat{x}, \hat{y})$ at a point, are related to the displacements (u, v), referred to the global coordinates system (x, y), by

$$\begin{Bmatrix} \hat{u} \\ \hat{v} \end{Bmatrix}_c = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}_c \quad \rightarrow \quad \hat{\mathbf{u}}_c = \mathbf{A}\mathbf{u}_c \tag{4.6.51a}$$

and the inverse relation is given by

$$\mathbf{u}_c = \mathbf{A}^{\mathrm{T}}\hat{\mathbf{u}}_c, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \tag{4.6.51b}$$

where the subscript "c" refers to the constrained degrees of freedom. Since we wish to express the global displacements at a given node in terms of the local displacements at a specific node, we construct the transformation of the whole (global) system as

$$[T] = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [A] & [0] \\ [0] & [0] & [I] \end{bmatrix} \tag{4.6.52}$$

Thus, all displacement degrees of freedom that are not constrained are unaffected and only global displacements that are constrained are transformed to the local displacements. Further, note that the transformation matrix $[A]$ is placed in $[T]$ in such a way that only the constrained degrees of freedom are transformed. We have

$$\begin{Bmatrix} \Delta^1 \\ \mathbf{u}_c \\ \Delta^2 \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \Delta^1 \\ \hat{\mathbf{u}} \\ \Delta^2 \end{Bmatrix} \quad \text{or } \boldsymbol{\Delta} = \mathbf{T}^{\mathrm{T}}\hat{\boldsymbol{\Delta}} \tag{4.6.53}$$

where $\mathbf{\Delta}^1$ and $\mathbf{\Delta}^2$ denote vectors of the global displacement components ahead and behind (in terms of numbering) the constrained displacement degrees of freedom $\mathbf{u}_c$ in the mesh. We also note that the transformed displacement vector $\hat{\mathbf{\Delta}}$ contains the global displacement vectors $\mathbf{\Delta}^1$ and $\mathbf{\Delta}^2$ and the local (constrained) displacement vector $\hat{\mathbf{u}}_c$.

The remaining steps of the procedure are the same as that described in Sections 4.6.2 and 4.6.3. Thus, we obtain

$$\hat{\mathbf{K}}\hat{\mathbf{\Delta}} = \hat{\mathbf{F}} \tag{4.6.54}$$

where the transformed global stiffness matrix $\hat{\mathbf{K}}$ and global force vector $\hat{\mathbf{F}}$ are known in terms of the assembled global stiffness matrix $\mathbf{K}$ and force vector $\mathbf{F}$ as [see Eq. (4.6.7)]

$$\hat{\mathbf{K}} = \mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T}, \quad \hat{\mathbf{F}} = \mathbf{T}^{\mathrm{T}}\mathbf{F} \tag{4.6.55}$$

Since the constrained displacements are a part of the global system of equations, we may impose the boundary conditions on them directly (such as $\hat{u}_n = 0$ at an inclined roller support, where n denotes the coordinate normal to the roller).

In summary, we may introduce a transformation of the displacements that facilitates the imposition of boundary conditions or inclusion of constraints on the displacements. Once the transformation $\mathbf{T}$ is identified, we may use Eq. (4.6.7) or (4.6.55) to obatin the modified equations that have the desired effect. We revisit the problems of Examples 4.6.3 and 4.6.4 to illustrate the ideas described here.

Example 4.6.5

As an example to illustrate the direct approach, consider the structure shown in Fig. 4.6.6(*d*). The assembled equations are

$$10^3\begin{bmatrix} 116.67 & 0 & -116.67 & 0 \\ 0 & 300.00 & 0 & -300.00 \\ -116.67 & 0 & 116.67 & 0 \\ 0 & -300.00 & 0 & 300.00 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^1 \\ Q_2^2 \end{Bmatrix} \tag{4.6.56}$$

The geometric constraint between the displacements U_1 and U_2 is

$$\frac{U_1}{0.3} = \frac{U_2}{0.5} \rightarrow U_1 = 0.6U_2 \tag{4.6.57}$$

In this case, we can introduce a transformation $\mathbf{T}$ between (U_1, U_2, U_3, U_4) and (U_2, U_3, U_4) (i.e., eliminate U_1 by means of the constraint equation $U_1 = 0.6U_2$) as

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{bmatrix} 0.6 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} \quad \text{or } \mathbf{\Delta} = \mathbf{T}\hat{\mathbf{\Delta}}$$

Hence, the transformed equations are

$$\hat{\mathbf{K}}\hat{\boldsymbol{\Delta}} = \hat{\mathbf{F}}, \quad \boldsymbol{\Delta} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}, \quad \hat{\boldsymbol{\Delta}} = \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix}$$

or, in explicit form,

$$10^3 \begin{bmatrix} 342.0 & -70.0 & -300.0 \\ -70.0 & 116.67 & 0 \\ -300.0 & 0 & 300.0 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0.6Q_1^1 + Q_1^2 \\ Q_2^1 \\ Q_2^2 \end{Bmatrix}$$

From the free-body diagram of the bar OABE, we find that $0.6F_{AC} + F_{BD} = 1.8F_2$; therefore, we have $0.6Q_1^1 + Q_1^2 = 1.8F_2 = 54$ kN and the condensed equation for the unknown U_2 is $(U_3 = U_4 = 0.0)$

$$342U_2 = 54, \quad U_2 = 0.15789 \text{ (mm)}, \quad U_1 = 0.6U_2 = 0.09474 \text{ (mm)}$$

The forces in the bars AC and BD are calculated using Eq. (4.6.56) $(U_3 = U_4 = 0.0)$

$$\begin{Bmatrix} Q_1^1 \\ Q_1^2 \end{Bmatrix} = 10^3 \begin{bmatrix} 116.6667 & 0 \\ 0 & 300 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 10^3 \begin{Bmatrix} 11.053 \\ 47.368 \end{Bmatrix}$$

The stresses are $\sigma_{AC} = 22.11$ MPa and $\sigma_{BD} = 79$ MPa.

Example 4.6.6

Here we consider the truss in Figure 4.6.7(a). The assembled global equations are given by Eq. (4.6.50). The transformation between the global degrees of freedom ($U_1 = u_1$, $U_2 = v_1$, $U_3 = u_2$, $U_4 = v_2$, $U_5 = u_3$, $U_6 = v_3$) and ($U_1 = u_1$, $U_2 = v_1$, $U_3 = u_2$, $U_4 = v_2$, $\hat{u}_3 = u_t$, $\hat{v}_3 = u_n$) is given by [i.e., Eq. (4.6.53) for the problem at hand takes the form]

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & 0 & 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ \hat{u}_3 \\ \hat{v}_3 \end{Bmatrix} \tag{4.6.58a}$$

where $\alpha = 45^\circ$. Using Eq. (4.6.55), we obtain the transformed equations

$$10^8 \begin{bmatrix} 0.630 & 0.630 & 0.000 & 0.00 & -0.891 & 0.000 \\ 0.630 & 1.890 & 0.000 & -1.26 & -0.891 & 0.000 \\ 0.000 & 0.000 & 1.260 & 0.00 & -0.891 & 0.891 \\ 0.000 & -1.260 & 0.000 & 1.26 & 0.000 & 0.000 \\ -0.891 & -0.891 & -0.891 & 0.00 & 1.890 & -0.630 \\ 0.000 & 0.000 & 0.891 & 0.00 & -0.630 & 0.630 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ \hat{u}_3 \\ \hat{v}_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 + Q_1^3 \\ Q_2^1 + Q_2^3 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ \hat{F}_{3t} \\ \hat{F}_{3n} \end{Bmatrix} \tag{4.6.58b}$$

where $\hat{F}_{3t}$ and $\hat{F}_{3n}$ are the reaction forces at node 3 in the tangential and normal directions, respectively.

Now applying the boundary conditions

$$U_1 = U_2 = U_4 = \hat{v}_3 = 0, \quad Q_3^1 + Q_1^2 = P = 10^6, \quad \hat{F}_{3t} = 0$$

we obtain the following condensed equations:

$$10^8 \begin{bmatrix} 1.26000 & -0.89095 \\ -0.89095 & 1.88990 \end{bmatrix} \begin{Bmatrix} U_3 \\ \hat{u}_3 \end{Bmatrix} = \begin{Bmatrix} 10^6 \\ 0 \end{Bmatrix} \tag{4.6.59}$$

The x component of displacement at node 2 and the tangential displacement $\hat{u}_3$ of node 3 are

$$U_3 = 11.905 \times 10^{-3}\ \text{m}, \quad \hat{u}_3 = 5.6122 \times 10^{-3}\ \text{m} \tag{4.6.60a}$$

The unknown reactions can be calculated from Eq. (4.6.58b) as

$$F_{1x} = -500\ \text{kN}, \quad F_{1y} = -500\ \text{kN}, \quad F_{2y} = 0\ \text{kN}, \quad \hat{F}_{3n} = 707.1\ \text{kN} \tag{4.6.60b}$$

These results are the same as those obtained by the penalty method in Example 4.6.4.

4.7 SUMMARY

In this chapter, finite element models of discrete systems have been developed and applications of finite element models to the solution of problems of heat transfer, fluid mechanics, and solid mechanics have been presented. To aid the reader, a brief review of the basic terminology and governing equations of each of the three fields has also been given. Analysis of plane trusses and inclusion of constraint relations between primary variables by two different methods is also discussed.

It has been shown that the secondary variables of a problem can be computed using either the global algebraic equations of the finite element mesh (i.e., condensed equations for the secondary variables) or by their original definition through finite element interpolation.

The former method gives more accurate results, which will satisfy the equilibrium at interelement nodes, whereas the latter gives less accurate results that are discontinuous at the nodes. The secondary variables computed using the linear elements are elementwise constant, while they are elementwise linear for the Lagrange quadratic elements. The magnitude of discontinuity can be reduced by refining the mesh (h or p refinement). The discontinuity of the secondary variables at nodes is due to the fact that the secondary variables are not made continuous across the elements.

In closing this chapter, the reader is reminded that the finite element method is a powerful tool for engineering analysis. The power of the method lies in transforming the traditional variational methods (e.g., Ritz, Galerkin, least-squares, and other weighted-residual methods) into a powerful computer-based technique by developing suitable approximation functions for complex problems. Without a good understanding of the method as well as the engineering background behind each problem, one is ill-equipped to analyze the problem.

PROBLEMS

Many of the following problems are designed for hand calculation while some are intended specifically for computer calculations using the program FEM1D (see Chapter 7 for details on how to use the program). The problem set should give the student deeper understanding of what is involved in the setting up of the finite element equations, imposition of boundary conditions, and identifying the condensed equations for the unknown primary and secondary variables of a given problem. When the number of equations to be solved is greater than three, the student should opt for a computer solution of the equations. The calculations can be verified, in most cases, by solving the same problem using FEM1D.

Discrete Elements

4.1 Consider the system of linear elastic springs shown in Fig. P4.1. Assemble the element equations to obtain the force-displacement relations for the entire system. Use the boundary conditions to write the condensed equations for the unknown displacements and forces.

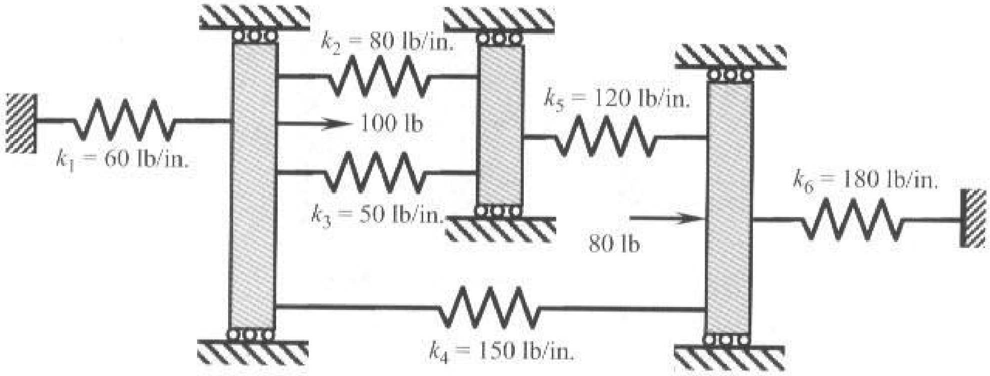


Figure P4.1

4.2 Repeat Problem 4.1 for the system of linear springs shown in Fig. P4.2.

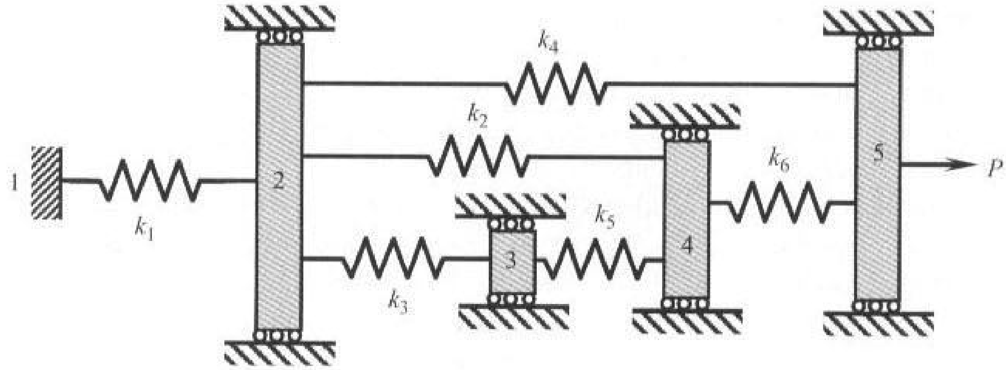


Figure P4.2

4.3 Consider the direct current electric network shown in Fig. P4.3. We wish to determine the voltages V and currents I in the network using the finite element method. Set up the algebraic equations (i.e., condensed equations) for the unknown voltages and currents.

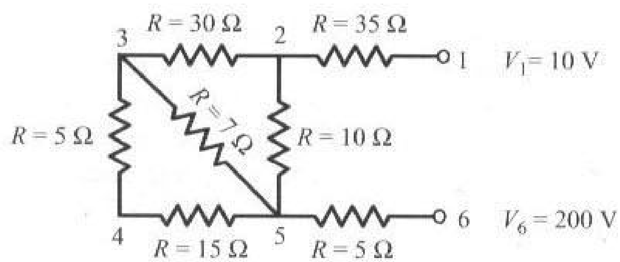


Figure P4.3

4.4 Repeat Problem 4.3 for the direct current electric network shown in Fig. P4.4.

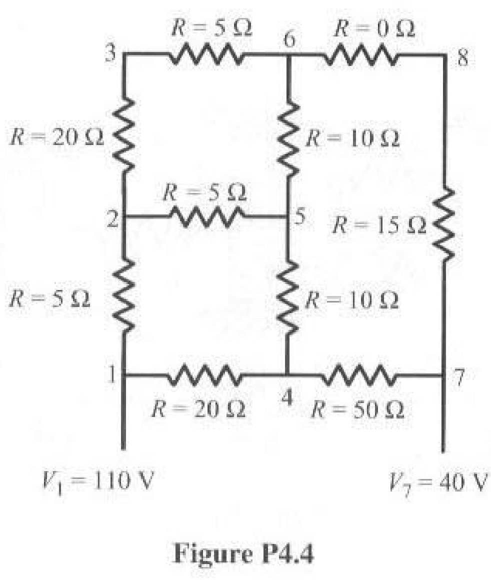


Figure P4.4

4.5 Write the condensed equations for the unknown pressures and flows (use the minimum number of elements) for the hydraulic pipe network shown in Fig. P4.5. *Answer:* $P_1 = \frac{39}{14}Qa$, $P_2 = \frac{12}{7}Qa$, and $P_3 = \frac{15}{14}Qa$.

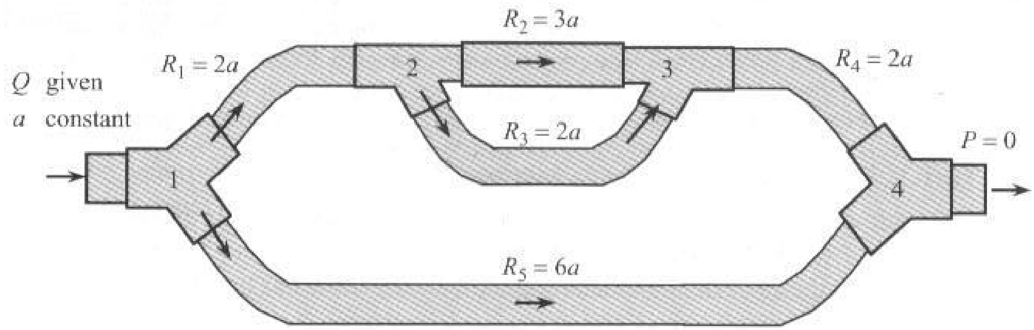


Figure P4.5

4.6 Consider the hydraulic pipe network (the flow is assumed to be laminar) shown in Fig. P4.6. Write the condensed equations for the unknown pressures and flows (use minimum number of elements.)

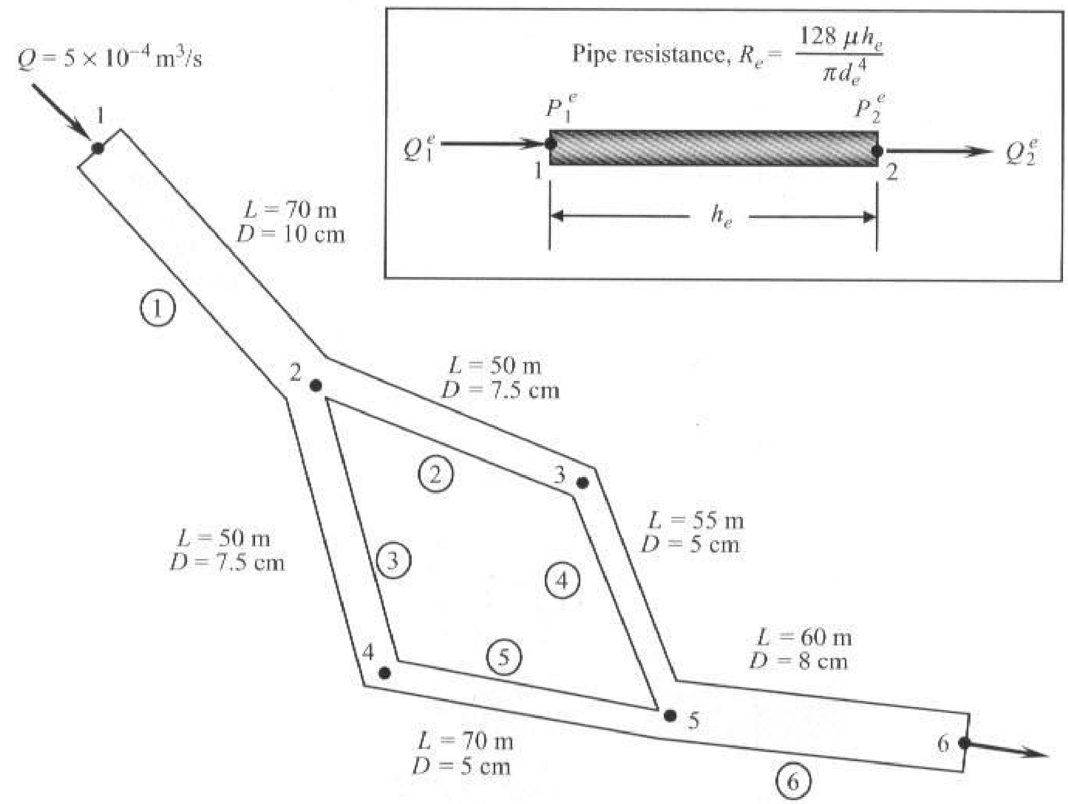


Figure P4.6

4.7 Determine the maximum shear stresses in the solid steel ($G_s = 12$ Msi) and aluminum ($G_a = 4$ Msi) shafts shown in Fig. P4.7.

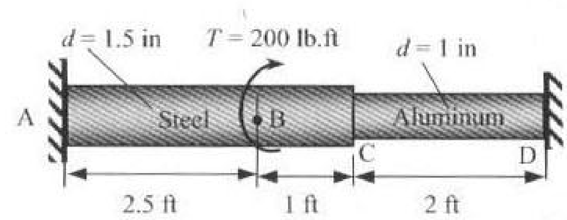


Figure P4.7

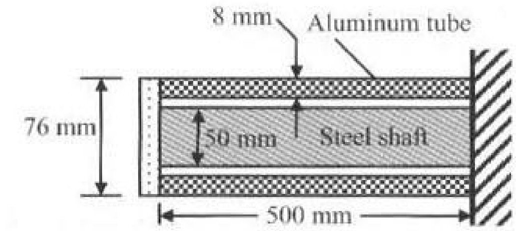


Figure P4.8

4.8 A steel shaft and an aluminum tube are connected to a fixed support and to a rigid disk, as shown in Fig. P4.8. If the torque applied at the end is equal to $T = 6,325$ N·m, determine the shear stresses in the steel shaft and aluminum tube. Use $G_s = 77$ GPa and $G_a = 27$ GPa.

Heat Transfer

4.9 Consider heat transfer in a plane wall of total thickness L. The left surface is maintained at temperature T_0 and the right surface is exposed to ambient temperature T_∞ with heat transfer coefficient β. Determine the temperature distribution in the wall and heat input at the left surface of the wall for the following data: $L = 0.1$ m, $k = 0.01$ W/(m · °C), $\beta = 25$ W/(m^2 · °C), $T_0 = 50$°C, and $T_\infty = 5$°C. Solve for nodal temperatures and the heat at the left wall using (*a*) two linear finite elements and (*b*) one quadratic element. *Answer:* (*a*) $U_2 = 27.59$°C, $U_3 = 5.179$°C, $Q_1^1 = 4.482$ W/m^2 $= -Q_2^2$.

4.10 An insulating wall is constructed of three homogeneous layers with conductivities k_1, k_2, and k_3 in intimate contact (see Fig. P4.10). Under steady-state conditions, the temperatures of the media in contact at the left and right surfaces of the wall are at ambient temperatures of T_∞^L and T_∞^R, respectively, and film coefficients β_L and β_R, respectively. Determine the temperatures on the left and right surfaces as well as at the interfaces. Assume that there is no internal heat generation and that the heat flow is one-dimensional ($\partial T/\partial y = 0$). *Answer:* $U_1 = 61.582$°C, $U_2 = 61.198$°C, $U_3 = 60.749$°C, $U_4 = 60.612$°C.

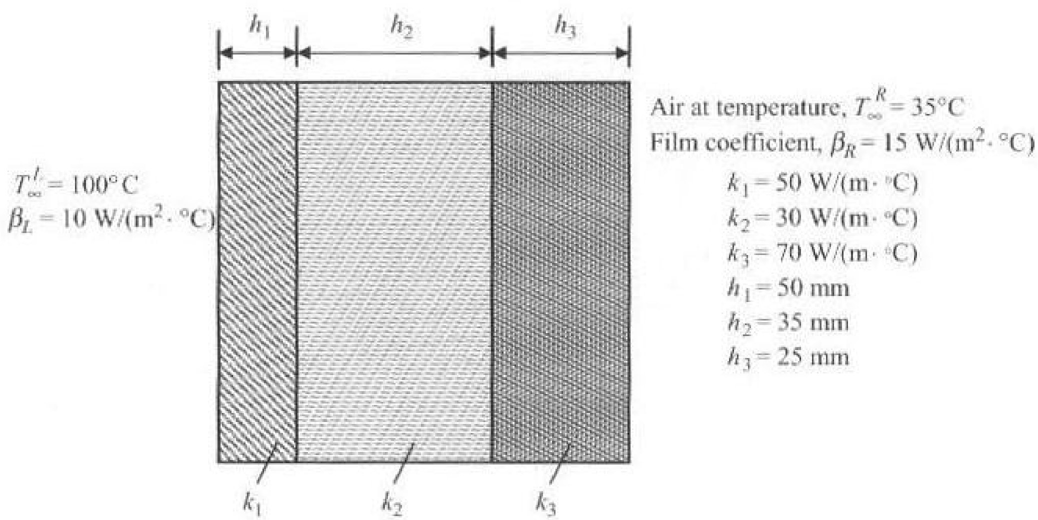


Figure P4.10

4.11 Rectangular fins are used to remove heat from the surface of a body by conduction along the fins and convection from the surface of the fins into the surroundings. The fins are 100 mm long, 5 mm wide, and 1 mm thick, and made of aluminum with thermal conductivity $k = 170\ \text{W/(m}\cdot{}^\circ\text{C)}$. The natural convection heat transfer coefficient associated with the surrounding air is $\beta = 35\ \text{W/(m}^2\cdot{}^\circ\text{C)}$ and the ambient temperature is $T_\infty = 20^\circ$C. Assuming that the heat transfer is one dimensional along the length of the fins and that the heat transfer in each fin is independent of the others, determine the temperature distribution along the fins and the heat removed from each fin by convection. Use (*a*) four linear elements, and (*b*) two quadratic elements.

4.12 Find the heat transfer per unit area through the composite wall shown in Fig. P4.12. Assume one-dimensional heat flow.

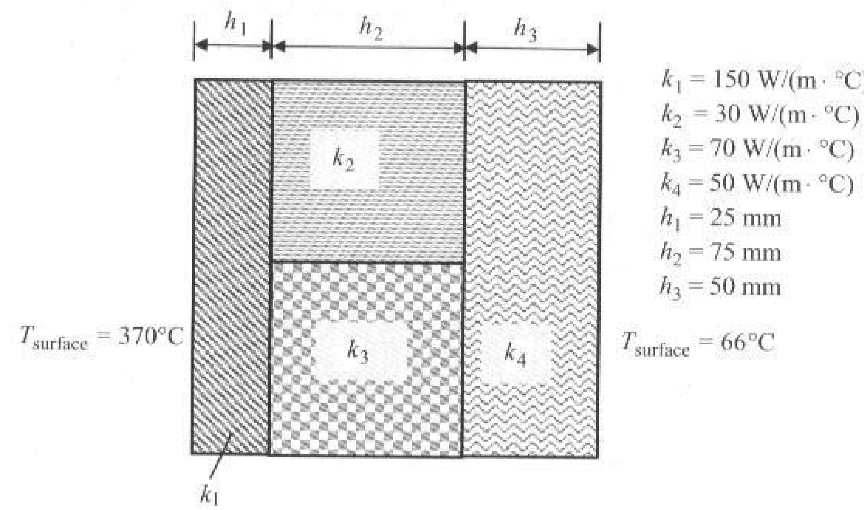


Figure P4.12

4.13 A steel rod of diameter $D = 2$ cm, length $L = 5$ cm, and thermal conductivity $k = 50\ \text{W/(m}\cdot{}^\circ\text{C)}$ is exposed to ambient air at $T_\infty = 20^\circ$C with a heat transfer coefficient $\beta = 100\ \text{W/(m}^2\cdot{}^\circ\text{C)}$. If the left end of the rod is maintained at temperature $T_0 = 320^\circ$C, determine the temperatures at distances 25 mm and 50 mm from the left end, and the heat at the left end. The governing equation of the problem is

$$-\frac{d^2\theta}{dx^2} + m^2\theta = 0 \quad \text{for} \quad 0 < x < L$$

where $\theta = T - T_\infty$, T is the temperature, and $m^2 = \beta P/Ak$. The boundary conditions are

$$\theta(0) = T(0) - T_\infty = 300^\circ\text{C}, \qquad \left(\frac{d\theta}{dx} + \frac{\beta}{k}\theta\right)\bigg|_{x=L} = 0$$

Use (*a*) two linear elements and (*b*) one quadratic element to solve the problem by the finite element method. Compare the finite element nodal temperatures against the exact values. *Answer:* (*a*) $U_1 = 300^\circ$C, $U_2 = 211.97^\circ$C, $U_3 = 179.24^\circ$C, $Q_1^1 = 3{,}521.1\ \text{W/m}^2$. (*b*) $U_1 = 300^\circ$C, $U_2 = 213.07^\circ$C, $U_3 = 180.77^\circ$C, $Q_1^1 = 4{,}569.9\ \text{W/m}^2$.

4.14 Find the temperature distribution in the tapered fin shown in Fig. P4.14. Assume that the temperature at the root of the fin is 250°F, the conductivity $k = 120\ \text{Btu/(h}\cdot\text{ft}\cdot{}^\circ\text{F)}$, and the film coefficient $\beta = 15\ \text{Btu/(h}\cdot\text{ft}^2\cdot{}^\circ\text{F)}$; use three linear elements. The ambient temperature at the top and bottom of the fin is $T_\infty = 75^\circ$F. *Answer:* $T_1(\text{tip}) = 166.23^\circ$F, $T_2 = 191.1^\circ$F, $T_3 = 218.89^\circ$F.

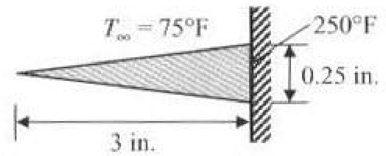


Figure P4.14

4.15 Consider steady heat conduction in a wire of circular cross section with an electrical heat source. Suppose that the radius of the wire is R_0, its electrical conductivity is K_e (Ω^{-1}/cm), and it is carrying an electric current density of I (A/cm^2). During the transmission of an electric current, some of the electrical energy is converted into thermal energy. The rate of heat production per unit volume is given by $q_e = I^2/K_e$. Assume that the temperature rise in the wire is sufficiently small that the dependence of the thermal or electric conductivity on temperature can be neglected. The governing equations of the problem are

$$-\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right) = q_e \quad \text{for} \quad 0 \leq r \leq R_0, \quad \left(rk\frac{dT}{dr}\right)\bigg|_{r=0} = 0, \quad T(R_0) = T_0$$

Determine the distribution of temperature in the wire using (a) two linear elements and (b) one quadratic element, and compare the finite element solution at eight equal intervals with the exact solution

$$T(r) = T_0 + \frac{q_e R_0^2}{4k}\left[1 - \left(\frac{r}{R_0}\right)\right]$$

Also, determine the heat flow, $Q = -2\pi R_0 k (dT/dr)|_{R_0}$, at the surface using (i) the temperature field and (ii) the balance equations.

4.16 Consider a nuclear fuel element of spherical form, consisting of a sphere of "fissionable" material surrounded by a spherical shell of aluminum "cladding" as shown in Fig. P4.16. Nuclear fission is a source of thermal energy, which varies nonuniformly from the center of the sphere to the interface of the fuel element and the cladding. We wish to determine the temperature distribution in the nuclear fuel element and the aluminum cladding.

The governing equations for the two regions are the same, with the exception that there is no heat source term for the aluminum cladding. We have

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2 k_1 \frac{dT_1}{dr}\right) = q \quad \text{for} \quad 0 \leq r \leq R_F$$

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2 k_2 \frac{dT_2}{dr}\right) = 0 \quad \text{for} \quad R_F \leq r \leq R_C$$

where subscripts 1 and 2 refer to the nuclear fuel element and cladding, respectively. The heat generation in the nuclear fuel element is assumed to be of the form

$$q_1 = q_0\left[1 + c\left(\frac{r}{R_F}\right)^2\right]$$

where q_0 and c are constants depending on the nuclear material. The boundary conditions are

$$kr^2\frac{dT_1}{dr} = 0 \quad \text{at} \quad r = 0$$

$$T_1 = T_2 \quad \text{at} \quad r = R_F, \quad \text{and} \quad T_2 = T_0 \quad \text{at} \quad r = R_C$$

Use two linear elements to determine the finite element solution for the temperature distribution, and compare the nodal temperatures with the exact solution

$$T_1 - T_0 = \frac{q_0 R_F^2}{6k_1}\left\{\left[1-\left(\frac{r}{R_F}\right)^2\right] + \frac{3}{10}c\left[1-\left(\frac{r}{R_F}\right)^4\right]\right\} + \frac{q_0 R_F^2}{3k_2}\left(1+\frac{3}{5}c\right)\left(1-\frac{R_F}{R_C}\right)$$

$$T_2 - T_0 = \frac{q_0 R_F^2}{3k_2}\left(1+\frac{3}{5}c\right)\left(\frac{R_F}{r}-\frac{R_F}{R_C}\right)$$

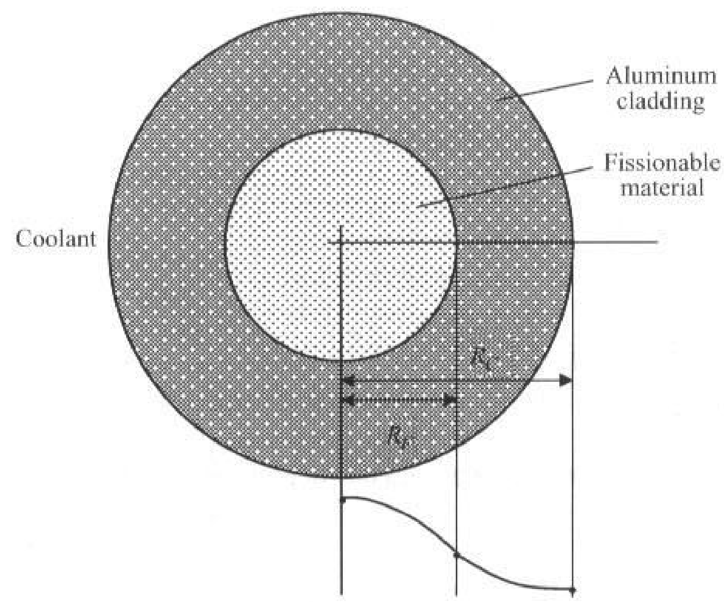


Figure P4.16

Fluid Mechanics

4.17 Consider the flow of a Newtonian viscous fluid on an inclined flat surface, as shown in Fig. P4.17. Examples of such flow can be found in wetted-wall towers and the application of coatings to wallpaper rolls. The momentum equation, for a fully developed steady laminar flow along the z coordinate, is given by

$$-\mu\frac{d^2w}{dx^2} = \rho g \cos\beta$$

where w is the z component of the velocity, μ is the viscosity of the fluid, ρ is the density, g is the acceleration due to gravity, and β is the angle between the inclined surface and the vertical.

The boundary conditions associated with the problem are that the shear stress is zero at $x = 0$ and the velocity is zero at $x = L$:

$$\left(\frac{dw}{dx}\right)\bigg|_{x=0} = 0, \quad w(L) = 0$$

Use (a) two linear finite elements of equal length and (b) one quadratic finite element in the domain $(0, L)$ to solve the problem and compare the two finite element solutions at four points

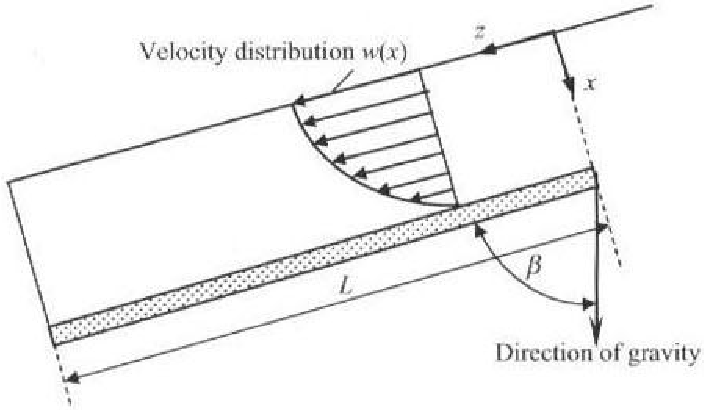


Figure P4.17

$x=0$, $\frac{1}{4}L$, $\frac{1}{2}L$, and $\frac{3}{4}L$ of the domain with the exact solution

$$w_e = \frac{\rho g L^2 \cos\beta}{2\mu}\left[1-\left(\frac{x}{L}\right)^2\right]$$

Evaluate the shear stress ($\tau_{xz} = -\mu\, dw/dx$) at the wall using (i) the velocity fields and (ii) the equilibrium equations, and compare with the exact value. *Answer:* (*a*) $U_1 = \frac{1}{2}f_0$, $U_2 = \frac{3}{8}f_0$, $f_0 = (\rho g\cos\beta)L^2/\mu$.

4.18 Consider the steady laminar flow of a viscous fluid through a long circular cylindrical tube. The governing equation is

$$-\frac{1}{r}\frac{d}{dr}\left(r\mu\frac{dw}{dr}\right) = \frac{P_0 - P_L}{L} \equiv f_0$$

where w is the axial (i.e., z) component of velocity, μ is the viscosity, and f_0 is the gradient of pressure (which includes the combined effect of static pressure and gravitational force). The boundary conditions are

$$\left(r\frac{dw}{dr}\right)\bigg|_{r=0} = 0, \quad w(R_0) = 0$$

Using the symmetry and (*a*) two linear elements and (*b*) one quadratic element, determine the velocity field and compare with the exact solution at the nodes:

$$w_e(r) = \frac{f_0 R_0^2}{4\mu}\left[1-\left(\frac{r}{R_0}\right)^2\right]$$

4.19 In the problem of the flow of a viscous fluid through a circular cylinder (Problem 4.18), assume that the fluid slips at the cylinder wall; i.e., instead of assuming that $w=0$ at $r=R_0$, use the boundary condition that

$$kw = -\mu\frac{dw}{dr} \quad \text{at} \quad r = R_0$$

in which k is the "coefficient of sliding friction." Solve the problem with two linear elements.

4.20 Consider the steady laminar flow of a Newtonian fluid with constant density in a long annular region between two coaxial cylinders of radii R_i and R_0 (see Fig. P4.20). The differential equation for this case is given by

$$-\frac{1}{r}\frac{d}{dr}\left(r\mu\frac{dw}{dr}\right)=\frac{P_1-P_2}{L}\equiv f_0$$

where w is the velocity along the cylinders (i.e., the z component of velocity), μ is the viscosity, L is the length of the region along the cylinders in which the flow is fully developed, and P_1 and P_2 are the pressures at $z=0$ and $z=L$, respectively (P_1 and P_2 represent the combined effect of static pressure and gravitational force).

The boundary conditions are

$$w=0 \quad \text{at} \quad r=R_0 \quad \text{and} \quad R_i$$

Solve the problem using (a) two linear elements and (b) one quadratic element, and compare the finite element solutions with the exact solution at the nodes:

$$w_e(r)=\frac{f_0R_0^2}{4\mu}\left[1-\left(\frac{r}{R_0}\right)^2+\frac{1-k^2}{\ln(1/k)}\ln\left(\frac{r}{R_0}\right)\right]$$

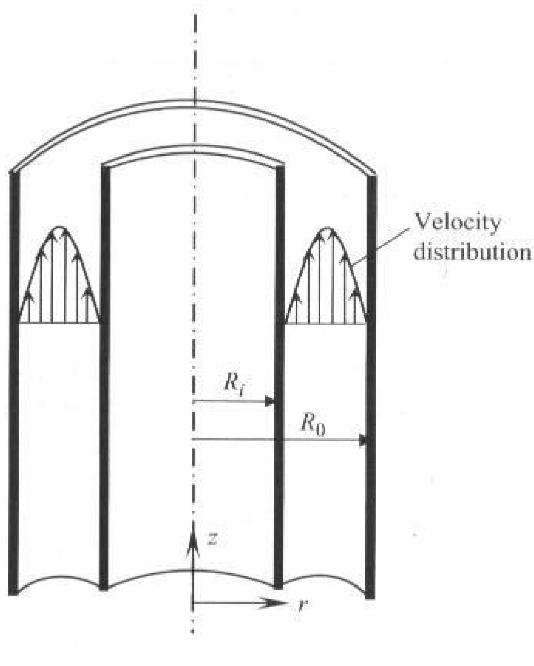


Figure P4.20

where $k=R_i/R_0$. Determine the shear stress $\tau_{rz}=-\mu\, dw/dr$ at the walls using (i) the velocity field and (ii) the equilibrium equations, and compare with the exact values. (Note that the steady laminar flow of a viscous fluid through a long cylinder or a circular tube can be obtained as a limiting case of $k\to 0$.)

4.21 Consider the steady laminar flow of two immiscible incompressible fluids in a region between two parallel stationary plates under the influence of a pressure gradient. The fluid rates are adjusted such that the lower half of the region is filled with fluid I (the denser and more viscous fluid) and the upper half is filled with fluid II (the less dense and less viscous fluid), as shown in Fig. P4.21. We wish to determine the velocity distributions in each region using the finite element method.

The governing equations for the two fluids are

$$-\mu_1 \frac{d^2 u_1}{dx^2} = f_0, \quad -\mu_2 \frac{d^2 u_2}{dx^2} = f_0$$

where $f_0 = (P_0 - P_L)/L$ is the pressure gradient. The boundary conditions are

$$u_1(-b) = 0 \quad u_2(b) = 0, \quad u_1(0) = u_2(0)$$

Solve the problem using four linear elements, and compare the finite element solutions with the exact solution at the nodes

$$u_i = \frac{f_0 b^2}{2\mu_i}\left[\frac{2\mu_i}{\mu_1+\mu_2} + \frac{\mu_1-\mu_2}{\mu_1+\mu_2}\frac{y}{b} - \left(\frac{y}{b}\right)^2\right] \quad (i = 1, 2)$$

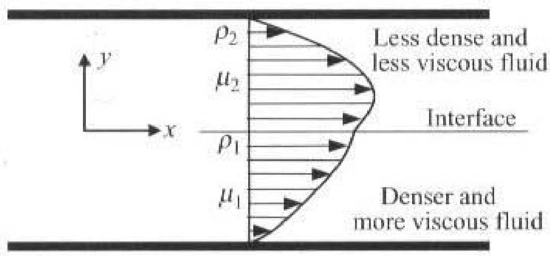


Figure P4.21

4.22 The governing equation for an unconfined aquifer with flow in the radial direction is given by the differential equation

$$-\frac{1}{r}\frac{d}{dr}\left(rk\frac{du}{dr}\right) = f$$

where k is the coefficient of permeability, f the recharge, and u the piezometric head. Pumping is considered to be a negative recharge. Consider the following problem. A well penetrates an aquifer and pumping is performed at $r = 0$ at a rate $Q = 150$ m^3/h. The permeability of the aquifer is $k = 25$ m^3/h. A constant head $u_0 = 50$ m exists at a radial distance $L = 200$ m. Determine the piezometric head at radial distances of 0, 10, 20, 40, 80, and 140 m (see Fig. P4.22). You are required to set up the finite element equations for the unknowns using a nonuniform mesh of six linear elements.

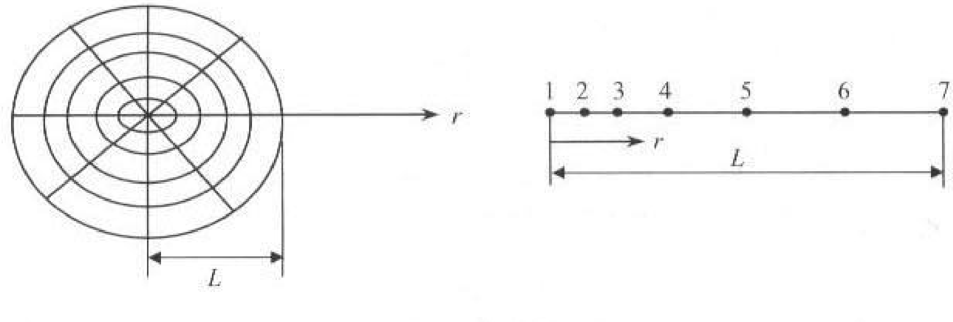


Figure P4.22

4.23 Consider a slow, laminar flow of a viscous substance (for example, glycerin solution) through a narrow channel under controlled pressure drop of 150 Pa/m. The channel is 5 m long (flow direction), 10 cm high, and 50 cm wide. The upper wall of the channel is maintained at 50°C while the lower wall is maintained at 25°C. The viscosity and density of the substance are temperature dependent, as given in Table P4.23. Assuming that the flow is essentially one dimensional (justified by the dimensions of the channel), determine the velocity field and mass flow rate of the fluid through the channel.

Table P4.23: Properties of the viscous substance of Problem 4.23.

y (m)	Temp. (°C)	Viscosity [kg/(m· s)]	Density (kg/m^3)
0.00	50	0.10	1233
0.02	45	0.12	1238
0.04	40	0.20	1243
0.06	35	0.28	1247
0.08	30	0.40	1250
0.10	25	0.65	1253

Solid and Structural Mechanics

4.24 The equation governing the axial deformation of an elastic bar in the presence of applied mechanical loads f and P and a temperature change T is

$$-\frac{d}{dx}\left[EA\left(\frac{du}{dx}-\alpha T\right)\right]=f \quad \text{for} \quad 0<x<L$$

where α is the thermal expansion coefficient, E the modulus of elasticity, and A the cross-sectional area. Using three linear finite elements, determine the axial displacements in a nonuniform rod of length 30 in., fixed at the left end and subjected to an axial force $P =$ 400 lb and a temperature change of 60°F. Take $A(x)=6-\frac{1}{10}x$ in.^{2} $E=30\times10^6$ lb/in.^{2}, and $\alpha=12\times10^{-6}$/(in. ·°F).

4.25 Find the stresses and compressions in each section of the composite member shown in Fig. P4.25. Use $E_s=30\times10^6$ psi, $E_a=10^7$ psi, $E_b=15\times10^6$ psi, and the minimum number of linear elements.

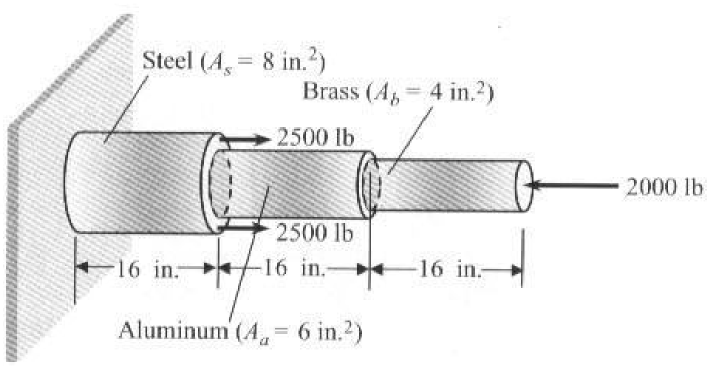


Figure P4.25

4.26 Find the three-element finite element solution to the stepped-bar problem. See Fig. P4.26 for the geometry and data. *Hint:* Solve the problem to see if the end displacement exceeds the gap. If it does, resolve the problem with modified boundary condition at $x=24$ in.

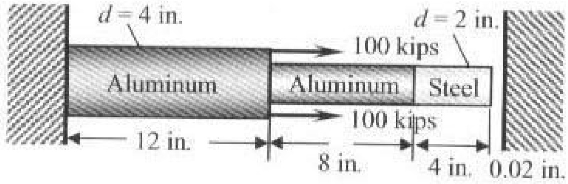


Figure P4.26

4.27 Analyze the stepped bar with its right end supported by a linear axial spring (see Fig. P4.27). The boundary condition at $x = 24$ in. is

$$EA\frac{du}{dx} + ku = 0$$

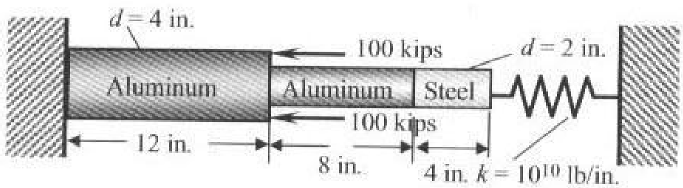


Figure P4.27

4.28 A solid circular brass cylinder $E_b = 15 \times 10^6$ psi, $d_b = 0.25$ in.) is encased in a hollow circular steel ($E_s = 30 \times 10^6$ psi, $d_s = 0.21$ in). A load of $P = 1330$ lb compresses the assembly, as shown in Fig. P4.28. Determine (a) the compression, and (b) compressive forces and stresses in the steel shell and brass cylinder. Use the minimum number of linear finite elements. Assume that the Poisson effect is negligible.

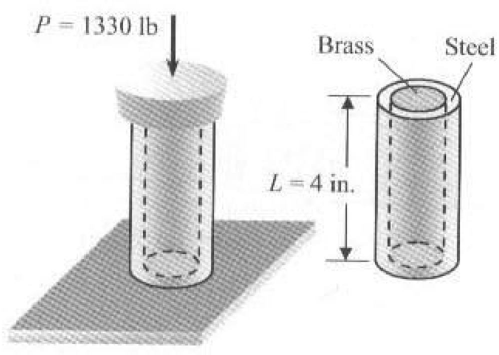


Figure P4.28

4.29 A rectangular steel bar ($E_s = 30 \times 10^6$ psi) of length 24 in. has a slot in the middle half of its length, as shown in Fig. 4.29. Determine the displacement of the ends due to the axial loads $P = 2000$ lb. Use the minimum number of linear elements.

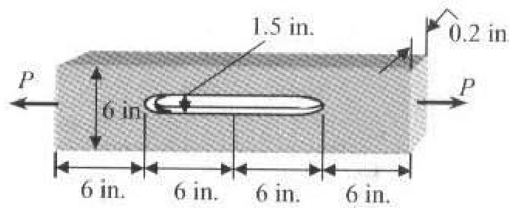
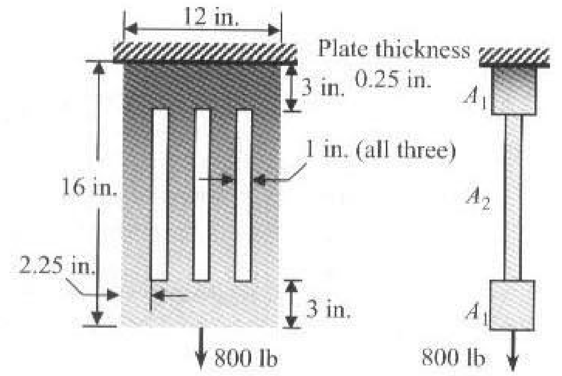


Figure P4.29



(*a*) Actual plate (*b*) Idealized plate

Figure P4.30

4.30 Repeat Problem 4.29 for the steel bar shown in Fig. P4.30.

4.31 The aluminum and steel pipes shown in Fig. P4.31 are fastened to rigid supports at ends A and B and to a rigid plate C at their junction. Determine the displacement of point C and stresses in the aluminum and steel pipes. Use the minimum number of linear finite elements.

4.32 A steel bar ABC is pin-supported at its upper end A to an immovable wall and loaded by a force F_1 at its lower end C, as shown in Fig. P4.32. A rigid horizontal beam BDE is pinned to the vertical bar at B, supported at point D, and carries a load F_2 at end E. Determine the displacements u_B and u_C at points B and C.

4.33 Repeat Problem 4.32 when point C is supported vertically by a spring ($k = 1000$ lb/in.).

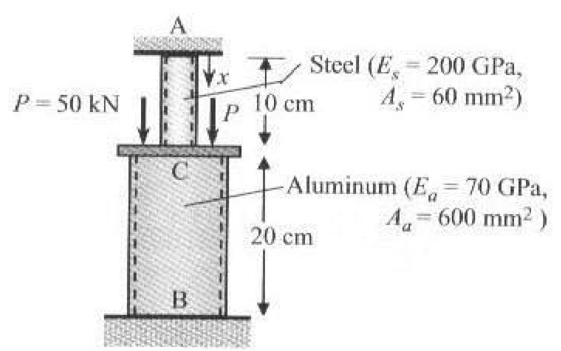


Figure P4.31

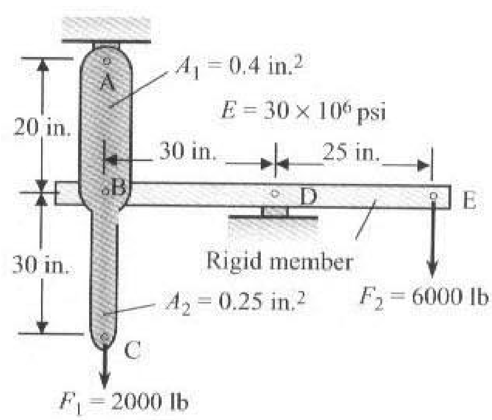


Figure P4.32

4.34 Consider the steel column (a typical column in a multi-storey building structure) shown in Fig. P4.34. The loads shown are due to the loads of different floors. The modulus of elasticity is $E = 30 \times 10^6$ psi and cross-sectional area of the column is $A = 40$ in^2. Determine the vertical displacements and axial stresses in the column at various floor-column connection points.

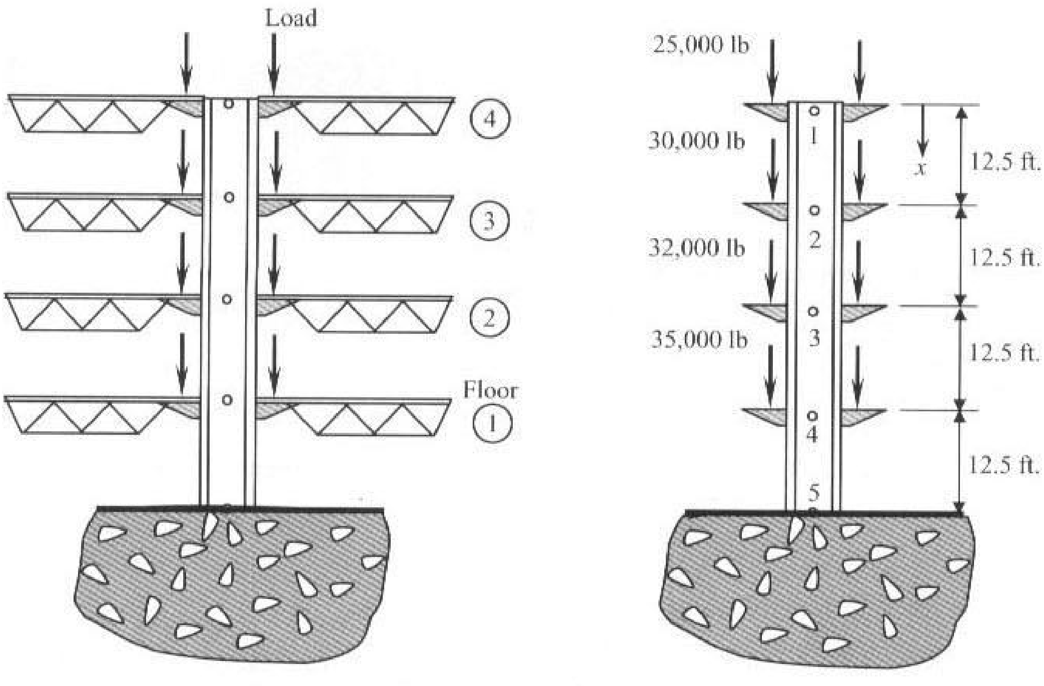


Figure P4.34

4.35 The bending moment (M) and transverse deflection (w) in a beam according to the Euler-Bernoulli beam theory are related by

$$-EI\frac{d^2w}{dx^2} = M(x)$$

For statically determinate beams, we can readily obtain the expression for the bending moment in terms of the applied loads. Thus, $M(x)$ is a known function of x. Determine the maximum deflection of the simply supported beam under uniform load (see Fig. P4.35) using the finite element method.

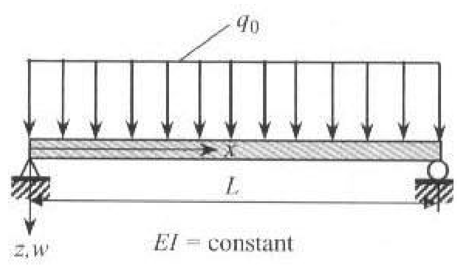


Figure P4.35

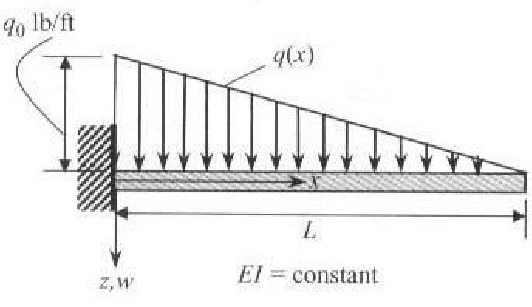


Figure P4.36

4.36 Repeat Problem 4.35 for the cantilever beam shown in Fig. P4.36.

4.37 Turbine disks are often thick near their hub and taper down to a smaller thickness at the periphery. The equation governing a variable-thickness $t = t(r)$ disk is

$$\frac{d}{dr}(rt\sigma_r) - t\sigma_\theta + t\rho\omega^2 r^2 = 0$$

where ω^2 is the angular speed of the disk and

$$\sigma_r = c\left(\frac{du}{dr} + \nu\frac{u}{r}\right), \quad \sigma_\theta = c\left(\frac{u}{r} + \nu\frac{du}{dr}\right), \quad c = \frac{E}{1-\nu^2}$$

(*a*) Construct the weak integral form of the governing equation such that the bilinear form is symmetric and the natural boundary condition involves specifying the quantity $tr\sigma_r$.

(*b*) Develop the finite element model associated with the weak form derived in part (*a*).

4.38–4.44 For the plane truss structures shown in Figs. P4.38–P4.44, give (*a*) the transformed element matrices, (*b*) the assembled element matrices, and (*c*) the condensed matrix equations for the unknown displacements and forces.

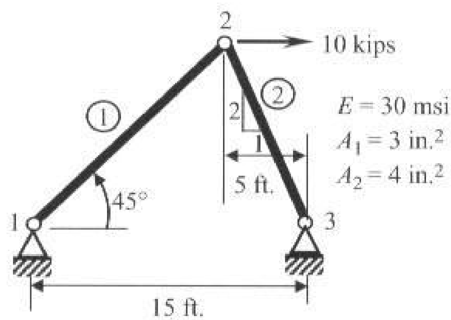


Figure P4.38

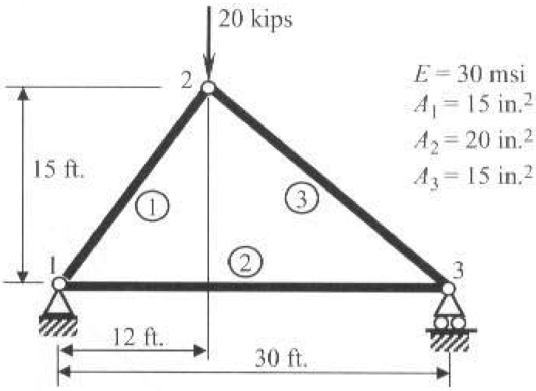


Figure P4.39

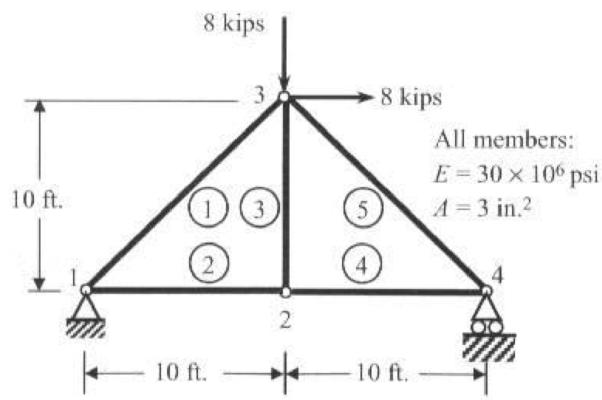


Figure P4.40

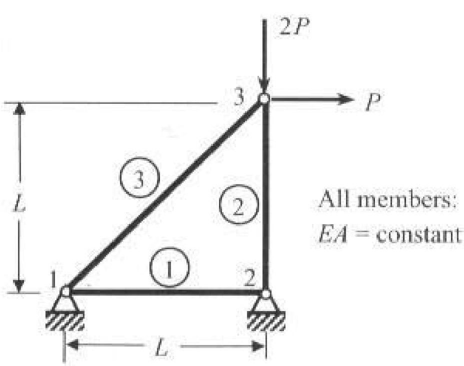


Figure P4.41

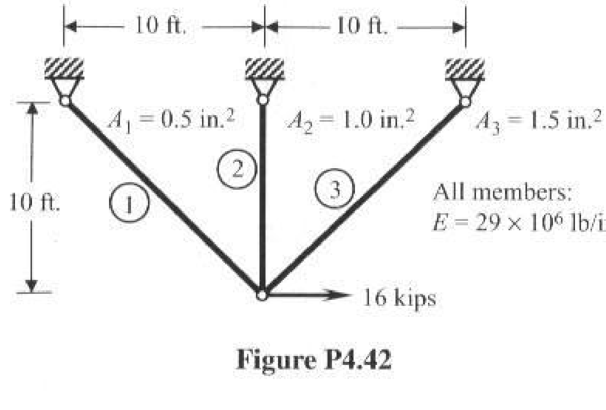


Figure P4.42

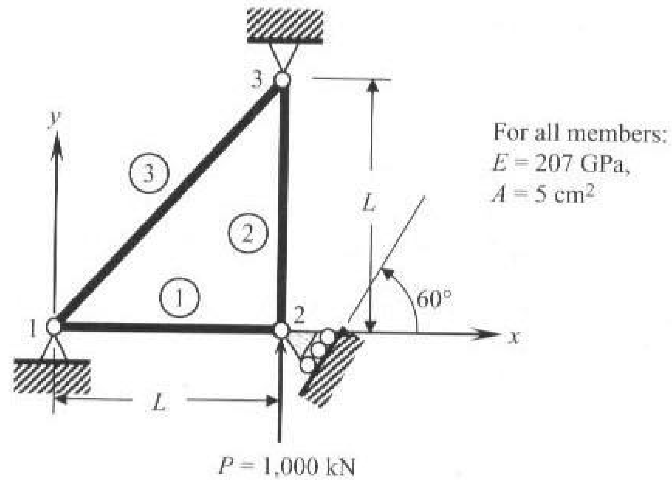


Figure P4.43

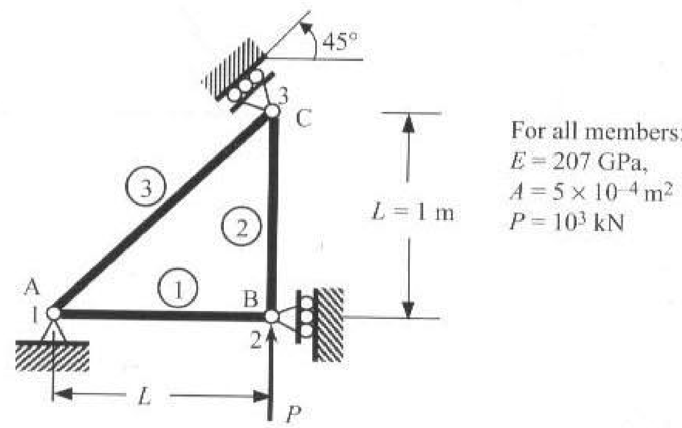


Figure P4.44

4.45 Determine the forces and elongations of each bar in the structure shown in Fig. P4.45. Also, determine the vertical displacements of points A and D.

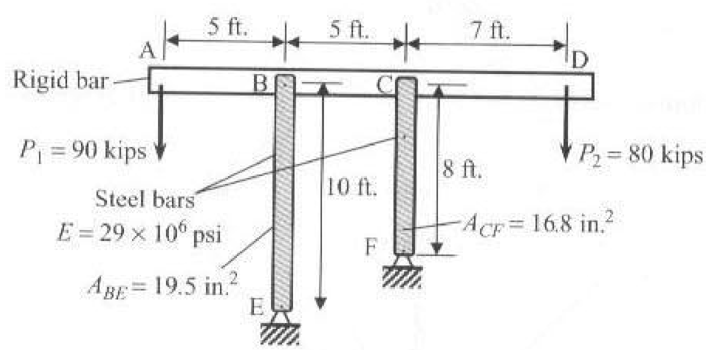


Figure P4.45

4.46 Determine the forces and elongations of each bar in the structure shown in Fig. P4.45 when end A is pinned to a rigid wall (and P_1 is removed).

REFERENCES FOR ADDITIONAL READING

1. Bird, R. B., Stewart, W. E. and Lightfoot, E. N., *Transport Phenomena*, John Wiley, New York, 1960.
2. Holman, J. P., *Heat Transfer*, 7th ed., McGraw-Hill, New York, 1990.
3. Kreith, F. and Bohn, M. S., *Principles of Heat Transfer*, 5th ed., West Publishing Company, St. Paul, MN, 1993.
4. Özisik, M. N., *Heat Conduction*, 2nd Ed., John Wiley, New York, 1993.
5. Reddy, J. N., *Energy Principles and Variational Methods in Applied Mechanics*, 2nd ed., John Wiley, New York, 2002.
6. Reddy, J. N. and Gartling, D. K., *The Finite Element Method in Heat Transfer and Fluid Dynamics*, 2nd ed, CRC Press, Boca Raton, FL, 2001.
7. Slaughter, W. S., *The Linearized Theory of Elasticity*, Birkhäuser, Boston, 2002.
8. Schlichting, H., *Boundary Layer Theory* (translated by J. Kestin), 7th ed., McGraw-Hill, New York (1979).
9. Timenshenko, S. P. and Goodier, J. N., *Theory of Elasticity*, McGraw-Hill, New York, 1970.