Chapter 3

SECOND-ORDER DIFFERENTIAL EQUATIONS IN ONE DIMENSION: FINITE ELEMENT MODELS

3.1 BACKGROUND

The traditional variational methods (e.g., the Ritz, Galerkin, and least-squares) described in Chapter 2 cease to be effective because of a serious shortcoming, namely, the difficulty in constructing the approximation functions. The approximation functions, apart from satisfying continuity, linear independence, completeness, and essential boundary conditions, are arbitrary; the selection becomes even more difficult when the given domain is geometrically complex. Since the quality of the approximation is directly affected by the choice of the approximation functions, it is discomforting to know that there exists no systematic procedure to construct them. Because of this shortcoming, despite the simplicity in obtaining approximate solutions, the traditional variational methods of approximation were never regarded as competitive computationally when compared with traditional finite difference schemes. The finite element method overcomes the shortcomings of the traditional variational methods by providing a systematic way of constructing the approximation functions.

Ideally speaking, an effective computational method should have the following features:

1. It should have a sound mathematical as well as physical basis (i.e., yield convergent solutions and be applicable to practical problems).
2. It should not have limitations with regard to the geometry, the physical composition of the domain, or the nature of the "loading."
3. The formulative procedure should be independent of the shape of the domain and the specific form of the boundary conditions.
4. It should be flexible enough to allow different degrees of approximation without reformulating the entire problem.
5. It should involve a systematic procedure that can be automated for use on digital computers.

The finite element method is a technique in which a given domain is represented as a collection of simple domains, called *finite elements*, so that it is possible to systematically construct the approximation functions needed in a variational or weighted-residual approximation of the solution of a problem over each element. Thus, the finite element method differs from the traditional Ritz, Galerkin, least-squares, collocation, and other weighted-residual methods in the manner in which the approximation functions are constructed. But this difference is responsible for the following three basic features of the finite element method:

1. *Division of whole domain into subdomains* that enable a systematic derivation of the approximation functions as well as representation of complex domains.
2. *Derivation of approximation functions* over each element. The approximation functions are often algebraic polynomials that are derived using interpolation theory. However, approximation functions need not be polynomials (like in meshless form of the finite element method).
3. *Assembly of elements* is based on continuity of the solution and balance of internal fluxes; the assemblage of elements results in a numerical analog of the mathematical model of the problem being analyzed.

These three features, which constitute three major steps of the finite element formulation, are closely related. The geometry of the elements used to represent the domain of a problem should be such that the approximation functions can be uniquely derived. The approximation functions depend not only on the geometry but also on the number and location of points, called nodes, in the element and the quantities to be interpolated (e.g., solution, or solution and its derivatives). Once the approximation functions have been derived, the procedure to obtain algebraic relations among the unknown coefficients (which give the values of the dependent variable at the nodes) is exactly the same as that used in the Ritz and weighted-residual methods. Hence, a study of Chapter 2, especially the weak-form development and the Ritz method, makes the present study easier.

The finite element method not only overcomes the shortcomings of the traditional variational methods, but it is also endowed with the features of an effective computational technique. The basic steps involved in the finite element analysis of a problem are given in Table 3.1.1.

In the sections that follow, our objective is to introduce many fundamental ideas that form the basis of the finite element method. In doing so, we postpone some issues of practical and theoretical complexity to later sections of this chapter and to Chapters 4–14. The basic steps of a finite element analysis are introduced via a model second-order differential equation.

3.2 BASIC STEPS OF FINITE ELEMENT ANALYSIS

3.2.1 Model Boundary Value Problem

Consider the problem of finding the function $u(x)$ that satisfies the differential equation

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f=0 \quad \text{for} \quad 0<x<L \tag{3.2.1}$$

Table 3.1.1 Steps involved in the finite element analysis of a typical problem.

1. Discretization (or representation) of the given domain into a collection of preselected finite elements. (This step can be postponed until the finite element formulation of the equation is completed.)
   (*a*) Construct the finite element mesh of preselected elements.
   (*b*) Number the nodes and the elements.
   (*c*) Generate the geometric properties (e.g., coordinates and cross-sectional areas) needed for the problem.
2. Derivation of element equations for all typical elements in the mesh.
   (*a*) Construct the variational formulation of the given differential equation over the typical element.
   (*b*) Assume that a typical dependent variable u is of the form

$$u = \sum_{i=1}^{n} u_i \psi_i$$

   and substitute it into Step 2*a* to obtain element equations in the form

$$[K^e]\{u^e\} = \{F^e\}$$

   (*c*) Select, if already available in the literature, or derive element interpolation functions ψ_i and compute the element matrices.
3. Assembly of element equations to obtain the equations of the whole problem.
   (*a*) Identify the interelement continuity conditions among the primary variables (relationship between the local degrees of freedom and the global degrees of freedom—connectivity of elements) by relating element nodes to global nodes.
   (*b*) Identify the "equilibrium" conditions among the secondary variables (relationship between the local source or force components and the globally specified source components).
   (*c*) Assemble element equations using Steps 3*a* and 3*b*.
4. Imposition of the boundary conditions of the problem.
   (*a*) Identify the specified global primary degrees of freedom.
   (*b*) Identify the specified global secondary degrees of freedom (if not already done in Step 3*b*).
5. Solution of the assembled equations.
6. Postprocessing of the results.
   (*a*) Compute the gradient of the solution or other desired quantities from the primary degrees of freedom computed in Step 5.
   (*b*) Represent the results in tabular and/or graphical form.

and the boundary conditions

$$u(0) = u_0, \quad \left(a\frac{du}{dx}\right)\bigg|_{x=L} = Q_0 \tag{3.2.2}$$

where $a = a(x)$, $c = c(x)$, $f = f(x)$, and u_0, and Q_0 are the *data* (i.e., known quantities) of the problem. Equation (3.2.1) arises in connection with the analytical description of many physical processes. For example, conduction and convection heat transfer in a plane wall or fin [see Fig. 3.2.1(*a*)], flow through channels and pipes, transverse deflection of cables, axial deformation of bars [see Fig. 3.2.1(*b*)], and many other physical processes are described by Eq. (3.2.1). A list of field problems described by Eq. (3.2.1) when $c(x) = 0$ are presented in Table 3.2.1 [see Reddy (2004)]. Thus, if we can develop a numerical procedure by which Eq. (3.2.1) can be solved for all possible boundary conditions, the procedure can be used to solve all field problems listed in Table 3.2.1, as well as many others. This fact provides us with the motivation to use (3.2.1) as the model second-order equation in one dimension. A step-by-step procedure for the formulation and solution of (3.2.1) by the

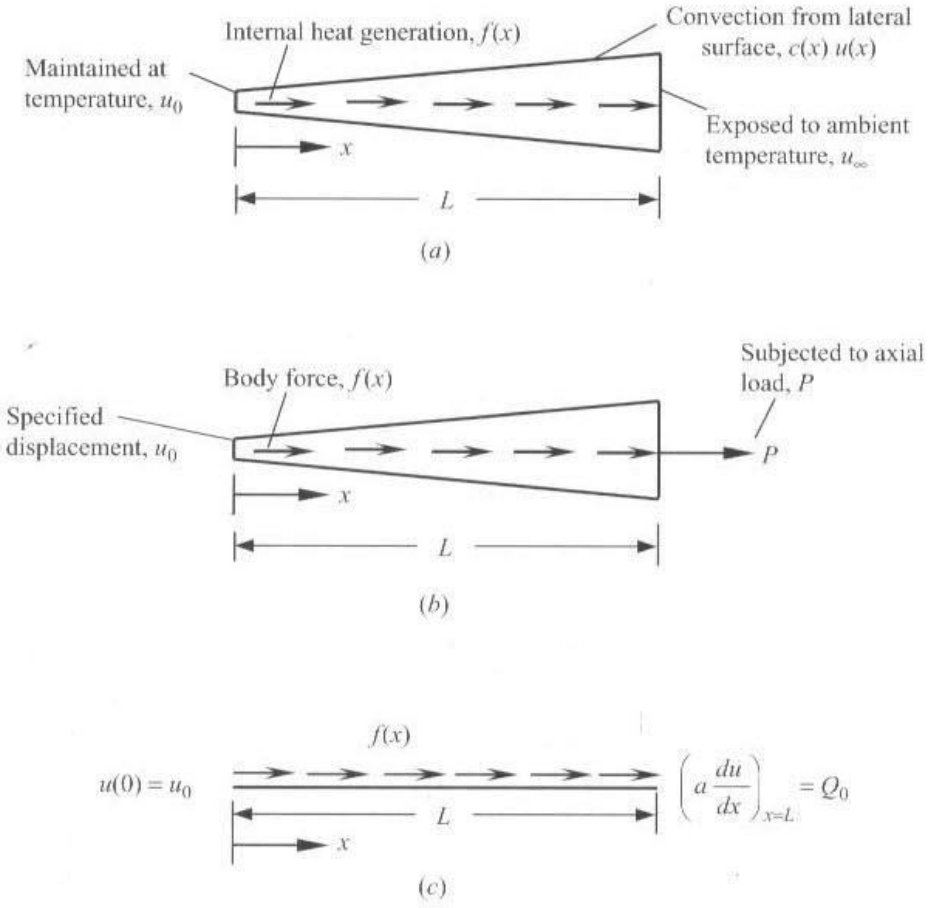

Figure 3.2.1 (*a*) Heat transfer in a fin. (*b*) Axial deformation of a bar. (*c*) Mathematical idealization of the problem in (*a*) or (*b*).

finite element method is summarized in Table 3.1.1. The mathematical problem consists of solving the differential equation (3.2.1) in one-dimensional domain $\Omega = (0, L)$ subject to a suitable set of specified boundary conditions at the boundary points $x = 0$ and $x = L$, as shown in Fig. 3.2.1(*c*). As already shown in Chapter 2, the type of boundary conditions associated with a differential equation emerges in a natural way during the weak-form development of the differential equation. A detailed discussion of these ideas is presented next.

3.2.2 Discretization of the Domain

In the finite element method, the domain Ω of the problem [Fig. 3.2.2(*a*)] is divided into a set of subintervals, i.e., line elements, called *finite elements*. A typical element is denoted Ω_e and it is located between points A and B with coordinates x_a and x_b (i.e., of length $h_e = x_b - x_a$). The collection of finite elements in a domain is called the *finite element mesh* of the domain [see Fig. 3.2.2(*b*)].

The reason for dividing a domain into a set of subdomains, i.e., finite elements, is two-fold. First, domains of most systems, by design, are a composite of geometrically and/or

Table 3.2.1 Some examples of engineering problems governed by the second-order equation (3.2.1) (see the footnote for the meaning of some parameters*).

Field of study	Primary variable u	Problem data a	Problem data c	Problem data f	Secondary variable Q
Heat transfer	Temperature $T - T_\infty$	Thermal conductance kA	Surface convection $AP\beta$	Heat generation f	Heat Q
Flow through porous medium	Fluid head ϕ	Permeability μ	0	Infiltraction f	Point source Q
Flow through pipes	Pressure P	Pipe resistance $1/R$	0	0	Point source Q
Flow of viscous fluids	Velocity v_x	Viscosity μ	0	Pressure gradient $-dP/dx$	Shear stress σ_{xz}
Elastic cables	Displacement u	Tension T	0	Transverse force f	Point force P
Elastic bars	Displacement u	Axial stiffness EA	0	Axial force f	Point force P
Torsion of bars	Angle of twist θ	Shear stiffness GJ	0	0	Torque T
Electrostatics	Electrical potential ϕ	Dielectric constant ϵ	0	Charge density ρ	Electric flux E

*k = thermal conductance; β = convective film conductance; p = perimeter; P = pressure or force; T_∞ = ambient temperature of the surrounding fluid medium; $R = 128\mu h/(\pi d^4)$ with μ being the viscosity, h the length, and d the diameter of the pipe; E = Young's modulus; A = area of cross section; J = polar moment of inertia.

materially different parts, and the solution on these subdomains is represented by different functions that are continuous at the interfaces of these subdomains. Therefore, it is appropriate to seek approximation of the solution over each subdomain. Second, approximation of the solution over each element of the mesh is simpler than its approximation over the entire domain. Approximation of the geometry of the domain in the present case is not a

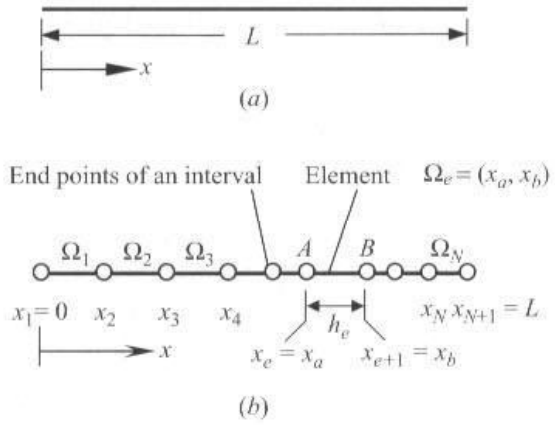

Figure 3.2.2 (*a*) Whole domain. (*b*) Finite element discretization (mesh).

concern, since it is a straight line. We must, however, seek a suitable approximation of the solution over each subdomain (i.e., finite element).

The number of elements into which the total domain is divided in a problem depends mainly on the geometry of the domain and on the desired accuracy of the solution. In the one-dimensional problem at hand, geometry is simple enough to represent it exactly. Since the exact solution is not known *a priori*, we may begin with a number of elements that are considered to be reasonable. Most often, the analyst has knowledge of the qualitative behavior of the solution, and this helps to choose a starting mesh. Whenever a problem is solved by the finite element method for the first time, we are required to investigate the convergence of the finite element solution by gradually *refining the mesh* (i.e., increasing the number of elements) and comparing the solution with those obtained by *higher-order* elements. The *order* of an element refers to the degree of polynomial used to represent the solution over the element. This is made clearer in the sequel.

3.2.3 Derivation of Element Equations

Next, we develop the algebraic equations among the unknown parameters, much the same way as we did in the Ritz and Galerkin methods discussed in Chapter 2 [see Eq. (2.5.5)]. The main difference here is that we work with a finite element (i.e., subdomain) as opposed to the total domain. This step results in a matrix equation of the form $[K^e]\{c^e\} = \{F^e\}$, which is called *the finite element model* of the original equation. Since the element is physically connected to its neighbors, the resulting algebraic equations will contain more unknowns (c^es as well as F^es are unknown in the element equations) than the number of algebraic equations. Then it becomes necessary to put the elements together (i.e., assembly) to eliminate the extra unknowns. This process is discussed in Section 3.2.5.

The derivation of finite element equations, i.e., algebraic equations among the unknown parameters of the finite element approximation, involves the following three steps:

1. Construct the weighted-residual or weak form of the differential equation.
2. Assume the form of the approximate solution over a typical finite element.
3. Derive the finite element equations by substituting the approximate solution into the weighted-residual or weak form.

A typical element $\Omega_e = (x_a, x_b)$ [see Fig. 3.2.3(*a*)], whose endpoints have the coordinates $x = x_a$ and $x = x_b$, is isolated from the mesh. We seek an approximate solution to the governing differential equation over the element. In principle, any method that allows the derivation of necessary algebraic relations among the nodal values of the dependent variable can be used. In this book we develop the algebraic equations using the Ritz method, which is based on the weak form of the differential equation. Other methods, such as the least-squares method, may also be used to construct the finite element equations. Apart from the method used to derive the algebraic equations, the steps presented in this book for the Ritz (or weak-form Galerkin) finite element models are the same for other methods. The three steps in the derivation of finite element equations associated with the model differential equation over a *typical element* of the mesh are discussed next.

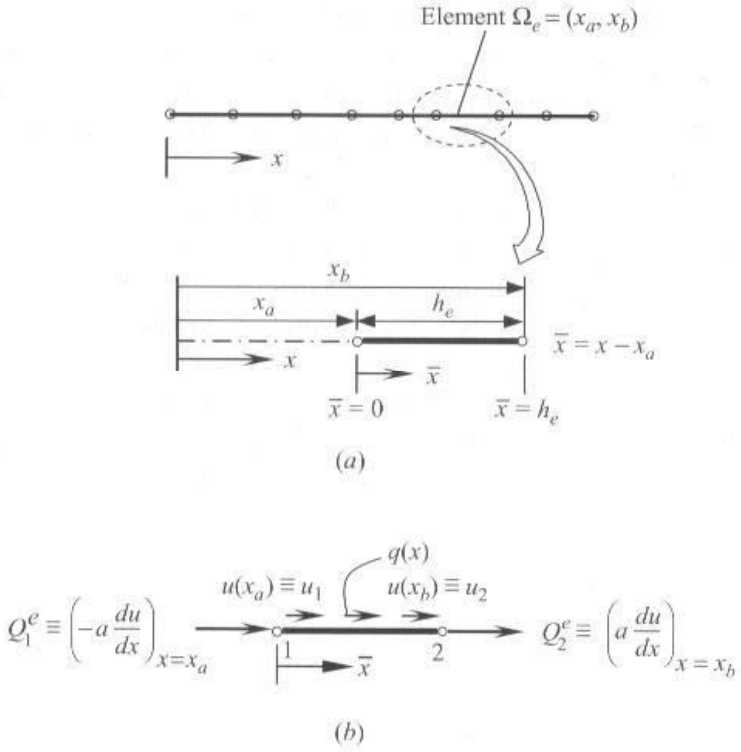

Figure 3.2.3 Finite element discretization of a one-dimensional domain for the model problem in (3.2.1). (*a*) A typical finite element from the finite element mesh. (*b*) A typical element, with the definition of the primary (u) and secondary (Q) variables at the element nodes.

Step 1. Weak Form and Minimum of a Quadratic Functional

In the finite element method, we seek an approximate solution to (3.2.1) over each finite element. The polynomial approximation of the solution within a typical finite element Ω_e is assumed to be of the form

$$u_h^e = \sum_{j=1}^{n} u_j^e \psi_j^e(x) \tag{3.2.3}$$

where u_j^e are the values of the solution $u(x)$ at the nodes of the finite element Ω_e, and ψ_j^e are the approximation functions over the element. The particular form in (3.2.3) will be derived in the next section. Note that the approximation in (3.2.3) differs from the one used in the Ritz method in that $c_j \phi_j(x)$ is now replaced with $u_j^e \psi_j^e$ (and $\phi_0 = 0$), and u_j^e plays the role of undetermined parameters and ψ_j^e the role of approximation functions. As we shall see later, writing the approximation in terms of the nodal values of the solution is necessitated by the fact that the continuity of $u(x)$ between elements can be readily imposed. The coefficients u_j^e are determined such that (3.2.1) is satisfied in a weighted-integral sense. As discussed in Chapter 2, the necessary and sufficient number of algebraic relations among the u_j^e can be obtained by recasting the differential equation (3.2.1) in a weighted-integral form:

$$0 = \int_{x_a}^{x_b} w \left[-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - f \right] dx \tag{3.2.4}$$

where $w(x)$ denotes the weight function and $\Omega_e = (x_a, x_b)$ is the domain of a typical element [see Fig. 3.2.3(*a*)]. For $u \approx u_h^e$ and each independent choice of w, we obtain an independent

algebraic equation relating all u_j^e of the element. A total of n independent equations are required to solve for the parameters u_j^e, $j = 1, 2, \ldots, n$. When w is selected to be ψ_i^e and (3.2.4) is used to obtain the ith equation of the required n equations, the resulting finite element model (i.e., system of algebraic equations among the nodal values) is termed the *Galerkin finite element model*. Since (3.2.4) contains the second derivative of u, the approximation functions ψ_j^e must be twice differentiable. In addition, if the secondary variables are to be included in the model, ψ_i^e must be at least cubic. Similar arguments apply for cases of the weighted-residual methods discussed in Chapter 2. For details of the weighted-residual finite element models, see Chapter 14 and Reddy (1986).

To weaken the continuity required of the functions $\psi_j^e(x)$, we trade the differentiation in (3.2.4) from u to w such that both u and w are differentiated equally—once each in the present case. The resulting integral form is termed the *weak form* of (3.2.1). This form is not only equivalent to (3.2.1) but it also contains the natural boundary conditions of the problem. The three-step procedure of constructing the weak form of (3.2.1) was presented in Chapter 2 and is revisited in the next few paragraphs.

The first step is to multiply the governing differential equation with a weight function w and integrate over a *typical element,* as given in (3.2.4). The second step is to trade differentiation from u to w using integration by parts:

$$0 = \int_{x_a}^{x_b} \left(a \frac{dw}{dx} \frac{du}{dx} + cwu - wf \right) dx - \left[wa \frac{du}{dx} \right]_{x_a}^{x_b} \tag{3.2.5}$$

The third and last step is to identify the primary and secondary variables of the weak form. This requires us to classify the boundary conditions of each differential equation into *essential* (or geometric) and *natural* (or force) boundary conditions. The classification is made uniquely by examining the boundary term appearing in the weak form (3.2.5),

$$\left[wa \frac{du}{dx} \right]_{x_a}^{x_b}$$

As a rule, the coefficient of the weight function w in the boundary expression is called a *secondary variable,* and its specification constitutes the *natural* or Neumann boundary condition. The dependent unknown u *in the same form as the weight function* w appearing in the boundary expression is termed a *primary variable,* and its specification constitutes the *essential* or Dirichlet boundary condition. For the model equation at hand, the primary and secondary variables are

$$u \quad \text{and} \quad a \frac{du}{dx} \equiv Q$$

The primary and secondary variables at the nodes are shown on the typical element in Fig. 3.2.3(b).

In writing the final form of the weighted-integral statement (i.e., weak form) we wish to use, we must address the fate of the boundary terms. For a typical line element, the end points (nodes 1 and 2) are the boundary points. At these points we have the following four conditions (none of them specified at the moment)

$$u_h^e(x_a) = u_1^e, \left(-a \frac{du}{dx} \right)_{x=x_a} = Q_1^e, u_h^e(x_b) = u_2^e, \left(a \frac{du}{dx} \right)_{x=x_b} = Q_2^e$$

If we select $u_h^e(x)$ in (3.2.3) such that it automatically satisfies the end conditions $u_h^e(x_a) = u_1^e$ and $u_h^e(x_b) = u_2^e$, then it remains that we include the remaining conditions

$$Q_1 = \left(-a\frac{du}{dx}\right)\bigg|_{x_a}, \quad Q_2 = \left(a\frac{du}{dx}\right)\bigg|_{x_b} \tag{3.2.6}$$

in the weak form (3.2.5). With the notation in (3.2.6), the weak form becomes

$$0 = \int_{x_a}^{x_b} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right) dx - w(x_a)Q_1 - w(x_b)Q_2 \tag{3.2.7}$$

This completes the three-step procedure of constructing the weak form of the model equation (3.2.1). The finite element model based on the weak form (3.2.7) is called the *weak form Galerkin finite element model.* It is clear that the weak form (3.2.7) admits approximation functions that are lower order than the weighted-residual statement (3.2.4).

Students of engineering recognize that Fig. 3.2.3(*b*) is the *free-body diagram* of a typical element. For axial deformation of bars, u denotes displacement, du/dx is the strain ε, $E\varepsilon$ is the stress σ, and $A\sigma$ denotes the force, where E is Young's modulus and A is the area of cross section of the bar; hence, $Q = EA(du/dx) = a(du/dx)$ has the meaning of force. The quantities Q_1^e and Q_2^e are the reaction forces at the left and right ends of the member; Q_1^e is a compressive force while Q_2^e is a tensile force [algebraically, both are positive, as shown in Fig. 3.2.3(*b*)]. For heat conduction problems, u denotes temperature, du/dx is the temperature gradient, $-k(du/dx)$ is the heat flux q, and Aq denotes the heat, where k is the thermal conductivity and A is the area of cross section of the bar; hence, $Q = kA(du/dx) = a(du/dx)$ has the meaning of heat; $Q_1^e = -kA(du/dx)_a$ is the heat *input* at node 1, while $Q_2^e = kA(du/dx)_b$ denote the heat *input* at node 2. Thus, the arrow on the second node should be reversed for heat transfer problems. For additional details on heat transfer, see Section 3.3.1.

The weak form in (3.2.7) contains two types of expressions: those containing both w and u (bilinear form) and those containing only w (linear form):

$$\begin{aligned} B^e(w,u) &= \int_{x_a}^{x_b} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu\right) dx \\ l^e(w) &= \int_{x_a}^{x_b} w\, f\, dx + w(x_a)Q_1 + w(x_b)Q_2 \end{aligned} \tag{3.2.8}$$

The weak form can be expressed as

$$B^e(w,u) = l^e(w) \tag{3.2.9}$$

which is called the *variational problem* associated with (3.2.1). As will be seen later, the bilinear form results directly in the element coefficient matrix, and the linear form leads to the right-hand-side column vector of the finite element equations. Derivation of the variational problem of the type in (3.2.9) is possible for all problems described by differential equations. However, the bilinear form $B^e(w,u)$ may not be linear in u, and it may not be symmetric in its arguments w and u.

Those who have a background in applied mathematics or solid and structural mechanics will appreciate the fact that the variational problem (3.2.9), when $B^e(w,u)$ is symmetric $B^e(w,u) = B^e(u,w)$ and $l^e(w)$ is linear in w, is the same as the statement of the minimum

of the quadratic functional $I^e(u)$, $\delta I^e = 0$, where

$$I^e(u) = \frac{1}{2} B^e(u, u) - l^e(u)$$
$$= \frac{1}{2} \int_{x_a}^{x_b} \left[a \left(\frac{du}{dx} \right)^2 + cu^2 \right] dx$$
$$- \int_{x_a}^{x_b} u f \, dx - u(x_a) Q_1 - u(x_b) Q_2 \tag{3.2.10}$$

Thus, the relationship between the weak form and the minimum of quadratic functional I^e is obvious [cf. (3.2.9)]:

$$0 = \delta I^e = B^e(w, u) - l^e(w), \quad w = \delta u$$

The statement $\delta I^e = 0$ in solid and structural mechanics is also known as the *principle of minimum total potential energy*. When (3.2.1) describes the axial deformation of a bar, $\frac{1}{2} B^e(u, u)$ represents the elastic strain energy stored in the bar element, $l^e(u)$ represents the work done by applied forces, and $I^e(u)$ is the total potential energy (Π^e) of the bar element. Thus, the finite element model can be developed using either the statement of the principle of minimum total potential energy of an element or the weak form of the governing equations of an element. However, this choice is restricted to those problems where the minimum of a quadratic finctional $I^e(u)$ corresponds to the governing equations. On the other hand, we can always construct a weak form of any set of differential equations, linear or not, of order 2 and higher. Finite element formulations do not require the existence of the functional $I^e(u)$; they only need weighted-integral statements or weak forms. However, when the functional $I^e(u)$ exists with an extremum (i.e., minimum or maximum principle), existence and uniqueness of solution to the variational problem and its discrete analog can be established. In all problems discussed in this book, the variational problem is derivable from a quadratic functional.

Step 2. Approximate Solution

Recall that in the traditional variational methods, approximate solutions are sought over the total domain $\Omega = (0, L)$ at once. Consequently, the approximate solution $[u(x) \approx U_N(x) = \sum c_j \phi_j(x) + \phi_0(x)]$ is required to satisfy the boundary conditions of the problem. This places severe restrictions on the derivation of the approximation functions $\phi_j(x)$ and $\phi_0(x)$, especially when discontinuities exist in the geometry, material properties, and/or loading of the problem (see Chapter 2 for details). The finite element method overcomes this shortcoming by seeking approximate solution (3.2.3) over each element. Obviously, geometry of the element should be simpler than that of the whole domain, and the geometry should allow a systematic derivation of the approximation functions, as we shall see shortly.

To put the elements back together into their original positions, i.e., connect the approximate solution from each element to form a continuous solution over the whole domain, we require the solution to be the same at points common to the elements. Therefore, we identify the end points of each line element as the *element nodes*, which play the role of interpolation points (or base points) in constructing the approximation functions over an element. Depending on the degree of polynomial approximation used to represent the solution, additional nodes may be identified inside the element.

Since the weak form over an element is equivalent to the differential equation and the natural boundary conditions (3.2.6), i.e., conditions on Q_i^e of the element, the approximate solution u_h^e of (3.2.3) is required to satisfy only the end conditions $u_h^e(x_a) = u_1^e$ and $u_h^e(x_b) = u_2^e$. We seek the approximate solution in the form of algebraic polynomials, although this is not always the case. The reason for this choice is two-fold: First, the interpolation theory of numerical analysis can be used to develop the approximation functions systematically over an element; second, numerical evaluation of integrals of algebraic polynomials is easy.

As in classical variational methods, the approximation solution u_h^e must fulfill certain conditions in order that it be convergent to the actual solution u as the number of elements is increased. These are:

1. It should be continuous over the element and differentiable, as required by the weak form.
2. It should be a *complete* polynomial, i.e., include all lower-order terms up to the highest order used.
3. It should be an interpolant of the primary variables at the nodes of the finite element (at least the nodes on the boundary of the element so that the continuity of the solution can be imposed across the interelement boundary).

The reason for the first condition is obvious; it ensures a nonzero coefficient matrix. The second condition is necessary in order to capture all possible states, i.e., constant, linear, and so on, of the actual solution. For example, if a linear polynomial without the constant term is used to represent the temperature distribution in a one-dimensional system, the approximate solution can never be able to represent a uniform state of temperature in the element should such a state occur. The third condition is necessary in order to enforce continuity of the primary variables at points common to the elements.

For the weak form in (3.2.7), the minimum polynomial order of u_h^e is linear. A complete linear polynomial is of the form

$$u_h^e(x) = c_1^e + c_2^e x \tag{3.2.11}$$

where c_1^e and c_2^e are constants. The phrase "complete polynomial" refers to the inclusion of all terms up to the order desired; omission of c_1^e would make it an incomplete linear polynomial. Similarly, $c_1^e + c_3^e x^2$ is an incomplete quadratic polynomial because the linear term is missing. The expression in (3.2.11) meets the first two conditions of an approximation. The third condition is satisfied if c_1^e and c_2^e meet the conditions

$$u_h^e(x_a) = c_1^e + c_2^e x_a \equiv u_1^e, \quad u_h^e(x_b) = c_1^e + c_2^e x_b \equiv u_2^e \tag{3.2.12}$$

Equation (3.2.12) provides two relations between (c_1^e, c_2^e) and (u_1^e, u_2^e), which can be expressed in matrix form as

$$\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{bmatrix} 1 & x_a \\ 1 & x_b \end{bmatrix} \begin{Bmatrix} c_1^e \\ c_2^e \end{Bmatrix} \tag{3.2.13}$$

Inverting (3.2.13), we obtain

$$\begin{aligned} c_1^e &= \frac{1}{h_e}(u_1^e x_b - u_2^e x_a) \equiv \frac{1}{h_e}(\alpha_1^e u_1^e + \alpha_2^e u_2^e) \\ c_2^e &= \frac{1}{h_e}(u_2^e - u_1^e) \equiv \frac{1}{h_e}(\beta_1^e u_1^e + \beta_2^e u_2^e) \end{aligned} \tag{3.2.14}$$

where $h_e = x_b - x_a$ and ($\alpha_1^e = x_b$, $\alpha_2^e = -x_a$, $\beta_1^e = 1$, and $\beta_2^e = -1$)

$$\alpha_i^e = (-1)^j x_j^e, \quad \beta_i^e = (-1)^j; \quad x_1^e = x_a, \quad x_2^e = x_b \tag{3.2.15}$$

In (3.2.15), i and j permute in a natural order:

$$\text{if } i = 1 \quad \text{then } j = 2; \quad \text{if } i = 2 \quad \text{then } j = 1$$

The α_i^e and β_i^e are introduced to show the typical form of the interpolation functions. Substituting c_1^e and c_2^e from (3.2.15) into (3.2.11), we obtain

$$\begin{aligned} u_h^e(x) &= \frac{1}{h_e}[(\alpha_1^e u_1^e + \alpha_2^e u_2^e) + (\beta_1^e u_1^e + \beta_2^e u_2^e)x] \\ &= \frac{1}{h_e}(\alpha_1^e + \beta_1^e x)u_1^e + \frac{1}{h_e}(\alpha_2^e + \beta_2^e x)u_2^e \\ &= \psi_1^e(x)u_1^e + \psi_2^e(x)u_2^e = \sum_{j=1}^{2} \psi_j^e(x)u_j^e \end{aligned} \tag{3.2.16}$$

where

$$\psi_1^e(x) = \frac{1}{h_e}(\alpha_1^e + \beta_1^e x) = \frac{x_b - x}{x_b - x_a}, \quad \psi_2^e(x) = \frac{1}{h_e}(\alpha_2^e + \beta_2^e x) = \frac{x - x_a}{x_b - x_a} \tag{3.2.17}$$

which are called the linear *finite element approximation functions.*

The approximation functions $\psi_i^e(x)$ have some interesting properties. First, note that

$$u_1^e \equiv u_h^e(x_a) = \psi_1^e(x_a)u_1^e + \psi_2^e(x_a)u_2^e$$

implies $\psi_1^e(x_a) = 1$ and $\psi_2^e(x_a) = 0$. Similarly,

$$u_2^e \equiv u_h^e(x_b) = \psi_1^e(x_b)u_1^e + \psi_2^e(x_b)u_2^e$$

gives $\psi_1^e(x_b) = 0$ and $\psi_2^e(x_b) = 1$. In other words, ψ_i^e is unity at the ith node and zero at the other node. This property is known as the interpolation property (i.e., u_h^e is an interpolant of $u(x)$ through nodes 1 and 2) of $\psi_i^e(x)$ and they are also called *interpolation functions.* When they are derived to interpolate function values only and not the derivatives of the function, they are known as the *Lagrange interpolation functions.* When the function and its derivatives are interpolated, the resulting interpolation functions are known as the *Hermite family of interpolation functions.* These will be discussed in connection with beam finite elements in Chapter 5.

Another property of $\psi_i^e(x)$ is that their sum is unity. To see this, consider a constant state of $u_h^e = c_0$. Then the nodal values u_1^e and u_2^e should be equal to each other and both equal to the constant c_0. Hence, we have

$$u_h^e(x) = \psi_1^e(x)u_1^e + \psi_2^e(x)u_2^e = c_0 \;\rightarrow\; 1 = \psi_1^e(x) + \psi_2^e(x)$$

This property of $\psi_i^e(x)$ is known as the *partition of unity*. In summary, we have

$$\psi_i^e(x_j^e) = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \tag{3.2.18a}$$

$$\sum_{i=1}^{n} \psi_i^e(x) = 1 \tag{3.2.18b}$$

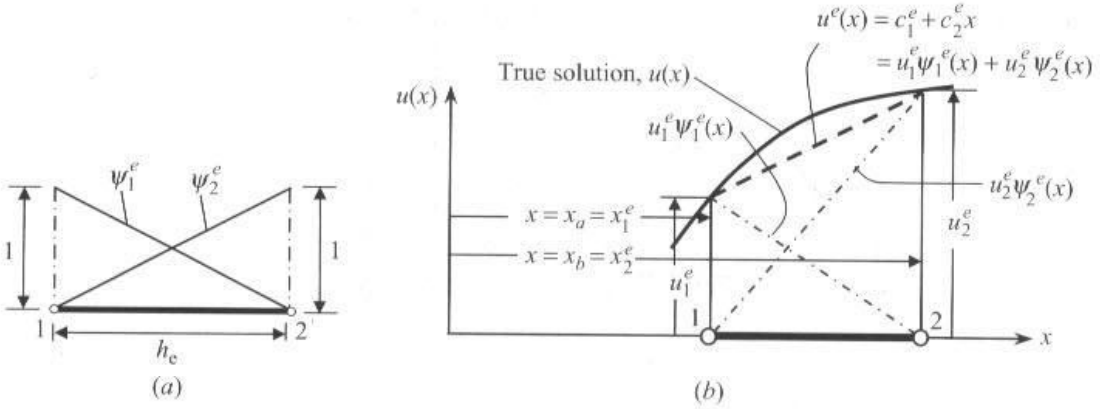

Figure 3.2.4 (*a*) Linear interpolation functions. (*b*) Linear finite element approximation.

where $x_1^e = x_a$ and $x_2^e = x_b$ [see Eq. (3.2.15)]. The functions ψ_i^e are shown in Fig. 3.2.4(*a*) and $u_h^e(x)$ is shown in Fig. 3.2.4(*b*). Although properties (3.2.18*a*) and (3.2.18*b*) are verified for the linear Lagrange interpolation functions, they hold for Lagrange interpolation functions of all degrees.

The element interpolation functions ψ_i^e in (3.2.17) were derived in terms of the *global coordinate* x [i.e., the coordinate appearing in the governing differential equation (3.2.1)], but they are defined only on the element domain $\Omega_e = (x_a, x_b)$. If we choose to express them in terms of a coordinate $\bar{x}$ (convenient in evaluating integrals of ψ_i^e), with the origin fixed at node 1 of the element Ω_e, the functions ψ_i^e of (3.2.17) in terms of $\bar{x}$ are

$$\psi_1^e(\bar{x}) = 1 - \frac{\bar{x}}{h_e}, \quad \psi_2^e(\bar{x}) = \frac{\bar{x}}{h_e} \tag{3.2.19}$$

The coordinate $\bar{x}$ is termed the *local* or *element coordinate* [see Fig. 3.2.3(*a*)].

The interpolation functions ψ_i^e are derived systematically: Starting with an assumed degree of a complete algebraic polynomial for the primary variable u and determining the coefficients c_i^e of the polynomial in terms of the values u_i^e of the primary variable u at the element nodes, the primary variable is finally expressed as a linear combination of approximation functions $\psi_i^e(x)$ and the nodal values u_i^e, which are called the element *nodal degrees of freedom*. The key point in this procedure is the selection of the number and the location of nodes in the element so that the geometry of the element is uniquely defined and interelement continuity may be easily imposed. The number of nodes must be sufficient to allow the rewriting of the coefficients in the assumed polynomial in terms of the primary variables. For a linear polynomial approximation, two nodes with one primary variable per node are sufficient to rewrite the polynomial in terms of the values of the primary variable at the two nodes and also define the geometry of the element, provided the two nodes are the end points of the element.

The degree (or order) of the polynomial approximation can be increased to improve the accuracy (see Fig. 3.2.5). To illustrate the derivation of the interpolation functions of higher order, we consider the quadratic approximation of $u(x)$

$$u_h^e(x) = c_1^e + c_2^e x + c_3^e x^2 \tag{3.2.20}$$

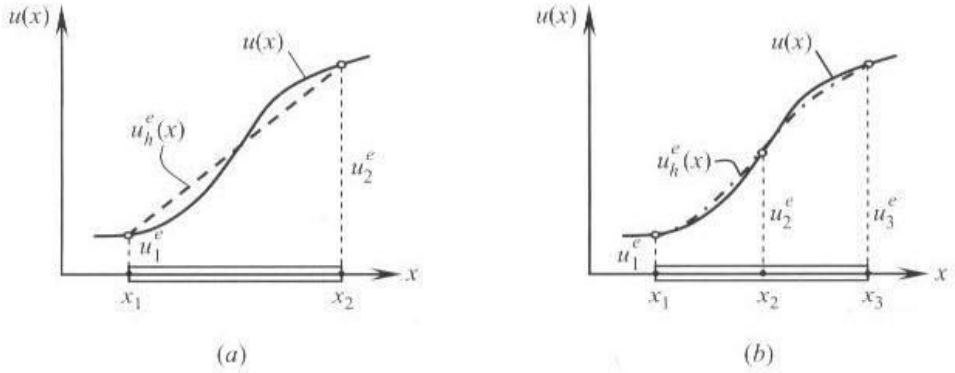

Figure 3.2.5 (*a*) Linear approximation. (*b*) Quadratic approximation.

Since there are three parameters c_i^e $(i = 1, 2, 3)$, we must identify three nodes in the element so that the three parameters can be expressed in terms of the three nodal values u_i^e. Two of the nodes are identified as the endpoints of the element to define its geometry, and the third node is taken interior to the element. In theory, the third node can be placed at any interior point. However, the midpoint of the element, being equidistant from the end nodes, is the best choice. Other choices are dictated by special considerations (e.g., to have a certain degree of singularity in the derivative of the solution), and we will not discuss such special cases here. Thus, we identify three nodes, two at the ends and one in the middle, of the element of length h_e [see Fig. 3.2.5(*b*)]. Following the procedure outlined for the linear polynomial, we eliminate c_i^e by rewriting $u_h^e(x)$ in terms of the three nodal values, (u_1^e, u_2^e, u_3^e). The three relations among c_i^e and u_i^e are

$$\begin{aligned} u_1^e &\equiv u_h^e(x_1^e) = c_1^e + c_2^e x_1^e + c_3^e (x_1^e)^2 \\ u_2^e &\equiv u_h^e(x_2^e) = c_1^e + c_2^e x_2^e + c_3^e (x_2^e)^2 \\ u_3^e &\equiv u_h^e(x_3^e) = c_1^e + c_2^e x_3^e + c_3^e (x_3^e)^2 \end{aligned} \tag{3.2.21a}$$

or, in matrix form,

$$\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \begin{bmatrix} 1 & x_1^e & (x_1^e)^2 \\ 1 & x_2^e & (x_2^e)^2 \\ 1 & x_3^e & (x_3^e)^2 \end{bmatrix} \begin{Bmatrix} c_1^e \\ c_2^e \\ c_3^e \end{Bmatrix} \tag{3.2.21b}$$

where x_i^e $(i = 1, 2, 3)$ is the global coordinate of the ith node of the element Ω_e $(x_1^e = x_a$, $x_2^e = x_a + 0.5h_e$, and $x_3^e = x_b)$. Inverting (3.2.21*b*), we obtain

$$\begin{aligned} c_1^e &= \frac{1}{D^e} \sum_{i=1}^{3} \alpha_i^e u_i^e, \quad \alpha_i^e = x_j^e (x_k^e)^2 - x_k^e (x_j^e)^2 \\ c_2^e &= \frac{1}{D^e} \sum_{i=1}^{3} \beta_i^e u_i^e, \quad \beta_i^e = (x_j^e)^2 - (x_k^e)^2 \\ c_3^e &= \frac{1}{D^e} \sum_{i=1}^{3} \gamma_i^e u_i^e, \quad \gamma_i^e = -(x_j^e - x_k^e), \quad D^e = \sum_{i=1}^{3} \alpha_i^e \end{aligned} \tag{3.2.22}$$

where D^e denotes the determinant of the matrix in (3.2.21*b*), and α_i^e, β_i^e, and γ_i^e are defined in (3.2.22) in terms of the nodal coordinates. The subscripts used in (3.2.22) permute in a natural order:

$$\begin{aligned} &\text{if} \quad i=1, \quad \text{then} \quad j=2 \quad \text{and} \quad k=3 \\ &\text{if} \quad i=2, \quad \text{then} \quad j=3 \quad \text{and} \quad k=1 \\ &\text{if} \quad i=3, \quad \text{then} \quad j=1 \quad \text{and} \quad k=2 \end{aligned} \tag{3.2.23}$$

For example, α_2^e, β_3^e, and γ_1^e are given by

$$\alpha_2^e = x_3^e (x_1^e)^2 - x_1^e (x_3^e)^2, \quad \beta_3^e = (x_1^e)^2 - (x_2^e)^2, \quad \gamma_1^e = x_3^e - x_2^e$$

Note that if x_2^e is very close to either x_1^e or x_3^e (i.e., node 2 is placed close to node 1 or node 3), making the rows of the coefficient matrix in (3.2.21) nearly the same, which makes the matrix nearly singular and not invertable.

Substituting for c_i^e from (3.2.22) into (3.2.20) and collecting the coefficients of u_1^e, u_2^e, and u_3^e separately, we obtain

$$u_h^e(x) = \psi_1^e(x) u_1^e + \psi_2^e(x) u_2^e + \psi_3^e(x) u_3^e = \sum_{j=1}^{3} \psi_j^e(x) u_j^e \tag{3.2.24}$$

where ψ_j^e are the *quadratic Lagrange interpolation functions,*

$$\psi_i^e(x) = \frac{1}{D^e}(\alpha_i^e + \beta_i^e x + \gamma_i^e x^2) \quad (i = 1, 2, 3) \tag{3.2.25}$$

Once again, the quadratic interpolation functions can be expressed in terms of the *local coordinate* $\bar{x}$. When the interior node, node 2, is placed at a distance $\bar{x} = \alpha h_e$, $0 < \alpha < 1$, the quadratic interpolation functions are given by

$$\begin{aligned} \psi_1^e(\bar{x}) &= \left(1 - \frac{\bar{x}}{h}\right)\left(1 - \frac{1}{\alpha}\frac{\bar{x}}{h}\right) \\ \psi_2^e(\bar{x}) &= \frac{1}{\alpha(1-\alpha)}\frac{\bar{x}}{h}\left(1 - \frac{\bar{x}}{h}\right) \\ \psi_3^e(\bar{x}) &= -\frac{\alpha}{(1-\alpha)}\frac{\bar{x}}{h}\left(1 - \frac{1}{\alpha}\frac{\bar{x}}{h}\right) \end{aligned} \tag{3.2.26}$$

For $\alpha = \frac{1}{2}$, i.e., when node 2 is placed at the midpoint of the element, the interpolation functions in (3.2.26) become

$$\begin{aligned} \psi_1^e(\bar{x}) &= \left(1 - \frac{\bar{x}}{h}\right)\left(1 - \frac{2\bar{x}}{h}\right) \\ \psi_2^e(\bar{x}) &= 4\frac{\bar{x}}{h}\left(1 - \frac{\bar{x}}{h}\right) \\ \psi_3^e(\bar{x}) &= -\frac{\bar{x}}{h}\left(1 - \frac{2\bar{x}}{h}\right) \end{aligned} \tag{3.2.27}$$

Plots of the quadratic interpolation functions are given in Fig. 3.2.6. The function ψ_i^e is equal to 1 at node i and zero at the other two nodes, but varies quadratically between the nodes.

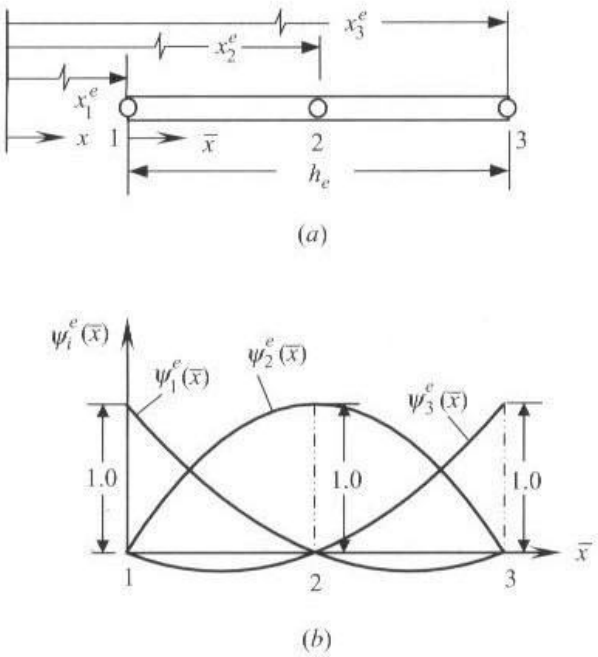

Figure 3.2.6 One-dimensional Lagrange quadratic element and its interpolation functions: (*a*) geometry of the element; (*b*) interpolation functions.

The interpolation properties (3.2.18*a*) and (3.2.18*b*) can be used to construct the Lagrange interpolation functions of any degree. For example, the quadratic interpolation functions (3.2.27) can be derived using the interpolation property (3.2.18*a*). Since $\psi_1^e(\bar{x})$ must vanish at nodes 2 and 3, i.e., at $\bar{x} = \frac{1}{2}h_e$ and $x = h_e$, respectively, $\psi_1^e(\bar{x})$ is of the form

$$\psi_1^e(\bar{x}) = C_1(\bar{x} - \frac{1}{2}h_e)(\bar{x} - h_e)$$

The constant C_1 is determined such that ψ_1^e is equal to 1 at $\bar{x} = 0$:

$$1 = C_1(0 - \frac{1}{2}h_e)(0 - h_e) \quad \text{or} \quad C_1 = 2/h_e^2$$

This gives

$$\psi_1^e(\bar{x}) = \frac{2}{h_e^2}(\bar{x} - \frac{1}{2}h_e)(\bar{x} - h_e) = \left(1 - \frac{\bar{x}}{h_e}\right)\left(1 - \frac{2\bar{x}}{h_e}\right)$$

which is identical to that in (3.2.27). The other two interpolations functions can be derived in a similar manner.

Although a detailed discussion is presented here on how to construct the Lagrange interpolation functions for one-dimensional elements, they are readily available in books on numerical analysis, and their derivation is independent of the physics of the problem to be solved. Their derivation depends only on the geometry of the element and the number and location of the nodes. The number of nodes must be equal to the number of terms in the polynomial. Thus, the interpolation functions derived above are useful not only in the finite element approximation of the problem at hand but also in all problems that admit

Lagrange interpolation of the variables, i.e., all problems for which the primary variables are the dependent unknowns—not their derivatives.

Step 3. Finite Element Model

The weak form (3.2.7) or (3.2.9) is equivalent to the differential equation (3.2.1) over the element Ω_e, and it also contains the natural boundary conditions (3.2.6). Further, the finite element approximations (3.2.16) and (3.2.24) are the interpolants of the solution. The substitution of (3.2.16) or (3.2.24) into (3.2.7) will give the necessary algebraic equations among the nodal values u_i^e and Q_i^e of the element Ω_e. In order to develop the finite element model based on the weak form (3.2.7), it is *not* necessary to decide *a priori* the degree of approximation of u_h^e. The finite element model can be developed for an arbitrary degree of interpolation:

$$u \approx u_h^e = \sum_{j=1}^{n} u_j^e \psi_j^e(x) \tag{3.2.28}$$

where ψ_j^e are the Lagrange interpolation functions of degree $n-1$. For $n > 2$, the weak form in (3.2.7) must be modified to include nonzero secondary variables, if any, at interior nodes. This modification is discussed next.

The integration by parts in Step 2 [see Eq. (3.2.5)] of the weak-form development for an element with interior nodes is carried out by intervals $(x_1^e, x_2^e), (x_2^e, x_3^e), \ldots, (x_{n-1}^e, x_n^e)$:

$$\begin{aligned}
0 &= \sum_{i=1}^{n-1} \left\{ \int_{x_i^e}^{x_{i+1}^e} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf \right) dx - \left[w(x) a \frac{du}{dx} \right]_{x_i^e}^{x_{i+1}^e} \right\} \\
&= \int_{x_1^e}^{x_n^e} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf \right) dx - w(x_1^e) \left(-a\frac{du}{dx} \right)_{x_1^e} - w(x_2^e) \left(a\frac{du}{dx} \right)_{x_2^e} \\
&\quad - w(x_2^e) \left(-a\frac{du}{dx} \right)_{x_2^e} - w(x_3^e) \left(a\frac{du}{dx} \right)_{x_3^e} - \cdots \\
&\quad - w(x_{n-1}^e) \left(-a\frac{du}{dx} \right)_{x_{n-1}^e} - w(x_n^e) \left(a\frac{du}{dx} \right)_{x_n^e} \\
&= \int_{x_1^e}^{x_n^e} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf \right) dx - w(x_1^e) \left(-a\frac{du}{dx} \right)_{x_1^e} \\
&\quad - w(x_2^e) \left[\left(a\frac{du}{dx} \right)_{x_2^{e-}} + \left(-a\frac{du}{dx} \right)_{x_2^{e+}} \right] - w(x_3^e) \left[\left(a\frac{du}{dx} \right)_{x_3^{e-}} + \left(-a\frac{du}{dx} \right)_{x_3^{e+}} \right] \\
&\quad \cdots - w(x_{n-1}^e) \left[\left(a\frac{du}{dx} \right)_{x_{n-1}^{e-}} + \left(-a\frac{du}{dx} \right)_{x_{n-1}^{e+}} \right] - w(x_n^e) \left(a\frac{du}{dx} \right)_{x_n^e} \\
&= \int_{x_1^e}^{x_n^e} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf \right) dx - w(x_1^e) Q_1^e - w(x_2^e) Q_2^e - w(x_3^e) Q_3^e \\
&\quad - \cdots - w(x_{n-1}^e) Q_{n-1}^e - w(x_n^e) Q_n^e
\end{aligned} \tag{3.2.29a}$$

where x_i^{e-} and x_i^{e+} denote the left and right sides, respectively, of node i, and

$$Q_1^e = \left(-a\frac{du}{dx}\right)_{x_1^e}, \quad Q_2^e = \left[\left(a\frac{du}{dx}\right)_{x_2^{e-}} + \left(-a\frac{du}{dx}\right)_{x_2^{e-}}\right]$$

$$\vdots \tag{3.2.29b}$$

$$Q_{n-1}^e = \left[\left(a\frac{du}{dx}\right)_{x_{n-1}^{e-}} + \left(-a\frac{du}{dx}\right)_{x_{n-1}^{e-}}\right], \quad Q_n^e = \left(a\frac{du}{dx}\right)_{x_n^e}$$

Thus, Q_i^e, $i = 2, 3, \ldots, n-1$, denotes the jump in the value of the secondary variable in going from the left to the right of the ith node. This value is zero if no external source is applied at the node. Thus, for an element with n nodes, the weak form becomes

$$0 = \int_{x_a}^{x_b} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu\right) dx - \int_{x_a}^{x_b} wq\, dx - \sum_{j=1}^{n} w(x_j^e) Q_j^e \tag{3.2.30}$$

Note that $Q_1^e = Q_a^e$ and $Q_n^e = Q_b^e$ represent the unknown point sources at the end nodes, and all other Q_i^e ($i = 2, 3, \ldots, n-1$) are the specified point sources, if any, at the interior nodes.

Next, we develop the finite element model of Eq. (3.2.1) when the $(n-1)$st-degree Lagrange polynomials are used to approximate $u(x)$. Following the Ritz procedure developed in Section 2.5.2, we substitute (3.2.28) for u and $w = \psi_1^e$, $w = \psi_2^e$, $\ldots$, $w = \psi_n^e$ into the weak form (3.2.30) to obtain n algebraic equations:

$$0 = \int_{x_a}^{x_b} \left[a\frac{d\psi_1^e}{dx}\left(\sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx}\right) + c\psi_1^e\left(\sum_{j=1}^{n} u_j^e \psi_j^e(x)\right) - \psi_1^e f\right] dx - \sum_{j=1}^{n} \psi_1^e(x_j^e) Q_j^e$$

(1st equation)

$$0 = \int_{x_a}^{x_b} \left[a\frac{d\psi_2^e}{dx}\left(\sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx}\right) + c\psi_2^e\left(\sum_{j=1}^{n} u_j^e \psi_j^e(x)\right) - \psi_2^e f\right] dx - \sum_{j=1}^{n} \psi_2^e(x_j^e) Q_j^e$$

(2nd equation)

$\vdots$

$$0 = \int_{x_a}^{x_b} \left[a\frac{d\psi_i^e}{dx}\left(\sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx}\right) + c\psi_i^e\left(\sum_{j=1}^{n} u_j^e \psi_j^e(x)\right) - \psi_i^e f\right] dx - \sum_{j=1}^{n} \psi_i^e(x_j^e) Q_j^e$$

(ith equation)

$\vdots$

$$0 = \int_{x_a}^{x_b} \left[a\frac{d\psi_n^e}{dx}\left(\sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx}\right) + c\psi_n^e\left(\sum_{j=1}^{n} u_j^e \psi_j^e(x)\right) - \psi_n^e f\right] dx - \sum_{j=1}^{n} \psi_n^e(x_j^e) Q_j^e$$

(nth equation)

The ith algebraic equation of the system of n equations can be written as

$$0 = \sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - Q_i^e \quad (i = 1, 2, \ldots, n) \tag{3.2.31a}$$

where

$$K_{ij}^e = B^e(\psi_i^e, \psi_j^e) = \int_{x_a}^{x_b} \left(a \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + c\psi_i^e \psi_j^e \right) dx$$

$$f_i^e = \int_{x_a}^{x_b} f \psi_i^e \, dx \tag{3.2.31b}$$

Note that the interpolation property (3.2.18a) is used to write

$$\sum_{j=1}^{n} \psi_i^e(x_j^e) Q_j^e = Q_i^e \tag{3.2.32}$$

Equations (3.2.31a) can be expressed in terms of the coefficients K_{ij}^e, f_i^e, and Q_i^e as

$$\begin{aligned} K_{11}^e u_1^e + K_{12}^e u_2^e + \cdots + K_{1n}^e u_n^e &= f_1^e + Q_1^e \\ K_{21}^e u_1^e + K_{22}^e u_2^e + \cdots + K_{2n}^e u_n^e &= f_2^e + Q_2^e \\ &\vdots \\ K_{n1}^e u_1^e + K_{n2}^e u_2^e + \cdots + K_{nn}^e u_n^e &= f_n^e + Q_n^e \end{aligned} \tag{3.2.33a}$$

In matrix notation, the linear algebraic equations (3.2.33a) can be written as

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\} \quad \text{or} \quad \mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e + \mathbf{Q}^e \tag{3.2.33b}$$

The matrix $\mathbf{K}^e$ is called the *coefficient matrix,* or *stiffness matrix* in structural mechanics applications. The column vector $\mathbf{f}^e$ is the *source vector,* or *force vector* in structural mechanics problems. Note that Eq. (3.2.33) contains $2n$ unknowns: $(u_1^e, u_2^e, \ldots, u_n^e)$ and $(Q_1^e, Q_2^e, \ldots, Q_n^e)$, called primary and secondary element *nodal degrees of freedom*; hence, it cannot be solved without having an additional n conditions. Some of these are provided by the boundary conditions and the remainder by "balance" or equilibrium of the secondary variables Q_i^e at nodes common to several elements. This balance can be implemented by putting the elements together (i.e., assembling the element equations). Upon assembly and imposition of boundary conditions, we shall obtain exactly the same number of algebraic equations as the number of unknown primary and secondary degrees of freedom. The ideas underlying the assembly procedure are discussed in the next section.

The coefficient matrix $[K^e]$, which is symmetric, and source vector $\{f^e\}$ can be evaluated for a given element and data (a, c, and f). For elementwise-constant values of a, c, and f (say, a_e, c_e, and f_e) the coefficients K_{ij}^e and f_i^e can easily be evaluated for a typical element, as discussed next.

Linear Element. For a mesh of linear elements, the element Ω_e is located between the global nodes $x_a = x_e$ and $x_b = x_{e+1}$ [see Fig. 3.2.2(b)]. Hence,

$$K_{ij}^e = \int_{x_e}^{x_{e+1}} \left(a_e \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + c_e \psi_i^e \psi_j^e \right) dx, \quad f_i^e = \int_{x_e}^{x_{e+1}} f_e \psi_i^e \, dx$$

or, in the local coordinate system $\bar{x}$,

$$K_{ij}^e = \int_0^{h_e} \left(a_e \frac{d\psi_i^e}{d\bar{x}} \frac{d\psi_j^e}{d\bar{x}} + c_e \psi_i^e \psi_j^e \right) d\bar{x}, \quad f_i^e = \int_0^{h_e} f_e \psi_i^e \, d\bar{x}$$

where $x = x_1^e + \bar{x}$ and

$$dx = d\bar{x}, \quad \frac{d\psi_i^e}{dx} = \frac{d\psi_i^e}{d\bar{x}}$$

The ψ_i^e can be expressed in terms of $\bar{x}$ as [see (3.2.19)]

$$\psi_1^e(\bar{x}) = 1 - \bar{x}/h_e, \quad \psi_2^e(\bar{x}) = \bar{x}/h_e$$

We can compute K_{ij}^e and f_i^e by evaluating the integrals. We have

$$\begin{aligned} K_{11}^e &= \int_0^{h_e} \left[a_e \left(-\frac{1}{h_e} \right) \left(-\frac{1}{h_e} \right) + c_e \left(1 - \frac{\bar{x}}{h_e} \right) \left(1 - \frac{\bar{x}}{h_e} \right) \right] d\bar{x} \\ &= \frac{a_e}{h_e} + \frac{1}{3} c_e h_e \\ K_{12}^e &= \int_0^{h_e} \left[a_e \left(-\frac{1}{h_e} \right) \frac{1}{h_e} + c_e \left(1 - \frac{\bar{x}}{h_e} \right) \frac{\bar{x}}{h_e} \right] d\bar{x} \\ &= -\frac{a_e}{h_e} + \frac{1}{6} c_e h_e = K_{21}^e \quad \text{(by symmetry)} \\ K_{22}^e &= \int_0^{h_e} \left(a_e \frac{1}{h_e} \frac{1}{h_e} + c_e \frac{\bar{x}}{h_e} \frac{\bar{x}}{h_e} \right) d\bar{x} = \frac{a_e}{h_e} + \frac{1}{3} c_e h_e \end{aligned}$$

Similarly,

$$f_1^e = \int_0^{h_e} f_e \left(1 - \frac{\bar{x}}{h_e} \right) d\bar{x} = \frac{1}{2} f_e h_e, \quad f_2^e = \int_0^{h_e} f_e \frac{\bar{x}}{h_e} \, d\bar{x} = \frac{1}{2} f_e h_e$$

Thus, for constant f_e, the total source $f_e h_e$ is equally distributed to the two nodes. The coefficient matrix and column vector are

$$[K^e] = \frac{a_e}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \{f^e\} = \frac{f_e h_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{3.2.34}$$

If $a = a_e x$ and $c = c_e$, the coefficient matrix $[K^e]$ can be evaluated as

$$[K^e] = \frac{a_e}{h_e} \left(\frac{x_e + x_{e+1}}{2} \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{3.2.35}$$

The reader should verify this. Note that when a is a linear function of x, this is equivalent to replacing a in the coefficient matrix with its average value [compare (3.2.34) with (3.2.35)]:

$$a_{\text{avg}} = \frac{1}{2}(x_e + x_{e+1}) a_e$$

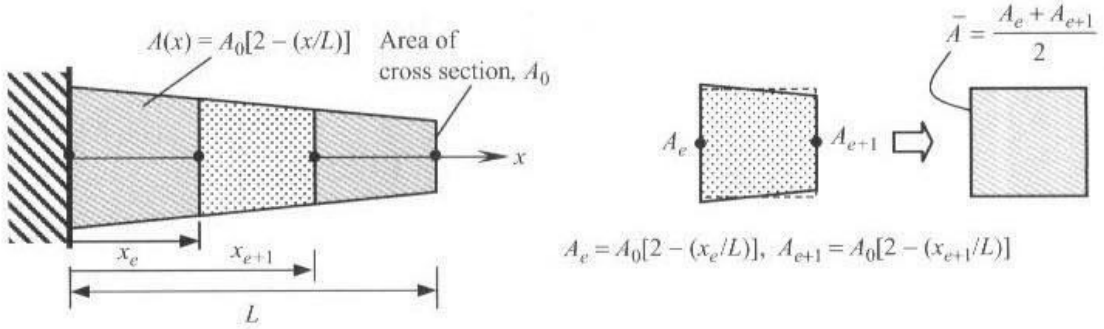

Figure 3.2.7 Approximation of an element with linearly varying cross section by an equivalent element of constant cross section.

For example, in the study of bars with linearly varying cross section (see Fig. 3.2.7)

$$a = EA(x) = E\left(A_e + \frac{A_{e+1} - A_e}{h_e}\bar{x}\right)$$

this amounts to replacing the linearly varying cross section with a constant cross section within each element, the cross-sectional area of the constant section being the average area of the linearly varying element. Here A_e denotes the cross-sectional area at x_e and A_{e+1} is the area at $x = x_{e+1}$.

Quadratic Element. For a mesh of quadratic elements, element Ω_e is located between global nodes $x_a = x_{2e-1}$ and $x_b = x_{2e+1}$. Hence,

$$K_{ij}^e = \int_{x_{2e-1}}^{x_{2e+1}} \left(a_e \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} + c_e \psi_i^e \psi_j^e\right) dx$$

$$= \int_0^{h_e} \left(a_e \frac{d\psi_i^e}{d\bar{x}}\frac{d\psi_j^e}{d\bar{x}} + c_e \psi_i^e \psi_j^e\right) d\bar{x} \tag{3.2.36a}$$

$$f_i^e = \int_{x_{2e-1}}^{x_{2e-1}} \psi_i^e f_e \, dx = \int_0^{h_e} \psi_i^e f_e \, d\bar{x} \tag{3.2.36b}$$

where $\psi_i^e(\bar{x})$ $(i = 1,2,3)$ are the Lagrange quadratic interpolation functions in (3.2.27). Evaluating the integrals in (3.2.36a) and (3.2.36b), we obtain

$$K_{11}^e = \int_0^{h_e} \left\{a_e \left(-\frac{3}{h_e} + \frac{4\bar{x}}{h_e^2}\right)\left(-\frac{3}{h_e} + \frac{4\bar{x}}{h_e^2}\right)\right.$$

$$\left. + c_e \left[1 - \frac{3\bar{x}}{h_e} + 2\left(\frac{\bar{x}}{h_e}\right)^2\right]\left[1 - \frac{3\bar{x}}{h_e} + 2\left(\frac{\bar{x}}{h_e}\right)^2\right]\right\} d\bar{x}$$

$$= \frac{7}{3}\frac{a_e}{h_e} + \frac{2}{15}c_e h_e$$

$$K_{12}^e = \int_0^{h_e} \left\{ a_e \left(-\frac{3}{h_e} + \frac{4\bar{x}}{h_e^2} \right) \left(\frac{4}{h_e} - \frac{8\bar{x}}{h_e} \right) + c_e \left[1 - \frac{3\bar{x}}{h_e} + 2\left(\frac{\bar{x}}{h_e} \right)^2 \right] \left[4\frac{\bar{x}}{h_e} \left(1 - \frac{\bar{x}}{h_e} \right) \right] \right\} d\bar{x}$$

$$= -\frac{8}{3}\frac{a_e}{h_e} + \frac{2}{30} c_e h_e = K_{21}^e$$

$$K_{22}^e = \int_0^{h_e} \left\{ a_e \left(\frac{4}{h_e} - \frac{8\bar{x}}{h_e} \right) \left(\frac{4}{h_e} - \frac{8\bar{x}}{h_e} \right) + c_e \left[4\frac{\bar{x}}{h_e} \left(1 - \frac{\bar{x}}{h_e} \right) \right] \left[4\frac{\bar{x}}{h_e} \left(1 - \frac{\bar{x}}{h_e} \right) \right] \right\} d\bar{x}$$

$$= \frac{7}{3}\frac{a_e}{h_e} + \frac{2}{15} c_e h_e$$

and so on. Similarly,

$$f_1^e = \int_0^{h_e} f_e \left[1 - \frac{3\bar{x}}{h_e} + 2\left(\frac{\bar{x}}{h_e} \right)^2 \right] d\bar{x} = \frac{1}{6} f_e h_e$$

$$f_2^e = \int_0^{h_e} f_e \left[4\frac{\bar{x}}{h_e} \left(1 - \frac{\bar{x}}{h_e} \right) \right] d\bar{x} = \frac{4}{6} f_e h_e$$

$$f_3^e = f_1^e \quad \text{(by symmetry)}$$

Thus, the element coefficient matrix and source vector for a quadratic element are

$$[K^e] = \frac{a_e}{3h_e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \tag{3.2.37a}$$

$$\{f^e\} = \frac{f_e h_e}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} \tag{3.2.37b}$$

Note that, for quadratic elements, the total value of the source $f_e h_e$ is not distributed equally between the three nodes. The distribution is *not* equivalent to that of two linear elements of length $\frac{1}{2}h_e$. The computation of f_i^e should be based on the interpolation functions of that element. The sum of f_i^e for *any element* should always be equal to the integral of $f(x)$ over the element:

$$\sum_{i=1}^{n} f_i^e = \int_{x_a}^{x_b} f(x)\, dx \tag{3.2.38}$$

In summary, for elementwise-constant values of the data a, c, and f, the element matrices of a linear and quadratic elements are

Linear Element

$$\left(\frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \frac{f_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{3.2.39}$$

Quadratic Element

$$\left(\frac{a_e}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}\right)\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \frac{f_e h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \end{Bmatrix} \tag{3.2.40}$$

When the coefficient $c_e = 0$, the corresponding contribution to the above equations should be omitted.

When a, c, and f are algebraic polynomials in x, the evaluation of K_{ij}^e and f_j^e is straightforward. When they are complicated functions of x, the integrals in $[K^e]$ and $\{f^e\}$ are evaluated using numerical integration. A complete discussion of numerical integration will be presented in Chapter 7.

3.2.4 Connectivity of Elements

In deriving the element equations, we isolated a typical element (the eth element) from the mesh and formulated the variational problem (or weak form) and developed its finite element model. The finite element model of a typical element contains n equations among $2n$ unknowns, $(u_1^e, u_2^e, \ldots, u_n^e)$ and $(Q_1^e, Q_2^e, \ldots, Q_n^e)$. Hence, they cannot be solved without using the equations from other elements to get rid of extra unknowns. From a physical point of view, this makes sense because we should not be able to solve the element equations without considering the complete set of elements and the boundary conditions of the total problem.

To obtain the finite element equations of the total problem, we must put the elements back into their original positions (undoing of what was done before formulating the discrete problem). In putting the elements with their nodal degrees of freedom back into their original positions, we must require that the solution $u(x)$ is uniquely defined (i.e., u is continuous) and their source terms Q_i^e are "balanced" at the points where elements are connected to each other. Of course, if the variable u is not continuous, we do not impose its continuity; but in all problems studied in this book, unless otherwise stated explicitly (like in the case of an internal hinge in the case of beam bending), the primary variables are assumed to be continuous. The continuity of the primary variables refers to the single-valued nature of the

solution; balance of secondary variables refers to the equilibrium of point sources. Thus, the assembly of elements is carried out by imposing the following two conditions:

1. If the node i of element Ω^e is connected to the node j of element Ω^f and node k of element Ω^g, the continuity of the primary variable u requires

$$u_i^{(e)} = u_j^{(f)} = u_k^{(g)} \tag{3.2.41}$$

When elements are connected in series, as shown in Fig. 3.2.8, the continuity of u requires

$$u_n^e = u_1^{e+1} \tag{3.2.42}$$

2. For the same three elements, the balance of secondary variables at connecting nodes requires

$$Q_i^{(e)} + Q_j^{(f)} + Q_k^{(g)} = Q_I \tag{3.2.43}$$

where I is the global node number assigned to the nodal point that is common to the three elements, and Q_I is the value of externally applied source, if any (otherwise zero), at this node [the sign of Q_I must be consistent with the sign of Q_i^e in Fig. 3.2.3(b)]. For the case shown in Fig. 3.2.8, we have

$$Q_n^e + Q_1^{e+1} = \begin{cases} 0 & \text{if no external point source is applied} \\ Q_I & \text{if an external point source of magnitude } Q_I \text{ is applied} \end{cases} \tag{3.2.44}$$

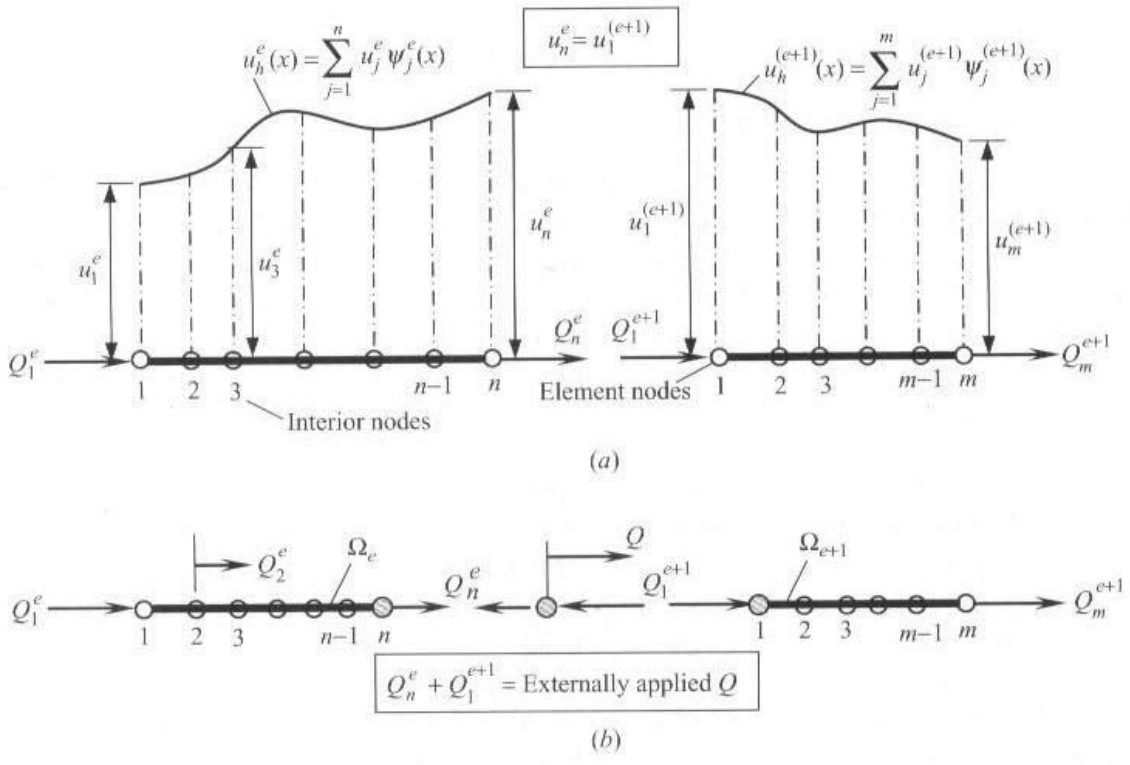

Figure 3.2.8 Assembly of two Lagrange elements: (a) continuity of the primary variable; (b) balance of the secondary variables.

The balance of secondary variables can be interpreted as the continuity of $a(du/dx)$ [not $a(du_h^e/dx)$] at the node (say, global node I) common to elements Ω_e and Ω_{e+1} when no change in adu/dx is imposed externally:

$$\left(a\frac{du}{dx}\right)_I^e = \left(a\frac{du}{dx}\right)_I^{e+1}$$

or

$$\left(a\frac{du}{dx}\right)_I^e + \left(-a\frac{du}{dx}\right)_I^{e+1} = 0 \rightarrow Q_n^e + Q_1^{e+1} = 0 \tag{3.3.45a}$$

If there is a discontinuity of magnitude Q_I in $a\frac{du}{dx}$ in going from one side of the node to the other side (in the positive x direction), we impose

$$\left(a\frac{du}{dx}\right)_I^e + \left(-a\frac{du}{dx}\right)_I^{e+1} = Q_I \rightarrow Q_n^e + Q_1^{e+1} = Q_I \tag{3.3.45b}$$

The interelement continuity of the primary variables can be imposed by simply renaming the variables of all elements connected to the common node. For the connection in Eq. (3.2.41), we simply use the name

$$u_i^{(e)} = u_j^{(f)} = u_k^{(g)} \equiv U_I \tag{3.2.46}$$

where I is the global node number at which the three elements are connected. For example, for a mesh of N linear finite elements ($n = 2$) connected in series [see Fig. 3.2.2(*b*)], we have

$$u_1^1 = U_1,\ u_2^1 = u_1^2 = U_2,\ u_2^2 = u_1^3 = U_3, \ldots, u_2^{N-1} = u_1^N = U_N,\ u_2^N = U_{N+1}$$

Figure 3.2.9 shows the connected finite element solution, u_h, composed of the element solutions u_h^e. From the connected solution, we can identify the *global interpolation functions* Φ_I, which can be defined in terms of the element interpolation functions ψ_i^e corresponding to the global node I, as shown in Fig. 3.2.9.

To enforce balance of the secondary variables, it is clear that we can set [see Eq. (3.2.44)] $Q_i^{(e)} + Q_j^{(f)} + Q_k^{(g)}$ equal to zero or to a specified value Q_I only if we have such an expression in the finite element equations. To obtain such expressions, it is clear that we must add the ith equation of the element Ω_e, the jth equation of element Ω_f, and the kth equation of element Ω_e. For the case shown in Fig. 3.2.8, the nth equation of element Ω_e must be added to the first equation of the element Ω_{e+1}, i.e., we add

$$\sum_{j=1}^{n} K_{nj}^e u_j^e = f_n^e + Q_n^e$$

and

$$\sum_{j=1}^{n} K_{1j}^{e+1} u_j^{e+1} = f_1^{e+1} + Q_1^{e+1}$$

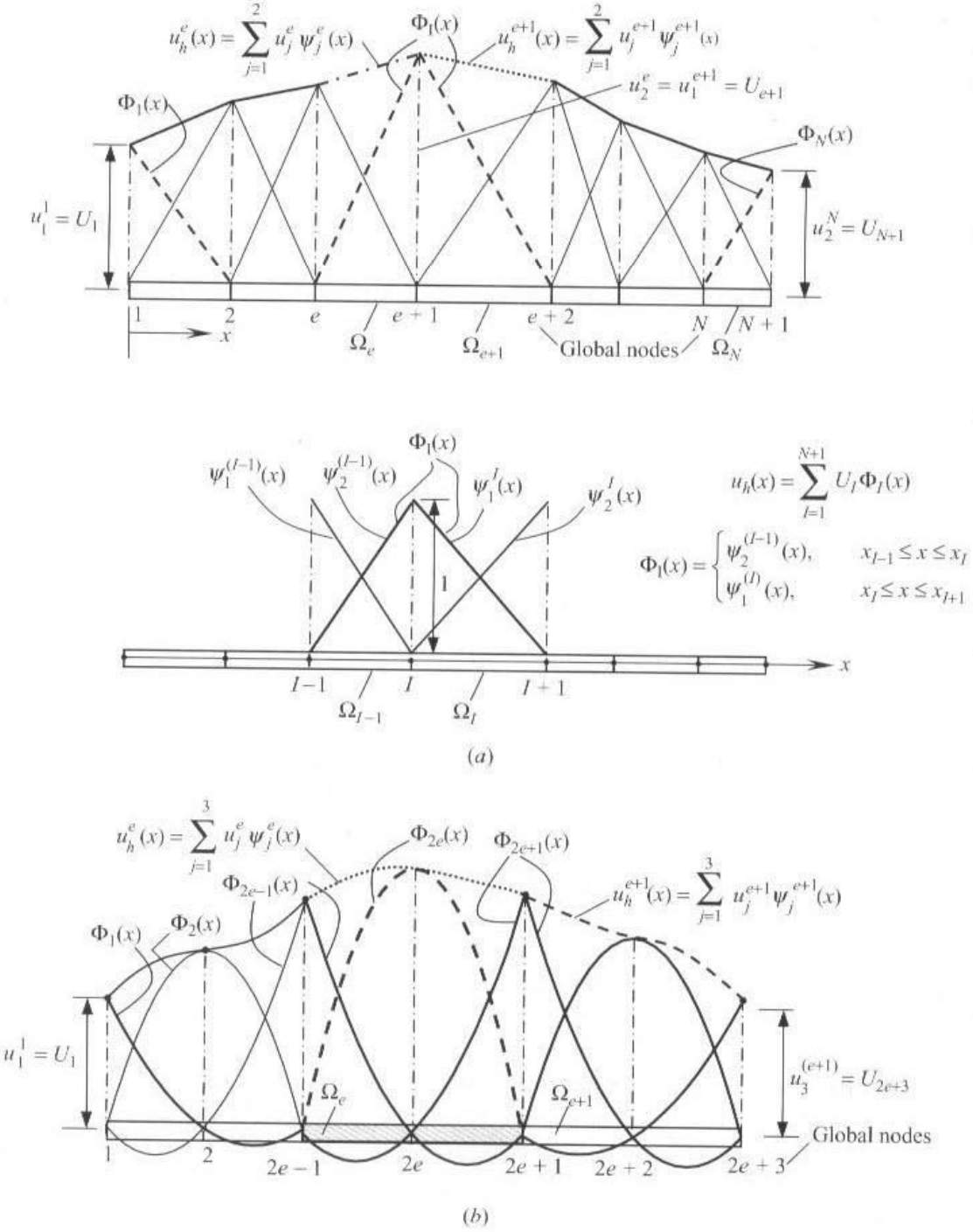

Figure 3.2.9 Global interpolation functions for the (*a*) linear elements and (*b*) quadratic elements.

to obtain

$$\sum_{j=1}^{n} \left(K_{nj}^e u_j^e + K_{1j}^{e+1} u_j^{e+1} \right) = f_n^e + f_1^{e+1} + \left(Q_n^e + Q_1^{e+1} \right)$$
$$= f_n^e + f_1^{e+1} + Q_I \tag{3.2.47}$$

This process reduces the number of equations from $2N$ to $N + 1$ in a mesh of N linear elements. The first equation of the first element and the last equation of the last element

will remain unchanged, except for renaming of the primary variables. The left-hand side of (3.2.47) can be written in terms of the global nodal values as

$$\begin{aligned}
&\left(K^e_{n1}\, u^e_1 + K^e_{n2}u^e_2 + \cdots + K^e_{nn}u^e_n\right) + \left(K^{e+1}_{11}u^{e+1}_1 + K^{e+1}_{12}u^{e+1}_2 + \cdots + K^{e+1}_{1n}u^{e+1}_n\right)\\
&\quad = \left(K^e_{n1}U_N + K^e_{n2}U_{N+1} + \cdots + K^e_{nn}U_{N+n-1}\right)\\
&\qquad + \left(K^{e+1}_{11}U_{N+n-1} + K^{e+1}_{12}U_{N+n} + \ldots + K^{e+1}_{1n}U_{N+2n-2}\right)\\
&\quad = K^e_{n1}U_N + K^e_{n2}U_{N+1} + \cdots + K^e_{n(n-1)}U_{N+n-2} + \left(K^e_{nn} + K^{e+1}_{11}\right)U_{N+n-1}\\
&\qquad + K^{e+1}_{12}U_{N+n} + \ldots + K^{e+1}_{1n}U_{N+2n-2}
\end{aligned} \tag{3.2.48}$$

where $N = (n-1)e + 1$. For a mesh of N linear elements ($n = 2$), we have

$$\begin{aligned}
K^1_{11}U_1 + K^1_{12}U_2 &= f^1_1 + Q^1_1 \quad \text{(unchanged)}\\
K^1_{21}U_1 + \left(K^1_{22} + K^2_{11}\right)U_2 + K^2_{12}U_3 &= f^1_2 + f^2_1 + Q^1_2 + Q^2_1\\
K^2_{21}U_2 + \left(K^2_{22} + K^3_{11}\right)U_3 + K^3_{12}U_4 &= f^2_2 + f^3_1 + Q^2_2 + Q^3_1\\
&\vdots\\
K^{N-1}_{21}U_{N-1} + \left(K^{N-1}_{22} + K^N_{11}\right)U_N + K^N_{12}U_{N+1} &= f^{N-1}_2 + f^N_1 + Q^{N-1}_2 + Q^N_1\\
K^N_{21}U_N + K^N_{22}U_{N+1} &= f^N_2 + Q^N_2 \quad \text{(unchanged)}
\end{aligned} \tag{3.2.49}$$

These are called the *assembled equations*. They contain the sum of coefficients and source terms at nodes common to two elements. Note that the numbering of the global equations corresponds to the numbering of the global primary degrees of freedom, U_I. This correspondence carries the symmetry of element matrices to the global matrix. Equations (3.2.49) can be expressed in matrix form as

$$\begin{bmatrix}
K^1_{11} & K^1_{12} & & & \\
K^1_{21} & K^1_{22} + K^2_{11} & K^2_{12} & \mathbf{0} & \\
 & K^2_{21} & K^2_{22} + K^3_{11} & & \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
 & \mathbf{0} & & K^{N-1}_{22} + K^N_{11} & K^N_{12}\\
 & & & K^N_{21} & K^N_{22}
\end{bmatrix}
\begin{Bmatrix} U_1\\ U_2\\ U_3\\ \vdots\\ U_N\\ U_{N+1} \end{Bmatrix}$$

$$= \begin{Bmatrix} f^1_1\\ f^1_2 + f^2_1\\ f^2_2 + f^3_1\\ \vdots\\ f^{N-1}_2 + f^N_1\\ f^N_2 \end{Bmatrix} + \begin{Bmatrix} Q^1_1\\ Q^1_2 + Q^2_1\\ Q^2_2 + Q^3_1\\ \vdots\\ Q^{N-1}_2 + Q^N_1\\ Q^N_2 \end{Bmatrix} \tag{3.2.50}$$

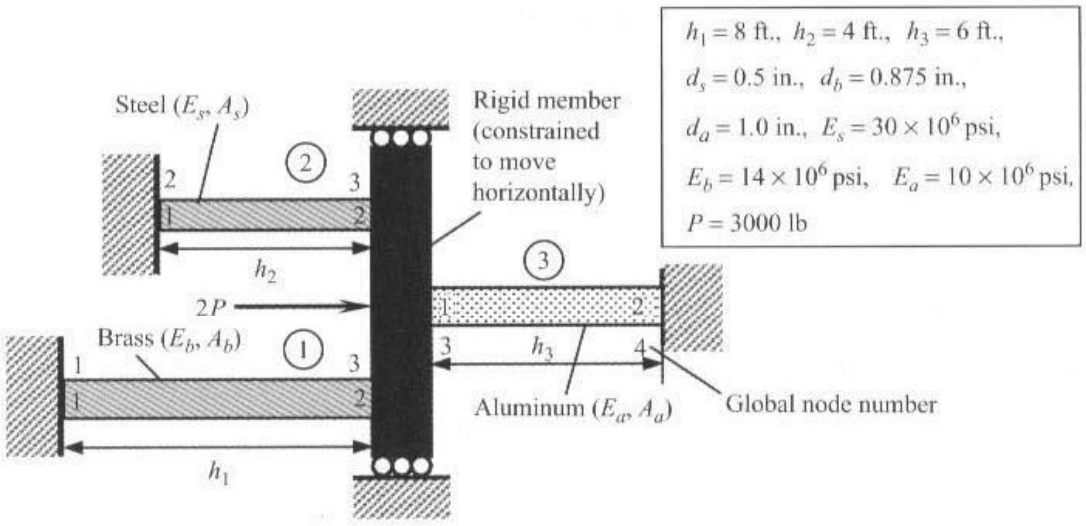

Figure 3.2.10 The geometry and finite element mesh of a three-bar structure.

Note that the discussion of assembly in Eqs. (3.2.49) and (3.2.50) is based on the assumption that elements are connected in series. In general, several elements can be connected at a global node, and the elements do not have to be consecutively numbered. In that case, the coefficients coming from all elements connected at that global node will add up. For example, consider the structure consisting of three bar elements shown in Fig. 3.2.10. Suppose that the connecting bar is rigid (i.e., not deformable) and is constrained to remain horizontal at all times. Then the continuity and force balance conditions for the structure are

$$u_2^1 = u_1^3 = u_2^2 \equiv U_3, \quad Q_2^1 + Q_1^3 + Q_2^2 = 2P \tag{3.2.51}$$

To enforce these conditions, we must add the second equation of element 1, the first equation of element 3, and the second equation of element 2:

$$\begin{aligned}\left(K_{21}^1 u_1^1 + K_{22}^1 u_2^1\right) + \left(K_{11}^3 u_1^3 + K_{12}^3 u_2^3\right) + \left(K_{21}^2 u_1^2 + K_{22}^2 u_2^2\right) \\ = f_2^1 + f_1^3 + f_2^2 + Q_2^1 + Q_1^3 + Q_2^2\end{aligned} \tag{3.2.52}$$

We note the following correspondence of local and global nodal values (see Fig. 3.2.10):

$$u_1^1 = U_1, \quad u_1^2 = U_2, \quad u_2^1 = u_1^3 = u_2^2 = U_3, \quad u_2^3 = U_4$$

Hence, (3.2.52) becomes

$$\begin{aligned}K_{21}^1 U_1 + K_{21}^2 U_2 + (K_{22}^1 + K_{11}^3 + K_{22}^2) U_3 + K_{12}^3 U_4 \\ = f_2^1 + f_1^3 + f_2^2 + Q_2^1 + Q_1^3 + Q_2^2 \\ = f_2^1 + f_1^3 + f_2^2 + 2P\end{aligned}$$

The other equations remain unchanged, except for renaming of the primary variables. The assembled equations are

$$\begin{bmatrix} K_{11}^1 & 0 & K_{12}^1 & 0 \\ 0 & K_{11}^2 & K_{12}^2 & 0 \\ K_{21}^1 & K_{21}^2 & \hat{K} & K_{12}^3 \\ 0 & 0 & K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_1^2 \\ f_2^1 + f_1^3 + f_2^2 \\ f_2^3 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^1 + Q_1^3 + Q_2^2 \\ Q_2^3 \end{Bmatrix} \tag{3.2.53}$$

where $\hat{K} = K_{22}^1 + K_{11}^3 + K_{22}^2$. Note that all f_i^e are zero for the problem in Fig. 3.2.10 because there is no distributed axial force ($f = 0$ for all elements).

The coefficients of the assembled matrix can be obtained directly. We note that the global coefficient K_{IJ} is a physical property of the system, relating global node I to global node J. For axial deformation of bars, K_{IJ} denotes the force required at node I to induce a unit displacement at node J, while the displacements at all other nodes are zero. Therefore, K_{IJ} is equal to the sum of all K_{ij}^e for which i corresponds to I and j corresponds to J, and i and j are the local nodes of the element Ω_e. Thus, if we have a correspondence between element node numbers and global node numbers, then the assembled global coefficients can readily be written in terms of the element coefficients. The correspondence can be expressed through a matrix $[B]$, called the *connectivity matrix,* whose coefficient b_{ij} has the following meaning: b_{ij} is the global node number corresponding to the jth node of element i. For example, for the structure shown in Fig. 3.2.10, the matrix $[B]$ is of order 3×2 (3 elements and 2 nodes per element):

$$[B] = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

This array can be used in a variety of ways—not only for assembly, but also in the computer implementation of finite element computations. The matrix $[B]$ is used to assemble coefficient matrices as follows:

$K_{11}^1 = K_{11}$ because local node 1 of element 1 corresponds to global node 1.
$K_{12}^1 = K_{13}$ because local nodes 1 and 2 of element 1 correspond to global nodes 1 and 3, respectively

and so on. When more than one element is connected at a global node, the element coefficients are to be added. For example, global node 3 appears in all three rows (i.e., elements) of the matrix $[B]$, implying that all three elements are connected to a global node 3. More specifically, it indicates that node 2 of element 1, node 2 of element 2, and node 1 of element 3 are the same as global node 3. Hence,

$$K_{22}^1 + K_{22}^2 + K_{11}^3 = K_{33}$$

Assembly by hand can be carried out by examining the finite element mesh of the problem. For the mesh shown in Fig. 3.2.10, we have

$K_{23} = K_{12}^2$ because global node 2 is the same as node 1 and global node 3 is the same as node 2 of element 2.

$K_{24} = 0$ because global nodes 2 and 4 do not belong to the same element
$K_{33} = K_{22}^1 + K_{22}^2 + K_{11}^3$

and so on. The problem can also be solved by numbering global nodes 1 and 2 to be the same node. This leads to a 3×3 global system of equations.

In summary, assembly of finite elements is carried out by imposing interelement continuity of primary variables and balance of secondary variables. Renaming the primary variables of an element in terms of the global primary variables and using the correspondence between the local and global nodes allows the assembly. When certain primary nodal values are not required to be continuous across elements (as dictated by physics or the variational formulation of the problem), such variables may be *condensed* out at the element level before assembling the elements.

3.2.5 Imposition of Boundary Conditions

The discussion in Section 3.2.1–3.2.4 is valid for any differential equation that is a special case of the model equation (3.2.1). Each problem differs from the other in the specification of the data (a, c, f) and boundary conditions on the primary and secondary variables (u, Q). Here we discuss how to impose the boundary conditions of a problem on the assembled set of finite element (algebraic) equations. To this end, we use the problem in Fig. 3.2.10. The boundary conditions of the problem are evident, at least for an engineering student, from the structure shown. The known primary degrees of freedom (i.e., displacements) are

$$u_1^1 = U_1 = 0, \quad u_1^2 = U_2 = 0, \quad u_2^3 = U_4 = 0 \tag{3.3.54a}$$

The known secondary degrees of freedom (i.e., forces) are

$$Q_2^1 + Q_2^2 + Q_1^3 = 2P \tag{3.3.54b}$$

The forces Q_1^1, Q_1^2, and Q_2^3 are unknown (reaction forces), and they can be determined after the primary degrees of freedom are determined.

Imposing the boundary conditions (3.2.54*a*) and (3.2.54*b*) on the assembled system of equations (3.2.53) with $f_i^e = 0$, we obtain

$$\begin{bmatrix} K_{11}^1 & 0 & K_{12}^1 & 0 \\ 0 & K_{11}^2 & K_{12}^2 & 0 \\ K_{21}^1 & K_{21}^2 & K_{22}^1 + K_{22}^2 + K_{11}^3 & K_{12}^3 \\ 0 & 0 & K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{Bmatrix} U_1 = 0 \\ U_2 = 0 \\ U_3 \\ U_4 = 0 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ 2P \\ Q_2^3 \end{Bmatrix} \tag{3.2.55}$$

This contains four equations in four unknowns: U_3, Q_1^1, Q_1^2, and Q_2^3.

3.2.6 Solution of Equations

As a standard procedure in finite element analysis, the unknown primary degrees of freedom are determined first by *considering the algebraic equations corresponding to the unknown primary variables*. Thus, in the present case, we consider the third equation in (3.2.55) to solve for U_3:

$$K_{21}^1 U_1 + K_{21}^2 U_2 + (K_{22}^1 + K_{22}^2 + K_{11}^3) U_3 + K_{12}^3 U_4 = 2P$$

or

$$(K_{22}^1 + K_{22}^2 + K_{11}^3)U_3 = 2P - (K_{21}^1 U_1 + K_{21}^2 U_2 + K_{12}^3 U_4) \tag{3.2.56}$$

Equation (3.2.56) is called the *condensed equation* for the unknown U_3. The term in parentheses on the right-hand side is zero because all specified displacements are zero in the present problem. Hence, the solution is given by

$$U_3 = \frac{2P}{K_{22}^1 + K_{22}^2 + K_{11}^3} \tag{3.2.57}$$

The unknown secondary variables are determined by considering the remaining equations of (3.2.55), i.e., those that contain the unknown secondary variables:

$$\begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^3 \end{Bmatrix} = \begin{bmatrix} K_{11}^1 & 0 & K_{12}^1 & 0 \\ 0 & K_{11}^2 & K_{12}^2 & 0 \\ 0 & 0 & K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} K_{12}^1 U_3 \\ K_{12}^2 U_3 \\ K_{21}^3 U_3 \end{Bmatrix} \tag{3.2.58}$$

because U_1, U_2, and U_4 are zero.

It is possible, although not common with commercial finite element programs, to move all the unknowns to the left-hand side of (3.2.53) and solve for them all at once. But this process destroys the symmetry of the coefficient matrix and requires more computational time in practical problems.

To obtain numerical values, we use the geometric and material data shown in Fig. 3.2.10. We obtain

$$K_{11}^1 = \frac{E_b A_b}{h_1} = \frac{(14 \times 10^6)(\pi d_b^2/4)}{8 \times 12} = 87,693 \text{ lb-in.}$$

$$K_{11}^2 = \frac{E_s A_s}{h_2} = \frac{(30 \times 10^6)(\pi d_s^2/4)}{4 \times 12} = 122,718 \text{ lb-in.}$$

$$K_{11}^3 = \frac{E_a A_a}{h_3} = \frac{(10 \times 10^6)(\pi d_a^2/4)}{6 \times 12} = 109,083 \text{ lb-in.}$$

The displacement of node 3 is $U_3 = 0.01878$ in., reaction at node 1 is $Q_1^1 = -1,647$ lbs, $Q_1^2 = -2,305$ lbs and reaction at node 4 is $Q_2^3 = -2,049$ lbs. Hence, the stresses in elements 1, 2, and 3 are $\sigma^1 = 2,739$ psi, $\sigma^2 = 11,739$ psi, and $\sigma^3 = -2,609$ psi, respectively.

In general, the assembled finite element equations can be partitioned according to the sets of specified and unspecified displacements into the following form:

$$\begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} \end{bmatrix} \begin{Bmatrix} \mathbf{U}^1 \\ \mathbf{U}^2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \end{Bmatrix} \tag{3.2.59}$$

where $\mathbf{U}^1$ is the column of known (i.e., specified) primary variables, $\mathbf{U}^2$ is the column of unknown primary variables, $\mathbf{F}^1$ is the column of unknown secondary variables, and $\mathbf{F}^2$ is the column of known secondary variables. Writing (3.2.59) as two matrix equations, we obtain

$$\mathbf{K}^{11}\mathbf{U}^1 + \mathbf{K}^{12}\mathbf{U}^2 = \mathbf{F}^1 \tag{3.2.60a}$$

$$\mathbf{K}^{21}\mathbf{U}^1 + \mathbf{K}^{22}\mathbf{U}^2 = \mathbf{F}^2 \tag{3.2.60b}$$

From (3.2.60*b*), we have

$$\mathbf{U}^2 = \left(\mathbf{K}^{22}\right)^{-1}\left(\mathbf{F}^2 - \mathbf{K}^{21}\mathbf{U}^1\right) \tag{3.2.60c}$$

Once $\mathbf{U}^2$ is known from (3.2.60*c*), the vector of unknown secondary variables, $\mathbf{F}^1$, can be computed using Eq. (3.2.60*a*).

In most finite element computer programs, element matrices are assembled as soon as they are generated and they are not stored in the memory of the computer. Thus, element equations are not available for postcomputation of the secondary variables. Also, due to the fact that the assembled coefficient matrix is modified (when operations are performed to invert a matrix) during the solution of equations, Eq. (3.2.60*a*) cannot be used. Therefore, secondary variables can only be computed using their definitions, as discussed next.

3.2.7 Postcomputation of the Solution

The solution of the finite element equations gives the nodal values U_I of the primary unknowns (e.g., displacement, velocity, or temperature). Once the nodal values of the primary variables are known, we can use the finite element approximation $u_h^e(x)$ to compute the desired quantities. The process of computing desired quantities in numerical form or graphical form from the known finite element solution is termed *postcomputation* or *postprocessing;* these phrases are meant to indicate that further computations are made after obtaining the solution of the finite element equations for the nodal values of the primary variables.

Postprocessing of the solution includes one or more of the following tasks:

1. Computation of the primary and secondary variables at points of interest; primary variables are known at nodal points.
2. Interpretation of the results to check whether the solution makes sense (an understanding of the physical process and experience are the guides when other solutions are not available for comparison).
3. Tabular and/or graphical presentation of the results.

To determine the solution u as a continuous function of position x, we return to the approximation (3.2.28) over each element:

$$u(x) \approx \begin{cases} u_h^1(x) = \displaystyle\sum_{j=1}^{n} u_j^1 \psi_j^1(x) \\ u_h^2(x) = \displaystyle\sum_{j=1}^{n} u_j^2 \psi_j^2(x) \\ \vdots \\ u_h^N(x) = \displaystyle\sum_{j=1}^{n} u_j^N \psi_j^N(x) \end{cases} \tag{3.2.61}$$

where N is the number of elements in the mesh. Depending on the value of x, the corresponding element equation from (3.2.61) is used. The derivative of the solution is obtained by differentiating (3.2.61):

$$\frac{du}{dx} \approx \begin{cases} \dfrac{du_h^1}{dx} = \displaystyle\sum_{j=1}^{n} u_j^1 \frac{d\psi_j^1}{dx} \\ \dfrac{du_h^2}{dx} = \displaystyle\sum_{j=1}^{n} u_j^2 \frac{d\psi_j^2}{dx} \\ \vdots \\ \dfrac{du_h^N}{dx} = \displaystyle\sum_{j=1}^{n} u_j^N \frac{d\psi_j^N}{dx} \end{cases} \tag{3.2.62}$$

Note that the derivative du_h^e/dx of the finite element approximation u_h^e based on Lagrange interpolation is discontinuous, for any order element, at the nodes connecting different elements because the continuity of the derivative of the finite element solution at the connecting nodes is not imposed. In the case of linear elements, the derivative of the solution is constant within each element (see Fig. 3.2.11).

The secondary variables Q_j^e can be computed in two different ways. In Eq. (3.2.58), we determined the unknown secondary variables Q_1^1, Q_1^2, and Q_2^3 from the assembled equations of the problem in Fig. 3.2.10. Since the assembled equations often represent the equilibrium relations of a system, the Q_I^e computed from them will be denoted by $(Q_I^e)_{\text{equil}}$. The Q_I^e can also be determined using the definitions in (3.2.6), replacing u with U. We shall denote Q_I^e computed in this way by $(Q_I^e)_{\text{def}}$. Since $(Q_I^e)_{\text{def}}$ are calculated using the approximate solution u_h^e, they are not as accurate as $(Q_I^e)_{\text{equil}}$. However, in finite element computer codes, $(Q_I^e)_{\text{def}}$ are calculated instead of $(Q_I^e)_{\text{equil}}$. This is primarily due to computational reasons. Recall that, in arriving at the results in (3.2.58), we used part of the assembled coefficient matrix. In the numerical solution of simultaneous algebraic equations in a computer, the original assembled coefficient matrix is often modified, and therefore the coefficients needed for the determination of the secondary variables are not available, unless they are saved in an additional array. For the problem in Fig. 3.2.10, we

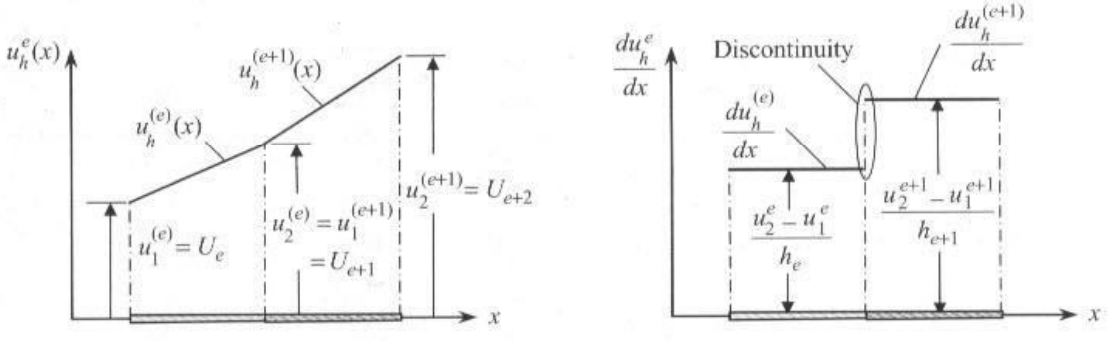

Figure 3.2.11 Gradient of the finite element solution.

have

$$(Q_1^1)_{\text{def}} = -\left(EA\frac{du_h^1}{dx}\right)\bigg|_{x=0} = -EA\frac{U_3 - U_1}{h_1} = -\frac{EA}{h_1}U_3 = K_{12}^1 U_3$$

$$(Q_1^2)_{\text{def}} = -\left(EA\frac{du_h^2}{dx}\right)\bigg|_{x=0} = K_{12}^2 U_3 \tag{3.2.63}$$

$$(Q_2^3)_{\text{def}} = \left(EA\frac{du_h^3}{dx}\right)\bigg|_{x=h_1+h_3} = EA\frac{U_4 - U_3}{h_3} = -\frac{EA}{h_3}U_3 = K_{21}^3 U_3$$

where h_1 and h_3 are the lengths of elements 1 and 3, respectively.

The secondary variables computed using the definitions (3.2.63) are the same as those derived from the assembled equations for the problem in Fig. 3.2.10. This equality is *not to be expected in general*. In fact, when the source vector f is not zero, the secondary variables computed from the definitions (3.2.6) will be in error compared with those computed from the assembled equations. The error decreases as the number of elements or the degree of interpolation is increased.

This completes the basic steps involved in the finite element analysis of the model equation (3.2.1). Next, we consider an example of application of the finite element method. Additional applications of the method to one-dimensional problems of heat transfer, fluid mechanics, and solid mechanics, are presented in Chapter 4.

Example 3.2.1

We wish to use the finite element method to solve the problem described by the following differential equation and boundary conditions (see Example 2.5.1):

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \quad \text{for} \quad 0 < x < 1 \tag{3.2.64}$$

$$u(0) = 0, \quad u(1) = 0 \tag{3.2.65}$$

The differential equation in (3.2.64) is a special case of the model equation (3.2.1) for the following data: $a = 1$, $c = -1$, and $f(x) = -x^2$. Hence, the coefficient matrix is given by

$$K_{ij}^e = \int_{x_b}^{x_a} \left(\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} - \psi_i^e\psi_j^e\right) dx$$

$$f_i^e = \int_{x_b}^{x_a} (-x^2)\psi_i^e \, dx$$

First we consider a mesh of four linear elements and next a mesh of two quadratic elements to solve the problem.

Linear Elements. The element coefficient matrix is given by [see Eq. (3.2.34), with $a_e = 1$, $c_e = -1$, $h_e = \frac{1}{4}$]

$$[K^e] = \frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix} = \begin{bmatrix} 3.9167 & -4.0417 \\ -4.0417 & 3.9167 \end{bmatrix}$$

The coefficients f_i^e are evaluated as

$$f_1^e = -\frac{1}{h_e}\left[\frac{x_b}{3}\left(x_b^3 - x_a^3\right) - \frac{1}{4}\left(x_b^4 - x_a^4\right)\right]$$

$$f_2^e = -\frac{1}{h_e}\left[\frac{1}{4}\left(x_b^4 - x_a^4\right) - \frac{x_a}{3}\left(x_b^3 - x_a^3\right)\right]$$

Element 1. $(h_1 = \frac{1}{4},\ x_a = 0,\ x_b = h_1 = \frac{1}{4})$:

$$f_1^1 = -0.001302,\ f_2^1 = -0.003906$$

Element 2. $(h_2 = \frac{1}{4},\ x_a = h_1 = \frac{1}{4},\ x_b = h_1 + h_2 = \frac{1}{2})$:

$$f_1^2 = -0.014323,\ f_2^2 = -0.022135$$

Element 3. $(h_3 = \frac{1}{4},\ x_a = h_1 + h_2 = \frac{1}{2},\ x_b = h_1 + h_2 + h_3 = \frac{3}{4})$:

$$f_1^3 = -0.042969,\ f_2^3 = -0.05599$$

Element 4. $(h_4 = \frac{1}{4},\ x_a = h_1 + h_2 + h_3 = \frac{3}{4},\ x_b = h_1 + h_2 + h_3 + h_4 = 1)$:

$$f_1^4 = -0.08724,\ f_2^4 = -0.10547$$

The assembled set of equations is

$$\begin{bmatrix} 3.9167 & -4.0417 & 0 & 0 & 0 \\ -4.0417 & 7.8333 & -4.0417 & 0 & 0 \\ 0 & -4.0417 & 7.8333 & -4.0417 & 0 \\ 0 & 0 & -4.0417 & 7.8333 & -4.0417 \\ 0 & 0 & 0 & -4.0417 & 3.9167 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

$$= -\begin{Bmatrix} 0.00130 \\ 0.01823 \\ 0.06510 \\ 0.14323 \\ 0.10547 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

Since $U_1 = 0$ and $U_5 = 0$, the condensed equations are obtained by omitting the first and fifth rows and columns of the assembled equations. The condensed equations are

$$\begin{bmatrix} 7.8333 & -4.0417 & 0 \\ -4.0417 & 7.8333 & -4.0417 \\ 0 & -4.0417 & 7.8333 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = -\begin{Bmatrix} 0.01823 \\ 0.06510 \\ 0.14323 \end{Bmatrix}$$

The solution is (obtained using a computer)

$$U_1 = 0.0,\ U_2 = -0.02323,\ U_3 = -0.04052,\ U_4 = -0.03919,\ U_5 = 0.0$$

The secondary variables can be computed using either the definition or from the element equations. We have

$$(Q_1^1)_{def} \equiv \left(-a\frac{du_h^1}{dx}\right)\bigg|_{x=0} = \frac{U_1 - U_2}{h} = 0.09293$$

$$(Q_2^4)_{def} \equiv \left(a\frac{du_h^4}{dx}\right)\bigg|_{x=1} = \frac{U_5 - U_4}{h} = 0.15676$$

$$(Q_1^1)_{equil} = K_{11}^1 U_1 + K_{12}^1 U_2 - f_1^1 = 0.09520$$

$$(Q_2^4)_{equil} = K_{21}^4 U_4 + K_{22}^4 U_5 - f_2^4 = 0.26386$$

Quadratic Elements. The element coefficient matrix is given by [see Eq. (3.2.37a), with $a_e = 1,\ c_e = -1,\ h_e = \frac{1}{2}$]

$$[K^e] = \frac{1}{60}\begin{bmatrix} 276 & -322 & 41 \\ -322 & 624 & -322 \\ 41 & -322 & 276 \end{bmatrix} = \begin{bmatrix} 4.6000 & -5.3667 & 0.6833 \\ -5.3667 & 10.4000 & -5.3667 \\ 0.6833 & -5.3667 & 4.6000 \end{bmatrix}$$

The coefficients f_i^e are evaluated as

$$f_1^e = -\frac{h_e}{60}\left(-h_e^2 + 10x_a^2\right)$$

$$f_2^e = -\frac{h_e}{15}\left(3h_e^2 + 10x_a^2 + 10x_a^2 h_e\right)$$

$$f_3^e = -\frac{h_e}{60}\left(9h_e^2 + 20x_a^2 + 20x_a h_e\right)$$

Element 1. ($h_1 = \frac{1}{2},\ x_a = 0,\ x_b = h_1 = \frac{1}{2}$):

$$f_1^1 = 0.00208,\ f_2^1 = -0.02500,\ f_3^1 = -0.01875$$

Element 2. ($h_2 = \frac{1}{2},\ x_a = h_1 = \frac{1}{2},\ x_b = h_1 + h_2 = 1$):

$$f_1^2 = -0.01875,\ f_2^2 = -0.19167,\ f_3^2 = -0.08125$$

The assembled set of equations are

$$\begin{bmatrix} 4.6000 & -5.3667 & 0.6833 & 0 & 0 \\ -5.3667 & 10.4000 & -5.3667 & 0 & 0 \\ 0.6833 & -5.3667 & 9.2000 & -5.3667 & 0.6833 \\ 0 & 0 & -5.3667 & 10.4000 & -5.3667 \\ 0 & 0 & 0.6833 & -5.3667 & 4.6000 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

$$= -\begin{Bmatrix} -0.00208 \\ 0.02500 \\ 0.03750 \\ 0.19167 \\ 0.08125 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_2^2 \\ Q_3^2 \end{Bmatrix}$$

Again, using $U_1 = 0$ and $U_5 = 0$, the condensed equations are obtained as

$$\begin{bmatrix} 10.4000 & -5.3667 & 0 \\ -5.3667 & 9.2000 & -5.3667 \\ 0 & -5.3667 & 10.4000 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = - \begin{Bmatrix} 0.02500 \\ 0.03750 \\ 0.19167 \end{Bmatrix}$$

The solution is

$$U_1 = 0.0,\ U_2 = -0.02345,\ U_3 = -0.04078,\ U_4 = -0.03947,\ U_5 = 0.0$$

The secondary variables can be computed using either the definition or from the element equations. We have

$$(Q_1^1)_{def} \equiv \left(-a\frac{du_h^1}{dx}\right)\bigg|_{x=0} = -\left[\frac{U_1}{h}\left(-3+4\frac{x}{h}\right) + \frac{U_2}{h}\left(4-8\frac{x}{h}\right) + \frac{U_3}{h}\left(-1+4\frac{x}{h}\right)\right]_{x=0}$$

$$= \left(\frac{1}{h}U_1 - \frac{4}{h}U_2 + \frac{1}{h}U_3\right) = 0.10602$$

Table 3.2.1 Comparison of the finite element results with the exact solution $(-10u)$ of eqs. (3.2.64) and (3.2.65).

	FEM* solution			Ritz† Solution	Exact Solution
x	4L	2Q	4Q		
0.0000	**0.0000**‡	**0.0000**	**0.0000**	0.0000	0.0000
0.0625	0.0581	0.0644	0.0602	0.0596	0.0598
0.1250	0.1162	0.1249	**0.1193**	0.1191	0.1192
0.1875	0.1743	0.1816	0.1771	0.1776	0.1775
0.2500	**0.2323**	**0.2345**	**0.2337**	0.2339	0.2337
0.3125	0.2756	0.2835	0.2876	0.2868	0.2866
0.3750	0.3188	0.3288	**0.3345**	0.3347	0.3345
0.4375	0.3620	0.3702	0.3745	0.3757	0.3755
0.5000	**0.4052**	**0.4078**	**0.4076**	0.4076	0.4076
0.5625	0.4019	0.4403	0.4298	0.4282	0.4283
0.6250	0.3986	0.4490	**0.4350**	0.4347	0.4350
0.6875	0.3952	0.4338	0.4231	0.4244	0.4246
0.7500	**0.3919**	**0.3947**	**0.3942**	0.3940	0.3942
0.8125	0.2939	0.3318	0.3421	0.3401	0.3402
0.8750	0.1960	0.2451	**0.2591**	0.2592	0.2590
0.9375	0.0980	0.1345	0.1450	0.1472	0.1470
1.0000	**0.0000**	**0.0000**	**0.0000**	0.0000	0.0000

* FEM, finite element method

† Three-parameter Ritz solution from Example 2.5.1.

‡ Numbers in bold are the nodal values; others are the interpolated values (4L = 4 linear elements; 2Q = 2 quadratic elements; 4Q = 4 quadratic elements).

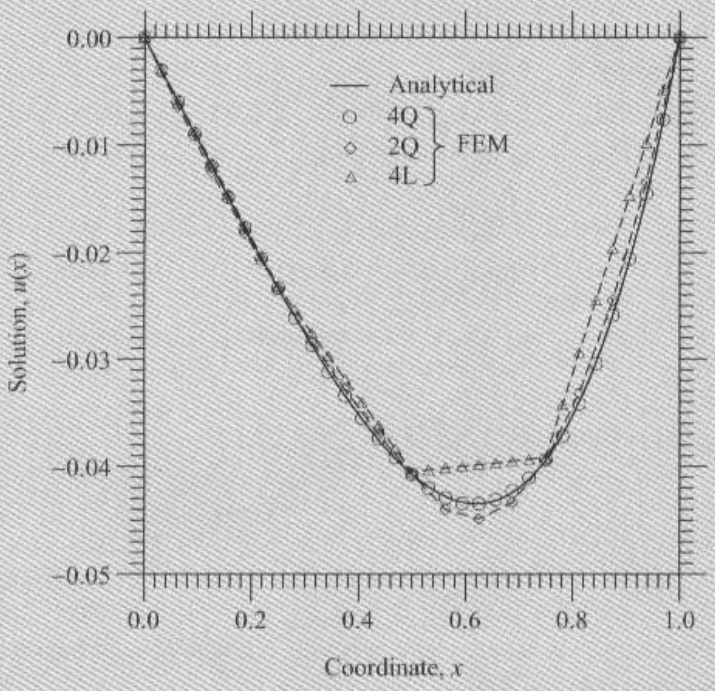

Figure 3.2.12 Comparison of the finite element solutions u_h with the exact and Ritz solutions.

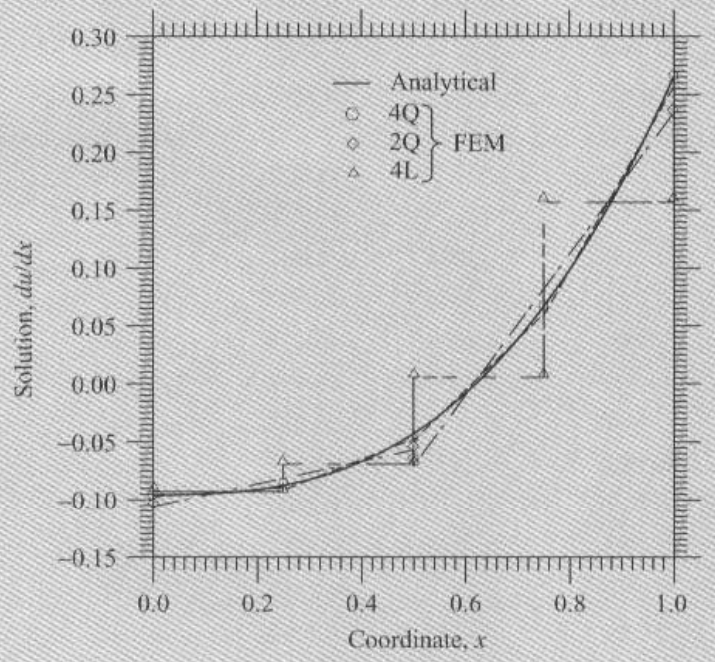

Figure 3.2.13 Comparison of the finite element solutions du_h/dx with the exact and Ritz solutions.

$$(Q_3^2)_{def} \equiv \left(a\frac{du_h^2}{dx}\right)\Big|_{x=1} = \left[\frac{U_3}{h}\left(-3+4\frac{\bar{x}}{h}\right)+\frac{U_4}{h}\left(4-8\frac{\bar{x}}{h}\right)+\frac{U_5}{h}\left(-1+4\frac{\bar{x}}{h}\right)\right]_{\bar{x}=h}$$

$$= \left(\frac{1}{h}U_3 - \frac{4}{h}U_4 + \frac{3}{h}U_5\right) = 0.23442$$

$$(Q_1^1)_{equil} = K_{11}^1 U_1 + K_{12}^1 U_2 + K_{13}^1 U_3 - f_1^1 = 0.10006$$

$$(Q_3^2)_{equil} = K_{13}^2 U_3 + K_{23}^2 U_4 + K_{33}^2 U_5 - f_3^2 = 0.26521$$

Table 3.2.1 contains a comparison of the finite element results against the three-parameter Ritz solution and the exact solution (see Example 2.5.1 and Table 2.5.1). The four linear-element and two quadratic-element solutions are not as accurate as the three-parameter Ritz solution. However, the four quadratic-element solution at the nodes (details are not given here) is virtually the same as the exact solution. Plots of u_h and du_h/dx are shown in Figs. 3.2.12 and 3.2.13, respectively. Note that the finite element solutions for the derivative [postcomputed from $u_h(x)$] are discontinuous between the elements; the value of the difference in du_h/dx between elements reduces as we move from linear to quadratic elements and from coarse to refined meshes.

3.3 SOME REMARKS

A few remarks are in order on the steps described for the model equation.

Remark 1. Although the Ritz method was used to set up the element equations, any other method, such as a weighted-residual (i.e., the least-squares or Galerkin) method could be used.

Remark 2. Steps 1–6 (see Table 3.1.1) are common for any problem. The derivation of interpolation functions depends only on the element geometry and on the number and position of nodes in the element. The number of nodes in the element and the degree of approximation used are related.

Remark 3. The finite element equations (3.2.33*b*) are derived for the linear operator equation

$$A(u) = f, \quad \text{where} \quad A = -\frac{d}{dx}\left(a\frac{d}{dx}\right) + c \tag{3.3.1}$$

Hence, they are valid for any physical problem that is described by the operator equation $A(u) = f$ or its special cases. One need only interpret the quantities appropriately. Examples of problems described by this operator are listed in Table 3.2.1. Thus, a computer program written for the finite element analysis of (3.2.1) can be used to analyze any of the problems in Table 3.2.1. Also, note that the data $a = a(x)$, $c = c(x)$, and $f = f(x)$ can be different in different elements.

Remark 4. Integration of the element matrices in (3.2.31*b*) can be implemented on a computer using numerical integration. When these integrals are algebraically complicated, we have no other choice but numerical integration (see Chapter 7).

Remark 5. As discussed in (3.2.45*a*) and (3.2.45*b*) the point sources at the nodes are included in the finite element model via the balance of sources at the nodes. Thus, in constructing finite element meshes, we should include nodes at the locations of point sources. If a point source does not occur at a node, it is possible to "distribute" it to the element nodes, consistent with the finite element approximation. Let Q_0 denote a point source at a point $\bar{x} = \bar{x}_0$, $\bar{x}_1^e \le \bar{x}_0 \le \bar{x}_n^e$, where $\bar{x}$ is the local coordinate. The point source Q_0 can be represented as a "function" with the help of the Dirac delta function

$$f(\bar{x}) = Q_0 \delta(\bar{x} - \bar{x}_0) \tag{3.3.2}$$

where the Dirac delta function $\delta(\cdot)$ is defined by

$$\int_{-\infty}^{\infty} F(\bar{x}) \delta(\bar{x} - \bar{x}_0)\, d\bar{x} = F(\bar{x}_0) \tag{3.3.3}$$

The contribution of the function $f(\bar{x})$ to the nodes of the element $\Omega_e = (\bar{x}_1, \bar{x}_n)$ is computed from [see Eq. (3.2.31*b*)]

$$f_i^e = \int_0^{h_e} f(\bar{x}) \psi_i^e(\bar{x})\, d\bar{x} = \int_0^{h_e} Q_0 \delta(\bar{x} - \bar{x}_0) \psi_i^e(\bar{x})\, d\bar{x} = Q_0 \psi_i^e(\bar{x}_0) \tag{3.3.4}$$

where ψ_i^e are the interpolation functions of the element Ω_e. Thus, the point source Q_0 is distributed to the element node i by the value $Q_0 \psi_i^e(\bar{x}_0)$. Equation (3.3.4) holds for any element, irrespective of the degree of the interpolation, the nature of the interpolation (i.e., Lagrange or Hermite polynomials), or the dimension (i.e., one-dimensional, two-dimensional, or three-dimensional) of the elements. For one-dimensional linear Lagrange interpolation functions, Eq. (3.3.4) yields

$$f_1^e = Q_0 \left(1 - \frac{\bar{x}_0}{h_e}\right), \quad f_2^e = Q_0 \left(\frac{\bar{x}_0}{h_e}\right)$$

Note that $f_1^e + f_2^e = Q_0$. When $\bar{x}_0 = h_e/2$, we have $f_1^e = f_2^e = Q_0/2$, as expected.

Remark 6. There are three sources of error that may contribute to the inaccuracy of the finite element solution of a problem:

1. *Domain approximation error,* which is due to the approximation of the domain.
2. *Computational errors,* which are due to inexact evaluation of the coefficients K_{ij}^e and f_i^e, or are introduced owing to the finite arithmetic in a computer.
3. *Approximation error,* which is due to approximation of the solution by piecewise polynomials.

For the structure shown in Fig. 3.2.10, the geometry of the problem is exactly represented and the linear approximation is able to represent the exact solution at the nodes when a is a constant, $c = 0$, and f is arbitrary [see Reddy (1986), p. 403; also, see Sec. 14.5]. Therefore, the first and third type of errors are zero. The only error that can be introduced into the final numerical results is possibly due to the computer evaluation of the coefficients K_{ij}^e and f_i^e and the solution of algebraic equations. However, in general, computational as well

as approximation errors exist even in one-dimensional problems. Additional discussion of the errors in the finite element approximation can be found in Oden and Carey (1983) and Reddy (1986); also, see Sec. 14.5.

Remark 7. The approach used in matrix methods of structural analysis to solve the problem in Fig. 3.2.10 is not much different than that presented here. The difference lies only in the derivation of the element equations (3.2.39) for the case $c_e = 0$. In matrix methods of structural analysis, the element equations are obtained directly from the definitions of stress and strain and their relationship. For example, consider the free-body diagram of a bar element [see Fig. 3.2.3(*b*)]. From a course on mechanics of deformable solids, we have

$$\begin{aligned} \text{strain, } \varepsilon &= \text{elongation/original length} \\ \text{stress, } \sigma &= \text{Young's modulus} \times \text{strain} \\ \text{load, } P &= \text{stress} \times \text{area of cross section} \end{aligned}$$

The strain defined above is the average (or engineering) strain. Mathematically, strain for one-dimensional problems is defined as $\varepsilon = du/dx$, u being displacement, which includes rigid body motion as well as elongation of the bar. Hence, the compressive force at the left end of the bar element is

$$P_1^e = A_e \sigma_1^e = A_e E_e \varepsilon_1^e = A_e E_e \frac{u_1^e - u_2^e}{h_e} = \frac{A_e E_e}{h_e}(u_1^e - u_2^e)$$

where E_e is Young's modulus of the bar element. Similarly, the force at the right end is

$$P_2^e = \frac{A_e E_e}{h_e}(u_2^e - u_1^e)$$

In matrix form, these relations can be expressed as

$$\frac{A_e E_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} P_1^e \\ P_2^e \end{Bmatrix} \tag{3.3.5a}$$

which is the same as (3.2.39) obtained for a linear finite element with $a_e = A_e E_e$, $c_e = 0$, and $P_i^e = Q_i^e + f_i^e$. Note that in deriving the element equations, we have used knowledge of the mechanics of materials and the assumption that the strain is constant (or the displacement is linear) over the length of the element. If a higher-order representation of the displacement is required, we cannot write the force-displacement relations (3.3.5*a*) directly, i.e., the element equations of a quadratic finite elements cannot be derived using the arguments presented above.

A similar approach can be used to develop the relations between the temperatures and heats at the ends of an insulated fin. From a course on basic heat transfer, we have

$$\begin{aligned} \text{temperature gradient} &= \text{difference in temperature/length} \\ \text{heat flux, } q &= \text{conductivity} \times (-\text{temperature gradient}) \\ \text{heat, } Q &= \text{heat flux} \times \text{area of cross section} \end{aligned}$$

Then, if the temperature is assumed to vary linearly between the ends of the fin and there is no internal heat generation, the heat input at the left end of the fin is

$$Q_1^e = A_e q_1^e = A_e k_e \frac{T_1^e - T_2^e}{h_e} = \frac{A_e k_e}{h_e}(T_1^e - T_2^e)$$

where k_e is thermal conductivity of the fin. Similarly, the heat *input* at the right end is

$$Q_2^e = A_e q_2^e = A_e k_e \frac{T_2^e - T_1^e}{h_e} = \frac{A_e k_e}{h_e}(T_2^e - T_1^e)$$

In matrix form, we have

$$\frac{A_e k_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{3.3.5b}$$

which are the same as those in (3.3.5*a*) with E_e replaced by k_e, T_i^e by u_i^e, and P_i^e by Q_i^e [and reverse the arrow at node 2 in Fig. 3.2.3(*b*)].

Equations of the type (3.3.5*a*) can also be derived for discrete systems (as opposed to a continuum) consisting of spring elements, pipe-flow elements, electrical resistor elements, and so on. These elements will be discussed in Chapter 4 using physical principles. However, such a direct approach cannot be used when the element data is not constant or when higher-order approximation of the dependent unknowns is used.

Remark 8. Another interpretation of (3.2.39) for $c_e = 0$ can be given in terms of the finite difference approximation. The axial force at any point x is given by $P(x) = EA\, du/dx$. Using the forward difference approximation, we approximate the derivative du/dx and write

$$-P_1^e \equiv P(x)|_{x_e} = E_e A_e [u(x_{e+1}) - u(x_e)]/h_e \tag{3.3.6a}$$

$$P_2^e \equiv P(x)|_{x_{e+1}} = E_e A_e [u(x_{e+1}) - u(x_e)]/h_e \tag{3.3.6b}$$

which are the same as (3.3.5*a*), with $u_1^e = u(x_e)$ and $u_2^e = u(x_{e+1})$. Note that no explicit approximation of $u(x)$ itself is assumed in writing (3.3.6*a*) and (3.3.6*b*), but the fact that we used values of the function from two consecutive points to define its slope implies that we assumed a linear approximation of the function. Thus, to compute the value of u at a point other than the nodes (or mesh points), linear interpolation must be used.

Remark 9. For the model problem considered, the element matrices $[K^e]$ in (3.2.31*b*) are symmetric: $K_{ij}^e = K_{ji}^e$. This enables one to compute K_{ij}^e $(I = 1, 2, \ldots, n)$ for $j \le i$ only. In other words, we need compute only the diagonal terms and the upper or lower diagonal terms. Because of the symmetry of the element matrices, the assembled global matrix will also be symmetric. Thus, we need to store only the upper triangle, including the diagonal, of the assembled matrix in a finite element computer program. Another property characteristic of the finite element method is the *sparseness* of the assembled matrix. Since $K_{IJ} = 0$, if global nodes I and J do not belong to the same finite element, the global coefficient matrix is *banded*, i.e., all coefficients beyond a certain distance from the diagonal are zero. The maximum of the distances between the diagonal element, including the latter, of a row and the last nonzero coefficient in that row is called the *half-bandwidth*. When a matrix is banded and symmetric, we need to store only entries in the upper or lower band of the matrix. Equation solvers written for the solution of banded symmetric equations are available for use in such cases (see Chapter 7 for additional discussion). The symmetry of the coefficient matrix depends on the type of the differential equation, its variational form, and the numbering of the finite element equations. The sparseness of the matrix is a result of the finite element interpolation functions, which have nonzero values only over an element of the domain (i.e., so-called "compactness" property of the approximation functions).

Remark 10. The balance (or "equilibrium") of the secondary variables Q_I^e at the interelement boundaries is expressed by (3.2.43). This amounts to imposing the condition that the secondary variable adu/dx at the node, where u is the actual solution, be continuous. However, this does not imply continuity of adu_h^e/dx, where u_h^e is the finite element solution. Thus, we have

$$Q_2^e + Q_1^{e+1} = 0 \quad \text{or} \quad Q_0 \tag{3.3.7}$$

but

$$\left(a\frac{du_h^e}{dx}\right)\Big|_{x_e} + \left(-a\frac{du_h^{e+1}}{dx}\right)\Big|_{x_e} \neq 0 \quad \text{or} \quad Q_0 \tag{3.3.8}$$

In most books on the finite element method, this point is not made clear to the reader. These books consider the quadratic form or weak form of the total problem and omit the sum of the interelement contributions (for linear elements),

$$\sum_{e=1}^{N}\left(\sum_{i=1}^{2} Q_i^e u_i^e\right) \tag{3.3.9}$$

in the quadratic form (or functional) of the problem. However, this amounts to imposing equilibrium conditions of the form (3.3.8). When the secondary variable is specified to be nonzero (say, Q_0) at an interelement boundary (say, at global node 2), we have

$$Q_2^1 + Q_1^2 = Q_0$$

In other books, Q_0 is included in the functional as Q_0U_2, where U_2 is the value of u at global node 2.

To fully understand the difference between the direct use of the global statement versus using the assembly of element equations, consider the model problem described by Eqs. (3.2.1) and (3.2.2) with $c = 0$. The variational form of these equations over the entire domain is given by

$$0 = \int_0^L \left(a\frac{dv}{dx}\frac{du}{dx} - vq\right) dx - v(L)Q_0 \tag{3.3.10}$$

When u is approximated by functions that are defined only on a local interval (which is the case in the finite element method), use of the above variational form implies the omission of the sum of the interelement contributions of (3.3.9).

Consider a mesh of three linear elements. Since $\psi_i^e (e = 1, 2, 3)$ is zero in any element Ω_f for $e \neq f$, the (global) finite element solution for the entire domain is given by

$$u_h(x) = \sum_{e=1}^{3}\left(\sum_{i=1}^{2} u_i^e \psi_i^e\right) \equiv \sum_{I=1}^{4} U_I \Phi_I(x) \tag{3.3.11}$$

where $\Phi_I(x)$ $(I = 1, 2, 3, 4)$ are the piecewise-continuous *global interpolation functions* (see Fig. 3.2.9),

$$\Phi_I(x) = \begin{cases} \psi_2^{(I-1)}(x) & \text{for } x_{I-1} \leq x \leq x_I \\ \psi_1^{(I)}(x) & \text{for } x_I \leq x \leq x_{I+1} \end{cases} \tag{3.3.12}$$

Substituting (3.3.11) for u and $v = \Phi_I$ into (3.3.10), we obtain

$$0 = \int_0^L \left[a \frac{d\Phi_I}{dx} \left(\sum_{J=1}^{4} U_J \frac{d\Phi_J}{dx} \right) - \Phi_I q \right] dx - \Phi_I(L) Q_0 \tag{3.3.13}$$

Since Φ_I is nonzero only between x_{I-1} and x_{I+1}, the integral becomes

$$0 = \int_{x_{I-1}}^{x_{I+1}} \left[a \frac{d\Phi_I}{dx} \left(U_{I-1} \frac{d\Phi_{I-1}}{dx} + U_I \frac{d\Phi_I}{dx} + U_{I+1} \frac{d\Phi_{I+1}}{dx} \right) - \Phi_I q \right] dx - \Phi_I(L) Q_0 \tag{3.3.14}$$

and we have

$$\begin{aligned}
I = 1 : 0 &= \int_{x_1=0}^{x_2} \left[a \frac{d\Phi_1}{dx} \left(U_1 \frac{d\Phi_1}{dx} + U_2 \frac{d\Phi_2}{dx} \right) - \Phi_1 q \right] dx - \Phi_1(L) Q_0 \\
I = 2 : 0 &= \int_{x_1=0}^{x_3} \left[a \frac{d\Phi_2}{dx} \left(U_1 \frac{d\Phi_1}{dx} + U_2 \frac{d\Phi_2}{dx} + U_3 \frac{d\Phi_3}{dx} \right) - \Phi_2 q \right] dx \\
&\quad - \Phi_2(L) Q_0 \\
I = 3 : 0 &= \int_{x_2}^{x_4=L} \left[a \frac{d\Phi_3}{dx} \left(U_2 \frac{d\Phi_2}{dx} + U_3 \frac{d\Phi_3}{dx} + U_4 \frac{d\Phi_4}{dx} \right) - \Phi_3 q \right] dx \\
&\quad - \Phi_3(L) Q_0 \\
I = 4 : 0 &= \int_{x_3}^{x_4=L} \left[a \frac{d\Phi_4}{dx} \left(U_3 \frac{d\Phi_3}{dx} + U_4 \frac{d\Phi_4}{dx} \right) - \Phi_4 q \right] dx - \Phi_4(L) Q_0
\end{aligned} \tag{3.3.15}$$

These equations, upon performing the integration, yield (3.2.50), with the last column (containing Qs) in the latter replaced by

$$\begin{Bmatrix} 0 \\ Q_0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.3.16}$$

Although this procedure, known as the *direct stiffness method* in structural mechanics, gives the assembled equations directly, it is algebraically complicated (especially for two-dimensional problems) and not amenable to simple computer implementation.

3.4 AXISYMMETRIC PROBLEMS

3.4.1 Model Equation

The equations governing physical processes in a cylindrical geometry are described analytically in terms of cylindrical coordinates (see Fig. 3.4.1; also see Fig. 1.4.5). When the geometry, loading, and boundary conditions are independent of the circumferential direction (i.e., θ-coordinate in Fig. 3.4.1), the problem is said to be axisymmetric and the governing equations become two-dimensional in terms of r and z. The equations are functions of only

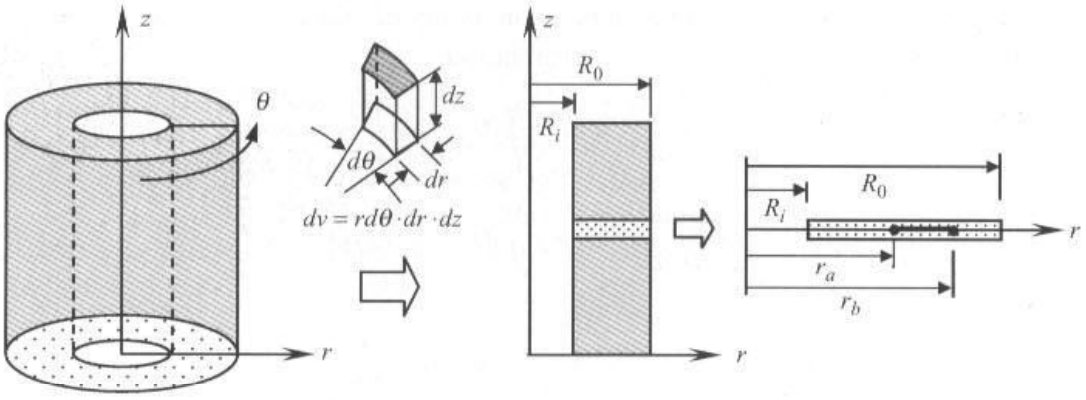

Figure 3.4.1 Volume element and computational domain of an axisymmetric problem.

the radial coordinate r if the problem geometry and data are independent of z. Here, we consider a model second-order equation in a single variable and formulate its finite element model.

Consider the differential equation [an analogue of (3.2.1)]

$$-\frac{1}{r}\frac{d}{dr}\left[a(r)\frac{du}{dr}\right] = f(r) \quad \text{for} \quad R_i < r < R_0 \tag{3.4.1}$$

where r is the radial coordinate, a and f are known functions of r, and u is the dependent variable. Such equations arise, for example, in connection with radial heat flow in a long circular cylinder of inner radius R_i and outer radius R_0. The radially symmetric conditions require that both $a = kr$ (k is the conductivity) and f (internal heat generation) be functions of only r. Since the cylinder is long, the temperature distribution at any section along its length (except perhaps at the ends) is the same, and it is sufficient to consider any cross section away from the ends, i.e., the problem is reduced from a three-dimensional problem to a two-dimensional one. Since a and f are independent of the circumferential direction θ, the temperature distribution along any radial line is the same, reducing the two-dimensional problem to a one-dimensional one, as described by (3.4.1).

3.4.2 Weak Form

In developing the weak form of (3.4.1), we multiply it with a weight function $w(r)$ and integrate over the volume of the cylinder of unit length (see Fig. 3.4.1)

$$\begin{aligned} 0 &= \int_V w\left[-\frac{1}{r}\frac{d}{dr}\left(a\frac{du}{dr}\right) - f\right] dv \\ &= \int_0^1 \int_0^{2\pi} \int_{r_a}^{r_b} w\left[-\frac{1}{r}\frac{d}{dr}\left(a\frac{du}{dr}\right) - f\right] r\, dr\, d\theta\, dz \\ &= 2\pi \int_{r_a}^{r_b} w\left[-\frac{1}{r}\frac{d}{dr}\left(a\frac{du}{dr}\right) - f\right] r\, dr \end{aligned} \tag{3.4.2}$$

where (r_a, r_b) is the domain of a typical element along the radial direction. Next, we carry out the remaining two steps of the weak formulation:

$$0 = 2\pi \int_{r_a}^{r_b} \left(a \frac{dw}{dr} \frac{du}{dr} - rwf \right) dr - 2\pi \left[wa \frac{du}{dr} \right]_{r_a}^{r_b}$$

$$0 = 2\pi \int_{r_a}^{r_b} \left(a \frac{dw}{dr} \frac{du}{dr} - rwf \right) dr - w(r_a) Q_1^e - w(r_b) Q_2^e \tag{3.4.3a}$$

where

$$Q_1^e \equiv -2\pi \left(a \frac{du}{dr} \right) \Big|_{r_a}, \quad Q_2^e \equiv 2\pi \left(a \frac{du}{dr} \right) \Big|_{r_b} \tag{3.4.3b}$$

3.4.3 Finite Element Model

The finite element model is obtained by substituting the approximation

$$u(r) \approx \sum_{j=1}^{n} u_j^e \psi_j^e(r) \tag{3.4.4}$$

and $w = \psi_1, \psi_2, \ldots, \psi_n$ into (3.4.3a). The finite-element model is given by

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\} \tag{3.4.5a}$$

where

$$K_{ij}^e = 2\pi \int_{r_a}^{r_b} a \frac{d\psi_i^e}{dr} \frac{d\psi_j^e}{dr} dr, \quad f_i^e = 2\pi \int_{r_a}^{r_b} \psi_i^e f r \, dr \tag{3.4.5b}$$

and ψ_i^e are the interpolation functions expressed in terms of the radial coordinate r. For example, the linear interpolation functions are of the form ($h_e = r_b - r_a$) [see Eq. (3.2.17)]

$$\psi_1^e(r) = \frac{r_b - r}{h_e}, \quad \psi_2^e(r) = \frac{r - r_a}{h_e} \tag{3.4.6}$$

The explicit forms of the coefficients K_{ij}^e and f_i^e for $a = a_e r$ and $f = f_e$ are given below (r_a denotes the global coordinate of node 1 of the element).

Linear Element

$$[K^e] = \frac{2\pi a_e}{h_e} (r_a + \frac{1}{2} h_e) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \{f^e\} = \frac{2\pi f_e h_e}{6} \begin{Bmatrix} 3r_a + h_e \\ 3r_a + 2h_e \end{Bmatrix} \tag{3.4.7}$$

Quadratic Element

$$[K^e] = \frac{2\pi a_e}{6h_e} \begin{bmatrix} 3h_e + 14r_a & -(4h_e + 16r_a) & h_e + 2r_a \\ -(4h_e + 16r_a) & 16h_e + 32r_a & -(12h_e + 16r_a) \\ h_e + 2r_a & -(12h_e + 16r_a) & 11h_e + 14r_a \end{bmatrix}$$

$$\{f^e\} = \frac{2\pi f_e h_e}{6} \begin{Bmatrix} r_a \\ 4r_a + 2h_e \\ r_a + h_e \end{Bmatrix} \tag{3.4.8}$$

Example 3.4.1

Equation (3.4.1) governs, for example, temperature distribution $u(r)$ in a *long* solid cylindrical bar of radius R and thermal conductivity k (i.e., $a(r)=rk$) that is heated by the passage of an electric current, which generates heat energy f_0. Heat is dissipated from the surface of the bar by convection into the surrounding medium at an ambient temperature of u_∞. We wish to determine the temperature distribution as a function of the radial distance. The boundary conditions for this case are

$$\frac{du}{dr}=0 \text{ at } r=0 \text{ (symmetry)}, \quad k\frac{du}{dr}+\beta(u-u_\infty)=0 \text{ at } r=R_0 \text{ (convection)} \tag{3.4.9}$$

Let us consider the following data for numerical computations:

$$R=0.05 \text{ m}, \quad k=40 \text{ W/(m}\cdot{}^\circ\text{C)}, \quad f_0=4\times 10^6 \text{ W/m}^3, \quad \beta=400 \text{ W/(m}^2\cdot{}^\circ\text{C)}, \quad u_\infty=20^\circ\text{C} \tag{3.4.10}$$

For the uniform mesh of four linear elements ($h=0.0125$ m), we have (omitting the common factor 2π) the following coefficient matrices.

Element 1. ($r_a=0$ and $r_b=h$):

$$[K^1]=\frac{40}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}, \quad \{f^1\}=\frac{4\times 10^6(0.0125)^2}{6}\begin{Bmatrix}1\\ 2\end{Bmatrix}$$

Element 2. ($r_a=h$ and $r_b=2h$):

$$[K^2]=\frac{40}{2}\begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}, \quad \{f^2\}=\frac{4\times 10^6(0.0125)^2}{6}\begin{Bmatrix}4\\ 5\end{Bmatrix}$$

Element 3. ($r_a=2h$ and $r_b=3h$):

$$[K^3]=\frac{40}{2}\begin{bmatrix}5 & -5\\ -5 & 5\end{bmatrix}, \quad \{f^3\}=\frac{4\times 10^6(0.0125)^2}{6}\begin{Bmatrix}7\\ 8\end{Bmatrix}$$

Element 4. ($r_a=3h$ and $r_b=4h$):

$$[K^2]=\frac{40}{2}\begin{bmatrix}7 & -7\\ -7 & 7\end{bmatrix}, \quad \{f^4\}=\frac{4\times 10^6(0.0125)^2}{6}\begin{Bmatrix}10\\ 11\end{Bmatrix}$$

The assembled equations are

$$\begin{bmatrix}20 & -20 & 0 & 0 & 0\\ -20 & 80 & -60 & 0 & 0\\ 0 & -60 & 160 & -100 & 0\\ 0 & 0 & -100 & 240 & -140\\ 0 & 0 & 0 & -140 & 140\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\\ U_5\end{Bmatrix}=104.166\begin{Bmatrix}1\\ 6\\ 12\\ 18\\ 11\end{Bmatrix}+\begin{Bmatrix}Q_1^1\\ Q_2^1+Q_1^2\\ Q_2^2+Q_1^3\\ Q_2^3+Q_1^4\\ Q_2^4\end{Bmatrix}$$

The boundary and balance conditions require

$$Q_1^1 = 0,\ Q_2^1 + Q_1^2 = 0,\ Q_2^2 + Q_1^3 = 0,\ Q_2^3 + Q_1^4 = 0,\ Q_2^4 = -R_0\beta(U_5 - u_\infty) = -20(U_5 - 20)$$

The condensed equations are (here, no equations are eliminated)

$$\begin{bmatrix} 20 & -20 & 0 & 0 & 0 \\ -20 & 80 & -60 & 0 & 0 \\ 0 & -60 & 160 & -100 & 0 \\ 0 & 0 & -100 & 240 & -140 \\ 0 & 0 & 0 & -140 & 160 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = 104.166 \begin{Bmatrix} 1 \\ 6 \\ 12 \\ 18 \\ 11 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 400 \end{Bmatrix}$$

The solution of these equations (with the help of a computer) is

$$U_1 = 334.68,\ U_2 = 329.47,\ U_3 = 317.32,\ U_4 = 297.53,\ U_5 = 270.00$$

The uniform mesh of two quadratic elements yields the nodal values

$$U_1 = 332.50,\ U_2 = 328.59,\ U_3 = 316.87,\ U_4 = 297.34,\ U_5 = 270.00$$

which coincide with the exact solution at the nodes

$$u(r) = u_\infty + \frac{f_0 R_0}{2\beta} + \frac{f_0 R_0^2}{4k}\left(1 - \frac{r^2}{R_0^2}\right) \tag{3.4.11}$$

3.5 SUMMARY

A systematic study of the steps involved in the finite element formulation of a model second-order differential equation in single variable is presented. The basic steps of the formulation and analysis of a typical equation are outlined in Table 3.1.1. The model equation is representative of the equations arising in various fields of engineering (see Table 3.2.1). Couple of numerical examples are presented to illustrate the steps in the finite element analysis of second-order differential equations. Additional examples, especially those arising in heat transfer, fluid mechanics, and solid mechanics will be presented in Chapter 4.

The finite element model is developed following three steps:

1. Weak (or variational) formulation of the differential equation over an element.
2. Finite element interpolation of the primary variables of the weak formulation.
3. Finite element model (i.e., a set of algebraic equations relating the "primary" and "secondary" variables) development over a typical element.

The weak formulation itself involves a three-step procedure, which enables identification of primary and secondary variables. The primary variables are required to be continuous throughout the domain, including the nodes at which elements are connected. The secondary variables are included in the weak form. The finite element interpolation functions have been developed systematically on the basis of continuity, completeness, and linear independence. The finite element model has been developed by substituting appropriate interpolation of the primary variable into the weak form of the differential equation.

PROBLEMS

For Problems 3.1–3.4, carry out the following tasks:

(a) Develop the *weak forms* of the given differential equation(s) over a typical finite element, which is a geometric subdomain located between $x = x_a$ and $x = x_b$. Note that there are no "specified" boundary conditions at the element level. Therefore, in going from Step 2 to Step 3 of the weak-form development, one must identify the secondary variable(s) at the two ends of the domain by some symbols (like Q_1^e and Q_2^e for the first problem) and complete the weak form.

(b) Assume an approximation(s) of the form

$$u(x) = \sum_{j=1}^{n} u_j^e \psi_j^e(x) \tag{1}$$

where u is a primary variable of the formulation, $\psi_j^e(x)$ are the interpolation functions, and u_j^e are the values of the primary variable(s) at the jth node of the element. Substitute the expression in (1) for the primary variable and ψ_i^e for the weight function into the weak form(s) and derive the finite element model. Be sure to define all coefficients of the model in terms of the problem data and ψ_i^e.

3.1 Develop the weak form and the finite element model of the following differential equation over an element:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + \frac{d^2}{dx^2}\left(b\frac{d^2u}{dx^2}\right) + cu = f \quad \text{for} \quad x_a < x < x_b$$

where a, b, c, and f are known functions of position x. Ensure that the element coefficient matrix $[K^e]$ is symmetric. What is the nature of the interpolation functions for the problem?

3.2 Construct the weak form and the finite element model of the differential equation

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) - b\frac{du}{dx} = f \quad \text{for} \quad 0 < x < L$$

over a typical element $\Omega_e = (x_a, x_b)$. Here a, b, and f are known functions of x, and u is the dependent variable. The natural boundary condition should *not* involve the function $b(x)$. What type of interpolation functions may be used for u?

3.3 Develop the weak forms of the following pair of coupled second-order differential equations over a typical element (x_a, x_b):

$$-\frac{d}{dx}\left[a(x)\left(u + \frac{dv}{dx}\right)\right] = f(x) \tag{1a}$$

$$-\frac{d}{dx}\left(b(x)\frac{du}{dx}\right) + a(x)\left(u + \frac{dv}{dx}\right) = q(x) \tag{1b}$$

where u and v are the dependent variables, and a, b, f, and q are known functions of x. Also identify the primary and secondary variables of the formulation.

3.4 Consider the following weak forms of a pair of coupled differential equations:

$$0 = \int_{x_a}^{x_b} \left(\frac{dw_1}{dx}\frac{dv}{dx} - w_1 f\right) dx - P_a w_1(x_a) - P_b w_1(x_b) \tag{1a}$$

$$0 = \int_{x_a}^{x_b} \left(\frac{dw_2}{dx}\frac{du}{dx} + c\, w_2 v - w_2 q\right) dx - Q_a w_2(x_a) - Q_b w_2(x_b) \tag{1b}$$

where $c(x)$ is a known function, w_1 and w_2 are weight functions, u and v are dependent variables (primary variables), and P_a, P_b, Q_a, and Q_b are the secondary variables of the formulation. Use the finite element approximations of the form

$$u(x) = \sum_{j=1}^{m} u_j^e \psi_j^e(x), \quad v(x) = \sum_{j=1}^{n} v_j^e \varphi_j^e(x) \tag{2}$$

and $w_1 = \psi_i$ and $w_2 = \varphi_i$ and derive the finite element equations from the weak forms. The finite element equations should be in the form

$$0 = \sum_{j=1}^{m} K_{ij}^{11} u_j^e + \sum_{j=1}^{n} K_{ij}^{12} v_j^e - F_i^1 \tag{3a}$$

$$0 = \sum_{j=1}^{m} K_{ij}^{21} u_j^e + \sum_{j=1}^{n} K_{ij}^{22} v_j^e - F_i^2 \tag{3b}$$

Define the coefficients K_{ij}^{11}, K_{ij}^{12}, K_{ij}^{21}, K_{ij}^{22}, F_i^1, and F_i^2 in terms of the interpolation functions, known data, and secondary variables.

3.5 Derive the Lagrange cubic interpolation functions for a four-node (one-dimensional) element (with equally spaced nodes) using the alternative procedure based on interpolation properties (3.2.18*a*) and (3.2.18*b*). Use the local coordinate $\bar{x}$ for simplicity.

3.6 Evaluate the element matrices $[K^{11}]$, $[K^{12}]$, and $[K^{22}]$ for the linear interpolation of $u(x)$ and $v(x)$ in Problem 3.4.

3.7 Evaluate the following coefficient matrices and source vector using the linear Lagrange interpolation functions:

$$K_{ij}^e = \int_{x_a}^{x_b} (a_0^e + a_1^e x) \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} dx, \quad M_{ij}^e = \int_{x_a}^{x_b} (c_0^e + c_1^e x) \psi_i^e \psi_j^e dx$$

$$f_i^e = \int_{x_a}^{x_b} (f_0^e + f_1^e x) \psi_i^e dx$$

where a_0^e, a_1^e, c_0^e, c_1^e, f_0^e, and f_1^e are constants.

3.8 (*Heat transfer in a rod*) The governing differential equation and convection boundary condition are of the form:

$$-\frac{d^2\theta}{dx^2} + c\theta = 0, \quad 0 < x < L \tag{1}$$

$$\theta(0) = T_0 - T_\infty, \quad \left[k\frac{d\theta}{dx} + \beta\theta \right]_{x=L} = 0 \tag{2}$$

where $\theta = T - T_\infty$, $c = \beta P/(Ak)$, β is the heat transfer coefficient, P is the perimeter, A is the area of cross section, and k is the conductivity. For a mesh of two linear elements (of equal length), give (*a*) the boundary conditions on the nodal variables (primary as well as secondary variables) and (*b*) the final condensed finite element equations for the unknowns (both primary and secondary nodal variables). Use the following data: $T_0 = 120°\text{C}$, $T_\infty = 20°\text{C}$, $L = 0.25$ m, $c = 256$, $\beta = 64$, and $k = 50$ (with proper units).

3.9 (*Axial deformation of a bar*) The governing differential equation is of the form (E and A are constant):

$$-\frac{d}{dx}\left[EA\frac{du}{dx} \right] = 0, \quad 0 < x < L \tag{1}$$

For the minimum number of linear elements, give (*a*) the boundary conditions on the nodal variables (primary as well as secondary variables) and (*b*) the final condensed finite element equations for the unknowns.

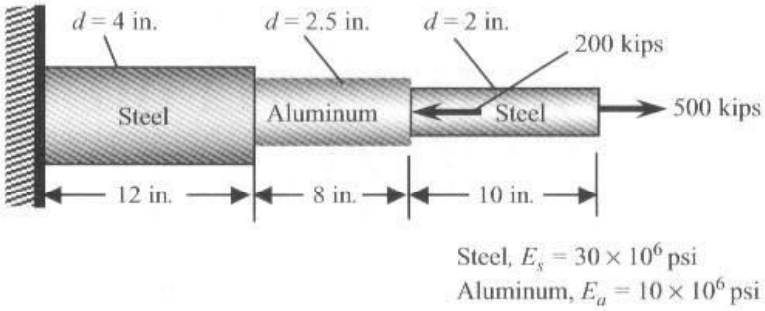

Figure P3.9

3.10 Resolve the problem in Example 3.2.1 using the uniform mesh of three linear finite elements.
Answer: $U_2 = -0.02999$, $U_3 = -0.04257$, $(Q_1^1)_{def} = 0.08998$, $(Q_2^3)_{def} = 0.12771$.

3.11 Solve the differential equation in Example 3.2.1 for the mixed boundary conditions

$$u(0) = 0, \quad \left(\frac{du}{dx}\right)\bigg|_{x=1} = 1$$

Use the uniform mesh of three linear elements. The exact solution is

$$u(x) = \frac{2\cos(1-x) - \sin x}{\cos(1)} + x^2 - 2$$

Answer: $U_2 = 0.4134$, $U_3 = 0.7958$, $U_4 = 1.1420$, $(Q_1^1)_{\text{def}} = -1.2402$.

3.12 Solve the differential equation in Example 3.2.1 for the *natural* (or Neumann) boundary conditions

$$\left(\frac{du}{dx}\right)\bigg|_{x=0} = 1, \quad \left(\frac{du}{dx}\right)\bigg|_{x=1} = 0$$

Use the uniform mesh of three linear finite elements to solve the problem. Verify your solution with the analytical solution

$$u(x) = \frac{\cos(1-x) + 2\cos x}{\sin(1)} + x^2 - 2$$

Answer: $U_1 = 1.0280$, $U_2 = 1.3002$, $U_4 = 1.4447$, $U_5 = 1.4821$.

3.13 Solve the problem described by the following equations

$$-\frac{d^2u}{dx^2} = \cos \pi x, \quad 0 < x < 1; \quad u(0) = 0, \quad u(1) = 0$$

Use the uniform mesh of three linear elements to solve the problem and compare against the exact solution

$$u(x) = \frac{1}{\pi^2}(\cos \pi x + 2x - 1)$$

3.14 Solve the differential equation in Problem 3.13 using the mixed boundary conditions

$$u(0) = 0, \quad \left(\frac{du}{dx}\right)\bigg|_{x=1} = 0$$

Use the uniform mesh of three linear elements to solve the problem and compare against the exact solution

$$u(x) = \frac{1}{\pi^2}(\cos \pi x - 1)$$

3.15 Solve the differential equation in Problem 3.13 using the Neumann boundary conditions

$$\left(\frac{du}{dx}\right)\bigg|_{x=0} = 0, \quad \left(\frac{du}{dx}\right)\bigg|_{x=1} = 0$$

Use the uniform mesh of three linear elements to solve the problem and compare against the exact solution

$$u(x) = \frac{\cos \pi x}{\pi^2}$$

Note: For Neumann boundary conditions, none of the primary dependent variables are specified, and therefore the solution can be determined within an arbitrary constant for this equation (i.e., when the cu term is not present, the coefficient matrix is singular and cannot be inverted). In such cases, one of the U_i should be set equal to a constant to remove the "rigid-body" mode (i.e., to determine the arbitrary constant in the solution).

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