Chapter 2

MATHEMATICAL PRELIMINARIES, INTEGRAL FORMULATIONS, AND VARIATIONAL METHODS

2.1 GENERAL INTRODUCTION

2.1.1 Variational Principles and Methods

This chapter is devoted to a review of some mathematical preliminaries that prove to be useful in the sequel and to a study of integral formulations and more commonly used variational methods such as the Ritz, Galerkin, collocation, and least-squares methods. Since the finite element method can be viewed as an elementwise application of a variational method (see Section 1.4), it is useful to learn how variational methods work. We begin with a discussion of the general meaning of the phrases "variational methods" and "variational formulations" used in the context of finite element formulations.

The phrase "direct variational methods" refers to methods that make use of variational principles, such as the principles of virtual work and the principle of minimum total potential energy in solid and structural mechanics, to determine approximate solutions of problems [see Oden and Reddy (1983) and Reddy (2002)]. In the classical sense, a *variational principle* has to do with finding the extremum (i.e., minimum or maximum) or stationary values of a functional with respect to the variables of the problem. The functional includes all the intrinsic features of the problem, such as the governing equations, boundary and/or initial conditions, and constraint conditions, if any. In solid and structural mechanics problems, the functional represents the total energy of the system, and in other problems, it is simply an integral representation of the governing equations.

Variational principles have always played an important role in mechanics (see the references at the end of the chapter). First, many problems of mechanics are posed in terms of finding the extremum (i.e., minima or maxima) and thus, by their nature, can be formulated in terms of variational statements. Second, there are problems that can be formulated by other means, such as the conservation laws (as illustrated in Chapter 1), but these can also be formulated by means of variational principles. Third, variational formulations form a powerful basis for obtaining approximate solutions to practical problems, many of which

are intractable otherwise. The principle of minimum total potential energy, for example, can be regarded as a substitute to the equations of equilibrium of an elastic body as well as a basis for the development of displacement finite element models that can be used to determine approximate displacement and stress fields in the body [Reddy (2002)]. Variational formulations can also serve to unify diverse fields, suggest new theories, and provide a powerful means to study the existence and uniqueness of solutions to problems. Similarly, Hamilton's principle [see Reddy (2002)] can be used in lieu of the equations governing dynamical systems, and the variational forms presented by Biot (1972) replace certain equations in linear continuum thermodynamics.

2.1.2 Variational Formulations

The classical use of the phrase "variational formulations" refers to the construction of a functional (whose meaning will be made clear shortly) or a variational principle that is equivalent to the governing equations of the problem. The modern use of the phrase refers to the formulation in which the governing equations are translated into equivalent weighted-integral statements that are not necessarily equivalent to a variational principle. Even those problems that do not admit variational principles in the classical sense (e.g., the Navier–Stokes equations governing the flow of viscous or inviscid fluids) can now be formulated using weighted-integral statements.

The importance of variational formulations of physical laws, in the modern or general sense of the phrase, goes far beyond its use as simply an alternate to other formulations [Oden and Reddy (1983)]. In fact, variational forms of the laws of continuum physics may be the only natural and rigorously correct way to think of them. While all sufficiently smooth fields lead to meaningful variational forms, the converse is not true: There exist physical phenomena which can be adequately modeled mathematically only in a variational setting; they are nonsensical when viewed locally.

The starting point for the discussion of the finite element method is differential equations governing the physical phenomena under study. As such, we shall first discuss why integral statements of the differential equations are needed.

2.1.3 Need for Weighted-Integral Statements

In almost all approximate methods used to determine the solution of differential and/or integral equations, we seek a solution in the form

$$u(\mathbf{x}) \approx U_N(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi_j(\mathbf{x}) \tag{2.1.1}$$

where u represents the solution of a particular differential equation and associated boundary conditions, and U_N is its approximation that is represented as a linear combination of unknown parameters c_j and known functions ϕ_j of position $\mathbf{x}$ in the domain Ω on which the problem is posed. We shall shortly discuss the conditions on ϕ_j. The approximate solution U_N is completely known only when c_j are known. Thus, we must find a means to determine c_j such that U_N satisfies the equations governing u. If somehow we can find U_N that satisfies the differential equation at every point $\mathbf{x}$ of the domain Ω

and conditions on the boundary Γ of Ω, then $U_N(\mathbf{x}) = u(\mathbf{x})$, which is the exact solution of the problem. Of course, approximate methods are not about problems for which exact solutions can be determined by some methods of mathematical analysis; the role of approximate methods is to find an approximate solution of problems that do not admit analytical solutions. When the exact solution cannot be determined, the alternative is to find a solution U_N that satisfies the governing equations in an approximate way. In the process of satisfying the governing equations approximately, we obtain (not accidentally but by planning) N algebraic relations among the N parameters $c_1, c_2, \cdots, c_N$. A detailed discussion of these ideas is given in the next few paragraphs in connection with a specific problem.

Consider the problem of solving the differential equation

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] + c(x)u = f(x) \quad \text{for } 0 < x < L \tag{2.1.2a}$$

subjected to the boundary conditions

$$u(0) = u_0, \quad \left[a(x)\frac{du}{dx}\right]_{x=1} = Q_0 \tag{2.1.2b}$$

where $a(x)$, $c(x)$, and $f(x)$ are known functions, u_0 and Q_0 are known parameters, and $u(x)$ is the function to be determined. The set $a(x)$, $c(x)$, $f(x)$, u_0, and Q_0 is called the problem *data*. An example of the above problem is given by the heat transfer in an uninsulated rod (see Example 1.2.2): here u denotes the temperature (θ), $f(x)$ is the internal heat generation per unit length (Ag), $a(x)$ is the thermal resistance (kA), $c = \beta P$, u_0 is the specified temperature (θ_0), and Q_0 is the specified heat.

We seek an approximate solution over the entire domain $\Omega = (0, L)$ in the form

$$U_N \equiv \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x) \tag{2.1.3}$$

where the c_j are coefficients to be determined and $\phi_j(x)$ and $\phi_0(x)$ are functions chosen such that the specified boundary conditions of the problem are satisfied by the N-parameter approximate solution U_N. Note that the particular form in (2.1.3) has two parts: one containing the unknowns ($\sum c_j \phi_j$) that is termed the homogeneous part and the other is the nonhomogeneous part (ϕ_0) that has the sole purpose of satisfying the specified boundary conditions of the problem. Since ϕ_0 satisfies the boundary conditions, the sum $\sum c_j \phi_j$ must satisfy, for arbitrary c_j, the homogeneous form of the boundary conditions ($Bu = \hat{u}$ is said to be a nonhomogeneous boundary condition when $\hat{u} \neq 0$, and it is termed a homogeneous boundary condition when $\hat{u} = 0$; here B denotes some operator). Thus, in the present case, the actual boundary conditions are both nonhomogeneous ($B = 1$ and $\hat{u} = u_0$ at $x = 0$, and $B = a(x)(d/dx)$ and $\hat{u} = Q_0$ at $x = L$). The particular form (2.1.3) is convenient in selecting ϕ_0 and ϕ_j. Thus, ϕ_0 and ϕ_j satisfy the conditions

$$B\phi_0 = \hat{u}, \quad B\phi_j = 0 \text{ for all } j = 1, 2, \cdots, n \tag{2.1.4}$$

To be more specific, let $L = 1, u_0 = 1, Q_0 = 0, a(x) = x, c(x) = 1, f(x) = 0$, and $N = 2$. Then we choose the approximate solution in the form

$$U_2 = c_1\phi_1 + c_2\phi_2 + \phi_0 \text{ with } \phi_0 = 1,\ \phi_1(x) = x^2 - 2x,\ \phi_2 = x^3 - 3x$$

that satisfies the boundary conditions (2.1.2*b*) of the problem for any values of c_1 and c_2 because

$$\phi_0(0)=1, \quad \left(x\frac{d\phi_0}{dx}\right)_{x=1}=0; \quad \phi_j(0)=0, \quad \left(\frac{d\phi_j}{dx}\right)_{x=1}=0 \text{ for } j=1,2 \tag{2.1.5}$$

To make U_2 satisfy the differential equation (2.1.2*a*), we must have

$$\begin{aligned}-\frac{dU_2}{dx}-x\frac{d^2U_2}{dx^2}+U_2 &= -2c_1(x-1)-3c_2(x^2-1)-2c_1x-6c_2x^2\\ &\quad +c_1(x^2-2x)+c_2(x^3-3x)+1=0\end{aligned} \tag{2.1.6}$$

Since this expression must be zero for any value of x, the coefficients of the various powers of x must be zero:

$$\begin{aligned}1+2c_1+3c_2&=0\\ -(6c_1+3c_2)&=0\\ c_1-9c_2&=0\\ c_2&=0\end{aligned}$$

The above relations are inconsistent; hence, there is *no solution* to the equations. On the other hand, we can require the approximate solution U_N to satisfy the differential equation (2.1.2*a*) in the weighted-integral sense,

$$\int_0^1 w(x)R\,dx=0 \tag{2.1.7}$$

where R denotes the left side of the equality in (2.1.6) and is called the *residual,*

$$R\equiv -\frac{dU_N}{dx}-x\frac{d^2U_N}{dx^2}+U_N$$

and $w(x)$ is called a *weight function.* From (2.1.7), we obtain as many linearly independent equations as there are linearly independent functions for $w(x)$. The number of linearly independent choices of w must be restricted to $N=2$ so that we have exactly the same number of equations as the number of unknown coefficients, c_j. For example, in the present example, if we take $w=1$ and $w=x$, we obtain

$$\begin{aligned}0&=\int_0^1 1\cdot R\,dx=(1+2c_1+3c_2)+\frac{1}{2}(-6c_1-3c_2)+\frac{1}{3}(c_1-9c_2)+\frac{1}{4}c_2\\ 0&=\int_0^1 x\cdot R\,dx=\frac{1}{2}(1+2c_1+3c_2)+\frac{1}{3}(-6c_1-3c_2)+\frac{1}{4}(c_1-9c_2)+\frac{1}{5}c_2\end{aligned}$$

or

$$\frac{2}{3}c_1+\frac{5}{4}c_2=1, \quad \frac{3}{4}c_1+\frac{31}{20}c_2=\frac{1}{2} \tag{2.1.8}$$

which provides two linearly independent equations for c_1 and c_2 (whose solution is $c_1=\frac{222}{23}$ and $c_2=-\frac{100}{23}$).

The above discussion clearly demonstrates the need for weighted-integral statements of the type in (2.1.7); they provide the means for obtaining as many algebraic equations

as there are unknown coefficients in the approximate solution. This chapter deals with the construction of different types of integral statements used in different variational methods. The variational methods differ from each other in the choice of the weight function w and/or the integral statement used, which in turn dictates the choice of the approximation functions ϕ_j. In the finite element method, a given domain is viewed as an assemblage of subdomains (i.e., finite elements), and an approximate solution is sought over each subdomain in the same way as in variational methods. Therefore, it is informative to study variational methods before we study the finite element method.

Our goal in this chapter is to illustrate the basic steps in the weighted-integral formulations and associated approximations of boundary value, eigenvalue, and initial value problems. Toward this goal, we first introduce the necessary terminology and notation.

2.2 SOME MATHEMATICAL CONCEPTS AND FORMULAE

2.2.1 Coordinate Systems and the Del Operator

In the analytical description of physical phenomena, a coordinate system in the chosen frame of reference is introduced, and various physical quantities involved in the description are expressed in terms of measurements made in that system. The vector and tensor quantities are expressed in terms of their components in that coordinate system. For example, a vector $\mathbf{A}$ in a three-dimensional space may be expressed in terms of its components (a_1, a_2, a_3) and *basis vectors* $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ ($\mathbf{e}_i$ are not necessarily unit vectors) as*

$$\mathbf{A} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \tag{2.2.1}$$

When the basis vectors of a coordinate system are constants, i.e., with fixed lengths and directions, the coordinate system is called a *Cartesian coordinate system*. The general Cartesian system is oblique. When the Cartesian system is orthogonal, it is called *rectangular Cartesian*. The rectangular Cartesian coordinates are denoted by

$$(x_1, x_2, x_3) \text{ or } (x, y, z) \tag{2.2.2}$$

The familiar rectangular Cartesian coordinate system is shown in Fig. 2.2.1. We shall always use a right-hand coordinate system. When the basis vectors are of unit lengths and mutually orthogonal, they are called *orthonormal*. In many situations an *orthonormal basis* simplifies calculations. We denote an orthonormal Cartesian basis by

$$(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \text{ or } (\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z) \tag{2.2.3}$$

For an orthonormal basis, the vector $\mathbf{A}$ can be written as

$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$

where $\hat{\mathbf{e}}_i$ $(i = 1, 2, 3)$ is the orthonormal basis and A_i are the corresponding *physical components* (i.e., the components have the same physical dimensions as the vector). Although the analytical description depends upon the chosen coordinate system and may appear different

*Vectors and matrices in this book are written with boldface letters.

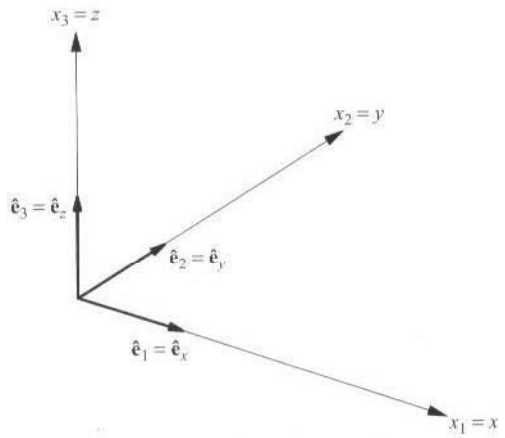


Figure 2.2.1 A rectangular Cartesian coordinate system, $(x_1, x_2, x_3) = (x, y, z)$; $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) = (\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$ are the unit basis vectors.

in another type of coordinate system, one must keep in mind that *the laws of nature are independent of the choice of coordinate system.*

It is useful to abbreviate a summation of terms by understanding that a repeated index means summation over all values of that index. Thus, the summation

$$\mathbf{A} = \sum_{i=1}^{3} A_i \mathbf{e}_i$$

can be shortened to

$$\mathbf{A} = A_i \mathbf{e}_i \tag{2.2.4}$$

The repeated index is a *dummy index* and thus can be replaced by *any other symbol that has not already been used.* Thus, we can also write

$$\mathbf{A} = A_i \mathbf{e}_i = A_m \mathbf{e}_m$$

and so on.

The "dot product" $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$ and "cross product" $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j$ of base vectors in a right-handed system are defined by

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \equiv \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \tag{2.2.5}$$

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j \equiv \varepsilon_{ijk} \hat{\mathbf{e}}_k \tag{2.2.6}$$

where δ_{ij} is the *Kronecker delta* and ε_{ijk} is the *alternating symbol* or *permutation symbol*

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } i, j, k \text{ are in cyclic order} \\ & \text{and not repeated } (i \neq j \neq k) \\ -1, & \text{if } i, j, k \text{ are not in cyclic order} \\ & \text{and not repeated } (i \neq j \neq k) \\ 0, & \text{if any of } i, j, k \text{ are repeated} \end{cases} \tag{2.2.7}$$

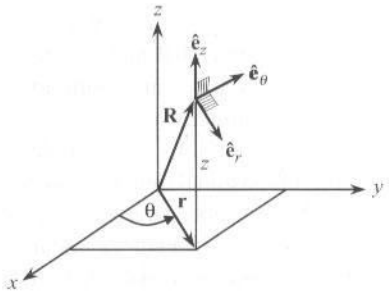


Figure 2.2.2 Cylindrical coordinate system.

Differentiation of vector functions with respect to the coordinates is common in science and engineering. Most of the operations involve the "del operator," denoted by ∇. In a rectangular Cartesian system, it has the form

$$\nabla \equiv \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \tag{2.2.8}$$

It is important to note that the del operator has some of the properties of a vector, but it does not have them all because it is an operator. The operation $\nabla\phi(\mathbf{x})$ is called the *gradient* of a scalar function ϕ whereas $\nabla \times \mathbf{A}(\mathbf{x})$ is called the *curl* of a vector function $\mathbf{A}$. The operator $\nabla^2 \equiv \nabla \cdot \nabla$ is called the Laplace operator. In a 3-D rectangular Cartesian coordinate system, it has the form

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.2.9}$$

We have the following relations between the rectangular Cartesian coordinates (x, y, z) and cylindrical coordinates (r, θ, z) (see Fig. 2.2.2):

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \tag{2.2.10}$$

$$\hat{\mathbf{e}}_r = \cos\theta\,\hat{\mathbf{e}}_x + \sin\theta\,\hat{\mathbf{e}}_y, \quad \hat{\mathbf{e}}_\theta = -\sin\theta\,\hat{\mathbf{e}}_x + \cos\theta\,\hat{\mathbf{e}}_y, \quad \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z \tag{2.2.11}$$

$$\frac{\partial \hat{\mathbf{e}}_r}{\partial\theta} = -\sin\theta\,\hat{\mathbf{e}}_x + \cos\theta\,\hat{\mathbf{e}}_y = \hat{\mathbf{e}}_\theta, \quad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial\theta} = -\cos\theta\,\hat{\mathbf{e}}_x - \sin\theta\hat{\mathbf{e}}_y = -\hat{\mathbf{e}}_r \tag{2.2.12}$$

and all other derivatives of the base vectors are zero. For more on vector calculus, see Reddy and Rasmussen (1982) and Reddy (2002), among other references.

2.2.2 Boundary Value, Initial Value, and Eigenvalue Problems

The objective of most analyses is to determine unknown functions, called *dependent variables,* that are governed by a set of differential equations posed in a given domain Ω and some conditions on the boundary Γ of the domain. Often, a domain not including its boundary is called an open domain. A domain Ω with its boundary Γ is called a closed domain and is denoted by $\bar{\Omega} = \Omega \cup \Gamma$.

A function u of several variables $(x, y, \cdots)$ is said to be of class $C^m(\Omega)$ in a domain Ω if all its partial derivatives with respect to $(x, y, \cdots)$ of order up to and including m exist

and are *continuous* in Ω. Thus, if u is of class C^0 in a two-dimensional domain Ω, then u is continuous in Ω (i.e., $\partial u/\partial x$ and $\partial u/\partial y$ exist but may not be continuous). Similarly, if u is of class C^1, then u, $\partial u/\partial x$, and $\partial u/\partial y$ exist and are continuous (hence, $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$, and $\partial^2 u/\partial y \partial x$ exist but may not be continuous).

When the dependent variables are functions of one independent variable (say, x), the domain is a line segment (i.e., one-dimensional) and the end points of the domain are called boundary points. When the dependent variables are functions of two independent variables (say, x and y), the domain is two-dimensional and the boundary is the closed curve enclosing it. In a three-dimensional domain, dependent variables are functions of three coordinates (say x, y, and z) and the boundary is a two-dimensional surface.

As discussed in Section 1.2, a differential equation is said to describe a *boundary value problem* over the domain Ω if the dependent variable and possibly its derivatives are required to take specified values on the boundary Γ of Ω. An *initial value problem* is one in which the dependent variable and possibly its derivatives are specified initially (i.e., at time $t = 0$). Initial value problems are generally time-dependent problems. Examples of boundary and initial value problems were discussed in Section 1.2. A problem can be both a boundary value and initial value problem if the dependent variable is subject to both boundary and initial conditions. Another type of problem we encounter is one in which a differential equation governing the dependent unknown also contains an unknown parameter and we are required to find both the dependent variable and the parameter such that the differential equation and associated boundary conditions are satisfied. Such problems are called *eigenvalue problems*. Examples of various types of problems we encounter in science and engineering are given below (the mathematical classification of differential equations into elliptic, parabolic, and hyperbolic is of no interest at the moment).

Boundary Value Problems. *Steady State Heat Transfer in a Fin and Axial Deformation of a Bar* [Fig. 2.2.3(*a*)]: Find $u(x)$ that satisfies the second-order differential equation and *boundary conditions*:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu = f \quad \text{for} \quad 0 < x < L \tag{2.2.13a}$$

$$u(0) = u_0, \quad \left(a\frac{du}{dx}\right)_{x=L} = q_0 \tag{2.2.13b}$$

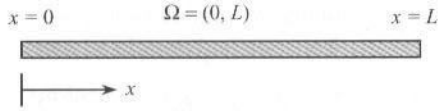
Bending of Elastic Beams under Transverse Load: Find $u(x)$ that satisfies the fourth-order differential equation and *boundary conditions*:

$$\frac{d^2}{dx^2}\left(b\frac{d^2u}{dx^2}\right) + cu = f \quad \text{for} \quad 0 < x < L \tag{2.2.14a}$$

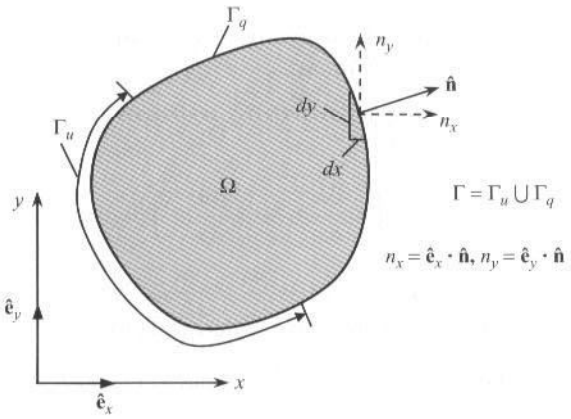
$$u(0) = u_0, \quad \left(\frac{du}{dx}\right)_{x=0} = d_0$$

$$\left[\frac{d}{dx}\left(b\frac{d^2u}{dx^2}\right)\right]_{x=L} = m_0, \quad \left(b\frac{d^2u}{dx^2}\right)_{x=L} = v_0 \tag{2.2.14b}$$

Steady Heat Conduction in a Two-Dimensional Region and *Transverse Deflections of a Membrane* [Fig. 2.2.3(*b*)]: Find $u(x, y)$ that satisfies the second-order partial differential



(*a*) A one-dimensional domain



(*b*) A two-dimensional domain

Figure 2.2.3 (*a*) One-dimensional domain. (*b*) Two-dimensional domain.

equation and *boundary conditions*:

$$-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]+cu=f \quad \text{in } \Omega \tag{2.2.15a}$$

$$u=u_0 \text{ on } \Gamma_u, \quad \left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial x}n_y\right)=q_0 \text{ on } \Gamma_q \tag{2.2.15b}$$

where (n_x, n_y) are the direction cosines of the unit normal vector $\hat{\mathbf{n}}$ to the boundary Γ_q.

Initial Value Problems. *A General First-Order Equation:* Find $u(t)$ that satisfies the first-order differential equation and *initial conditions*:

$$a\frac{du}{dt}+cu=f \quad \text{for} \quad 0<t\le T \tag{2.2.16a}$$

$$u(0)=u_0 \tag{2.2.16b}$$

A General Second-Order Equation: Find $u(t)$ that satisfies the second-order differential equation and *initial conditions*:

$$a\frac{du}{dt}+b\frac{d^2u}{dt^2}+cu=f \quad \text{for} \quad 0<t\le T \tag{2.2.17a}$$

$$u(0)=u_0, \left(b\frac{du}{dt}\right)_{t=0}=v_0 \tag{2.2.17b}$$

Boundary and Initial Value Problems. *Unsteady Heat Transfer in a Rod:* Find $u(x, t)$ that satisfies the partial differential equation and *initial* and *boundary conditions*:

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\rho\frac{\partial u}{\partial t}=f(x,t) \quad \text{for} \quad 0<x<L,\ 0<t\leq T \tag{2.2.18a}$$

$$u(0,t)=d_0(t), \quad \left(a\frac{du}{dx}\right)_{x=L}=q_0(t), \quad u(x,0)=u_0(x) \tag{2.2.18b}$$

Unsteady Motion of a Membrane: Find $u(x, y, t)$ that satisfies the partial differential equation and *initial* and *boundary conditions*:

$$-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]+\rho\frac{\partial^2 u}{\partial t^2}=f(x,y,t) \quad \text{in } \Omega,\ 0<t\leq T \tag{2.2.19a}$$

$$u=u_0(t) \text{ on } \Gamma_u, \quad \left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial y}n_y\right)=q_0(t) \text{ on } \Gamma_q \tag{2.2.19b}$$

$$u(x,y,0)=d_0, \quad \dot{u}(x,y,0)=v_0 \tag{2.2.19c}$$

where the superposed dot indicates a derivative with respect to time t.

Eigenvalue Problems. *Axial Vibrations of a Bar:* Find $u(x)$ and λ that satisfy the differential equation and *boundary conditions*:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)-\lambda u=0 \quad \text{for} \quad 0<x<L \tag{2.2.20a}$$

$$u(0)=0, \quad \left(a\frac{du}{dx}\right)_{x=L}=0 \tag{2.2.20b}$$

Transverse Vibrations of a Membrane: Find $u(x, y)$ and λ that satisfy the partial differential equation and *boundary conditions*:

$$-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]-\lambda u=0 \quad \text{in } \Omega \tag{2.2.21a}$$

$$u=0 \text{ on } \Gamma_u, \quad \left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial y}n_y\right)=0 \text{ on } \Gamma_q \tag{2.2.21b}$$

The values of λ are called *eigenvalues,* and the associated functions u are called *eigenfunctions*.

The set of specified functions and parameters (e.g., a, b, c, ρ, f, u_0, d_0, q_0, v_0, and so on) are called the *data* of the problem. Differential equations in which the right-hand side f is zero are called *homogeneous differential equations,* and boundary (initial) conditions in which the specified data is zero are called homogeneous boundary (initial) conditions. The *exact solution* of a differential equation is the function that identically satisfies the differential equation at every point of the domain and for all times $t > 0$, and satisfies the specified boundary and/or initial conditions.

2.2.3 Integral Identities

Integration by parts is frequently used in the integral formulation of differential equations. In two- and three-dimensional cases, integration by parts is carried out with the help of

the gradient and divergence theorems. In this section, we derive some useful identities for future use.

Integration-by-Parts Formulae. Let u, v, and w be sufficiently differentiable functions of the coordinate x. Then the following integration-by-parts formula holds:

$$\int_a^b w\,\frac{dv}{dx}\,dx = \int_a^b w\,dv = -\int_a^b v\,dw + [wv]_a^b$$
$$= -\int_a^b v\,\frac{dw}{dx}\,dx + w(b)\,v(b) - w(a)\,v(a) \tag{2.2.22}$$

This identity can easily be established. First, note the following identity from the product rule of differentiation:

$$\frac{d}{dx}(wv) = \frac{dw}{dx}\,v + w\,\frac{dv}{dx}$$

Therefore,

$$w\frac{dv}{dx} = \frac{d}{dx}(wv) - \frac{dw}{dx}\,v$$

Integrating both sides over the interval (a, b), we obtain the identity in (2.2.22)

$$\int_a^b w\frac{dv}{dx}\,dx = \int_a^b \left[\frac{d}{dx}(wv) - \frac{dw}{dx}\,v\right]dx$$
$$= \int_a^b \frac{d}{dx}(wv)\,dx - \int_a^b \frac{dw}{dx}\,v\,dx$$
$$= [wv]_a^b - \int_a^b \frac{dw}{dx}\,v\,dx$$

Next, consider the expression

$$\int_a^b w\,\frac{d^2u}{dx^2}\,dx = \int_a^b w\,\frac{d}{dx}\left(\frac{du}{dx}\right)dx = \int_a^b w\,\frac{dv}{dx}\,dx$$

where $v \equiv \frac{du}{dx}$. Using (2.2.22), we obtain

$$\int_a^b w\,\frac{d^2u}{dx^2}\,dx = -\int_a^b v\,\frac{dw}{dx}\,dx + w(b)\,v(b) - w(a)\,v(a)$$
$$= -\int_a^b \frac{du}{dx}\,\frac{dw}{dx}dx + w(b)\frac{du}{dx}\bigg|_b - w(a)\frac{du}{dx}\bigg|_a \tag{2.2.23a}$$

or

$$-\int_a^b \frac{du}{dx}\,\frac{dw}{dx}\,dx = \int_a^b w\,\frac{d^2u}{dx^2}\,dx + w(a)\frac{du}{dx}\bigg|_a - w(b)\frac{du}{dx}\bigg|_b \tag{2.2.23b}$$

Similarly,

$$\int_a^b v\,\frac{d^4w}{dx^4}\,dx = \int_a^b v\,\frac{d^2}{dx^2}\,\frac{d^2w}{dx^2}\,dx$$
$$= \int_a^b v\,\frac{d^2u}{dx^2}\,dx$$

where $u \equiv \frac{d^2w}{dx^2}$. Using (2.2.23$a$) with $w = v$, we can write the right-hand side as

$$-\int_a^b \frac{du}{dx}\frac{dv}{dx}\,dx + v(b)\frac{du}{dx}\bigg|_b - v(a)\frac{du}{dx}\bigg|_a \tag{2.2.24a}$$

We use (2.2.23b) with $w = u$ and $u = v$ to write (2.2.24a) as

$$\int_a^b u\frac{d^2v}{dx^2}\,dx + u(a)\frac{dv}{dx}\bigg|_a - u(b)\frac{dv}{dx}\bigg|_b + v(b)\frac{du}{dx}\bigg|_b - v(a)\frac{du}{dx}\bigg|_a \tag{2.2.24b}$$

and, finally, replacing u by its actual value $u = d^2w/dx^2$, we arrive at

$$\int_a^b v\frac{d^4w}{dx^4}\,dx = \int_a^b \frac{d^2w}{dx^2}\frac{d^2v}{dx^2}\,dx + \frac{d^2w}{dx^2}\bigg|_a\frac{dv}{dx}\bigg|_a - \frac{d^2w}{dx^2}\bigg|_b\frac{dv}{dx}\bigg|_b + v(b)\frac{d^3w}{dx^3}\bigg|_b - v(a)\frac{d^3w}{dx^3}\bigg|_a \tag{2.2.25}$$

Gradient and Divergence Theorems. Let ∇ and ∇^2 denote, respectively, the gradient operator and the Laplace operator in the two-dimensional Cartesian rectangular coordinate system (x, y):

$$\nabla = \hat{\mathbf{e}}_x\frac{\partial}{\partial x} + \hat{\mathbf{e}}_y\frac{\partial}{\partial y}, \quad \nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.2.26}$$

where $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ denote the unit basis vectors along the x and y coordinates, respectively. If $F(x, y)$ and $G(x, y)$ are scalar functions of class $C^0(\Omega)$ in the two-dimensional domain Ω shown in Fig. 2.2.3(b), the following gradient and divergence theorems hold.

Gradient Theorem

$$\int_\Omega \text{grad}F\,dxdy \equiv \int_\Omega \nabla F\,dxdy = \oint_\Gamma \hat{\mathbf{n}}F\,ds$$
$$\int_\Omega \left(\hat{\mathbf{e}}_x\frac{\partial F}{\partial x} + \hat{\mathbf{e}}_y\frac{\partial F}{\partial y}\right)dxdy = \oint_\Gamma (n_x\hat{\mathbf{e}}_x + n_y\hat{\mathbf{e}}_y)F\,ds \tag{2.2.27a}$$

The second equation implies, because two vectors are equal if and only if their components are equal, that the following relations hold:

$$\int_\Omega \frac{\partial F}{\partial x}\,dxdy = \oint_\Gamma n_x F\,ds, \quad \int_\Omega \frac{\partial F}{\partial y}\,dxdy = \oint_\Gamma n_y F\,ds \tag{2.2.27b}$$

Divergence Theorem

$$\int_\Omega \text{div}\mathbf{G}\,dxdy \equiv \int_\Omega \nabla\cdot\mathbf{G}\,dxdy = \oint_\Gamma \hat{\mathbf{n}}\cdot\mathbf{G}\,ds \tag{2.2.28a}$$

$$\int_\Omega \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}\right)dxdy = \oint_\Gamma (n_x G_x + n_y G_y)ds \tag{2.2.28b}$$

Here the dot denotes the scalar product of vectors, $\hat{\mathbf{n}}$ denotes the unit vector normal to the surface Γ of the domain Ω; n_x and n_y (G_x and G_y) are the rectangular components of $\hat{\mathbf{n}}(\mathbf{G})$; and the circle on the boundary integral indicates that the integration is taken over the entire

boundary [see Fig. 2.2.3(*b*)]. The direction cosines n_x and n_y of the unit vector $\hat{\mathbf{n}}$ can be written as

$$n_x = \cos(x, \hat{\mathbf{n}}) = \hat{\mathbf{e}}_x \cdot \hat{\mathbf{n}}, \quad n_y = \cos(y, \hat{\mathbf{n}}) = \hat{\mathbf{e}}_y \cdot \hat{\mathbf{n}} \tag{2.2.29}$$

where $\cos(x, \hat{\mathbf{n}})$, for example, is the cosine of the angle between the positive x direction and the unit vector $\hat{\mathbf{n}}$.

The following identities, which can be derived using the gradient and divergence theorems, will be useful in the sequel. Let w and G be scalar functions defined in a two-dimensional domain Ω. Then

$$\int_\Omega (\nabla G)\, w \, dxdy = -\int_\Omega (\nabla w)\, G \, dxdy + \int_\Gamma \hat{\mathbf{n}}\, w\, G \, ds \tag{2.2.30a}$$

and

$$-\int_\Omega \left(\nabla^2 G\right) w \, dxdy = \int_\Omega \nabla w \cdot \nabla G \, dxdy - \oint_\Gamma \frac{\partial G}{\partial n} w \, ds \tag{2.2.30b}$$

where $\partial/\partial n$ denotes the normal derivative operator,

$$\frac{\partial}{\partial n} = \hat{\mathbf{n}} \cdot \nabla = n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y} \tag{2.2.31}$$

The following component form of (2.2.30a), with an appropriate change of variables, is useful in the sequel:

$$\int_\Omega w \frac{\partial G}{\partial x} \, dxdy = -\int_\Omega \frac{\partial w}{\partial x} G \, dxdy + \oint_\Gamma n_x\, w\, G \, ds \tag{2.2.32a}$$

$$\int_\Omega w \frac{\partial G}{\partial y} \, dxdy = -\int_\Omega \frac{\partial w}{\partial y} G \, dxdy + \oint_\Gamma n_y\, w\, G \, ds \tag{2.2.32b}$$

Equations (2.2.32*a*) and (2.2.32*b*) can easily be established by means of (2.2.27*b*).

2.2.4 Linear and Bilinear Functionals

Consider the integral expression of the form

$$I(u) = \int_a^b F(x, u, u')\, dx, \quad u = u(x), \quad u' = \frac{du}{dx} \tag{2.2.33}$$

where the integrand $F(x, u, u')$ is a given function of the coordinate x (independent variable), dependent variable u, and its derivative du/dx. For a given real function $u = u(x)$, $I(u)$ is a real number. Therefore, I can be viewed as an operator that transforms functions $u(x)$ into real numbers, and such operators are called *functionals*. We shall use the term "functional" to describe functions defined by integrals whose arguments themselves are functions. Thus, loosely speaking, a functional is a "function of functions." A formal definition from functional analysis is that a functional is an operator I mapping functions u from a linear vector space into real number field. The following integrals qualify as functionals:

$$I(u) = \int_a^b \left(p(x)\frac{du}{dx} + q(x)u^2\right) dx + Pu(a)$$

$$I(u, v) = \int_\Omega \left(p(x, y)\frac{du}{dx}\frac{dv}{dx} + q(x, y)v\right) dxdy + \int_\Gamma Q\, u \, ds \tag{2.2.34}$$

where u and v are the dependent variables and all other parameters are either constants or functions of position.

A functional $I(u)$ is said to be *linear* in u if and only if it satisfies the relation

$$I(\alpha u + \beta v) = \alpha I(u) + \beta I(v) \tag{2.2.35}$$

for any real numbers α and β and dependent variables u and v. Examples of linear functionals are provided by

$$I(u) = \int_a^b f(x)\, u\, dx + q\, u(b), \quad I(u, v) = \int_\Omega (f(x, y)\, u + q(x, y) v)\, dxdy$$

Note that the functionals

$$I_1(u) = \int_a^b u \frac{du}{dx}\, dx, \quad I_2(u) = \int_a^b f(x)\, u\, dx + c, \quad c \text{ is a constant}$$

do not qualify as linear functionals (why?).

A functional $B(u, v)$ is said to be *bilinear* if it is linear in each of its arguments u and v:

$$B(\alpha u_1 + \beta u_2, v) = \alpha B(u_1, v) + \beta B(u_2, v) \text{ linear in the first argument} \tag{2.2.36a}$$

$$B(u, \alpha v_1 + \beta v_2) = \alpha B(u, v_1) + \beta B(u, v_2) \text{ linear in the second argument} \tag{2.2.36b}$$

where u, u_1, u_2, v, v_1, and v_2 are dependent variables and α and β are real numbers. Note that a bilinear functional necessarily contains two arguments (dependent variables), and it must be linear with respect to each argument. Examples of bilinear forms are

$$\begin{aligned} B_1(u, v) &= \int_a^b \left(p(x) uv + q(x) \frac{du}{dx} \frac{dv}{dx} \right) dx + ku(a)v(a) \\ B_2(\mathbf{u}, \mathbf{v}) &= \int_a^b \left(p(x) \mathbf{u} \cdot \mathbf{v} + q(x) \frac{d\mathbf{u}}{dx} \cdot \mathbf{v} \right) dx \\ B_3(\mathbf{u}, \mathbf{v}) &= \int_\Omega (p(\mathbf{x}) \mathbf{u} \cdot \mathbf{v} + q(\mathbf{x})\, \nabla \mathbf{u} \cdot \nabla \mathbf{v})\, d\mathbf{x} \end{aligned} \tag{2.2.37}$$

The functionals

$$\begin{aligned} I_1(u, v) &= \int_a^b \left[p(x) u^2 + q(x) \left(\frac{dv}{dx} \right)^2 \right] dx \\ I_2(\mathbf{u}, \mathbf{v}) &= \int_a^b \left(p(x) \mathbf{u} \cdot \mathbf{u} + q(x) \frac{d\mathbf{v}}{dx} \cdot \frac{d\mathbf{v}}{dx} \right) dx \\ I_3(u, v) &= \int_\Omega \left[p(x, y) \left(\frac{\partial u}{\partial x} \right)^2 v + q(x, y) u \right] dxdy \end{aligned} \tag{2.2.38}$$

are not linear in their arguments.

A bilinear form $B(u, v)$ is said to be *symmetric* in its arguments u and v if

$$B(u, v) = B(v, u) \tag{2.2.39}$$

for all u and v. Examples of symmetric bilinear forms are provided by $B_1(u, v)$ and $B_3(\mathbf{u}, \mathbf{v})$ listed above. Note that $B_2(\mathbf{u}, \mathbf{v})$ is not symmetric.

A *quadratic* functional $Q(u)$ is one that satisfies the relation

$$Q(\alpha u) = \alpha^2 Q(u) \tag{2.2.40}$$

for all real numbers α.

2.3 ELEMENTS OF CALCULUS OF VARIATIONS

2.3.1 Introduction

Calculus of variations is a branch of mathematics that deals with extrema and stationary behavior of functionals. As discussed in the introduction to this chapter, there are many problems that can be posed only as finding the extremum of functionals [see, for example, the *Brachistochrone problem, geodesic problem,* and *isoperimetric problem* discussed on pages 129 and 130 of Reddy (2002)]. The field of solid and structural mechanics heavily depends on the use of energy principles that are stated in terms of finding the extrema or stationary values of functionals to construct finite element models (e.g., the principle of minimum total potential energy, the principle of maximum total complementary energy, and the Hu–Washizu and Hellinger–Reissner mixed variational principles). The direct variational methods like the Ritz method use variational principles to obtain approximate solutions directly, bypassing the derivation of the governing equations.

In this section we study the concept of variational operator and its properties, first variation of functionals, and Euler equations resulting from the condition of vanishing of the first variation of a functional. In the interest of keeping the scope of the study within reasonable limits, only necessary concepts are covered.

Although the material covered in this section is not absolutely necessary for an understanding of the finite element method, it is deemed useful, at least for those readers who have solid and structural mechanics background, to understand the concepts from calculus of variations. Other readers may skip the section and go directly to Section 2.4.

2.3.2 Variational Operator and First Variation

Consider the function $F(x, u, u')$. For an arbitrarily fixed value of the independent variable x, F depends on u and u'. The change ϵv in u, where ϵ is a constant and v is a function, is called the *variation* of u and is denoted by δu (see Fig. 2.3.1):

$$\delta u \equiv \epsilon v \tag{2.3.1}$$

The operator δ is called the *variational operator*. The variation δu of a function u represents an admissible change in the function $u(x)$ at a *fixed* value of the independent variable x. If u is specified at a point (usually on the boundary), the variation of u is zero there because the specified value cannot be varied. Thus, the variation of a function u should satisfy the homogeneous form of the boundary conditions for u. The variation δu represents a *virtual* but admissible change in u. Associated with this change in u (i.e., when u is changed to $u + \epsilon v$), there is a change in F,

$$\Delta F = F(x, u + \epsilon v, u' + \epsilon v') - F(x, u, u')$$

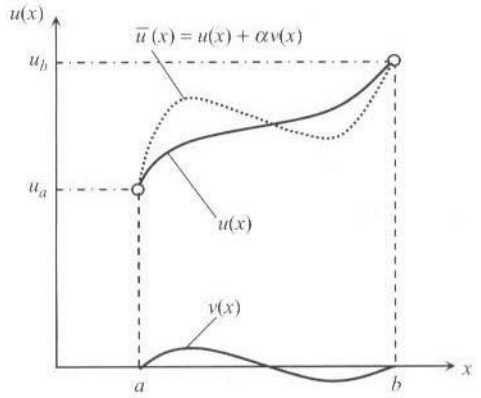


Figure 2.3.1 The variations of $u(x)$.

Expanding in powers of ϵ gives (treating $u+\epsilon v$ and $u'+\epsilon v'$ as dependent functions)

$$\begin{aligned}\Delta F &= F(x,u,u') + \epsilon v\frac{\partial F}{\partial u} + \epsilon v'\frac{\partial F}{\partial u'} + \frac{(\epsilon v)^2}{2!}\frac{\partial^2 F}{\partial u^2} \\ &\quad + \frac{(\epsilon v)(\epsilon v')}{2!}\frac{\partial^2 F}{\partial u\partial u'} + \frac{(\epsilon v')^2}{2!}\frac{\partial^2 F}{\partial u'^2} + \cdots - F(x,u,u') \\ &= \epsilon v\frac{\partial F}{\partial u} + \epsilon v'\frac{\partial F}{\partial u'} + \epsilon R_1(\epsilon) \end{aligned} \tag{2.3.2}$$

where $lim_{\epsilon\to 0} R_1(\epsilon) = 0$. The *first variation* of F is defined by

$$\begin{aligned}\delta F &= \epsilon\left[\lim_{\epsilon\to 0}\frac{F(x,u+\epsilon v,u'+\epsilon v') - F(x,u,u')}{\epsilon}\right] = \epsilon\left[\lim_{\epsilon\to 0}\frac{\Delta F}{\epsilon}\right] \\ &\equiv \epsilon\left[\frac{d}{d\epsilon}\left(F(u+\epsilon v)\right)\right]_{\epsilon=0} \\ &= \epsilon\left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right) = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \end{aligned} \tag{2.3.3}$$

Thus, the first variation of F can be written in terms of the variations of the dependent variable u and its derivatives. Note that in the special case when $F = u$, Eq. (2.3.3) gives the result in Eq. (2.3.1). Further, note the analogy between the first variation, (2.3.3), and the total differential of F,

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du' \tag{2.3.4}$$

Since x is *not varied* during the variation of u to $u+\delta u$, $dx = 0$ and the analogy between δF and dF becomes apparent, i.e., *δ acts as a differential operator with respect to dependent variables.*

The above discussion can be extended to two dimensions and to functions F that depend on more than one dependent variable in two or more dimensions. Let $F = F(x, y, u, v, u_x, v_x, u_y, v_y)$, where $u = u(x, y)$ and $v = v(x, y)$ are dependent variables,

and $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, and so on. The first variation of F is given by

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial v_x}\delta v_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial v_y}\delta v_y$$

It can easily be verified that the laws of variation of sums, products, ratios, powers, and so forth are completely analogous to the corresponding laws of differentiation. For example, if $F_1 = F_1(u)$ and $F_2 = F_2(u)$, then

$$\begin{aligned} \delta(F_1 \pm F_2) &= \delta F_1 \pm \delta F_2 \\ \delta(F_1\, F_2) &= \delta F_1\, F_2 + F_1 \delta F_2 \\ \delta\left(\frac{F_1}{F_2}\right) &= \frac{\delta F_1\, F_2 - F_1\, \delta F_2}{F_2^2} \\ \delta(F_1)^n &= n(F_1)^{n-1}\delta F_1 \end{aligned} \tag{2.3.5}$$

If $G = G(u, v, w)$ is function of several dependent variables u, v, and w (and possibly their derivatives), the total variation is the sum of the partial variations:

$$\delta G = \delta_u G + \delta_v G + \delta_w G \tag{2.3.6}$$

Furthermore, the variational operator can commute with differential and integral operators (as long as the coordinates x and y are the fixed coordinates):

$$\frac{d}{dx}(\delta u) = \frac{d}{dx}(\epsilon v) = \epsilon\frac{dv}{dx} = \epsilon v' = \delta u' = \delta\left(\frac{du}{dx}\right) \tag{2.3.7a}$$

$$\delta\int_a^b u(x)\,dx = \int_a^b \delta u(x)\,dx \tag{2.3.7b}$$

The first variation of a functional can now be computed readily. Consider the functional in Eq. (2.2.33). The first variation of $I(u)$ is

$$\begin{aligned} \delta I(u) &= \delta\int_a^b F(x, u, u')\,dx = \int_a^b \delta F(x, u, u')\,dx \\ &= \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right)dx \end{aligned} \tag{2.3.8}$$

More specific functionals are considered in the next example.

Example 2.3.1

Consider the functionals in (2.2.34). We wish to find their first variations. We have

$$\begin{aligned} \delta I(u) &= \int_a^b \left(p(x)\frac{d\delta u}{dx} + 2q(x)u\delta u\right)dx + P\delta u(a) \\ \delta I(u, v) &= \int_\Omega \left[p(x, y)\left(\frac{d\delta u}{dx}\frac{dv}{dx} + \frac{du}{dx}\frac{d\delta v}{dx}\right) + q(x, y)\delta v\right]dxdy \\ &\quad + \int_\Gamma Q\,\delta u\,ds \end{aligned} \tag{2.3.9}$$

Note that functions of position, p and q, do not undergo variation because they are not functions of the dependent variables.

2.3.3 Fundamental Lemma of Variational Calculus

The *fundamental lemma of calculus of variations* can be stated as follows: *for any integrable function $G(x)$, if the statement*

$$\int_a^b G(x)\,\eta(x)\,dx = 0 \tag{2.3.10}$$

holds for any arbitrary continuous function $\eta(x)$, for all x in (a, b), then it follows that $G(x) = 0$ in (a, b). A simple proof of the lemma follows from setting $\eta(x)$, which is arbitrary, equal to G. We have

$$\int_a^b [G(x)]^2\,dx = 0$$

Since an integral of a positive function, G^2, is positive, the above statement implies that $G(x) = 0$ in the domain $\Omega = (a, b)$.

A more general statement of the fundamental lemma is as follows: If $\eta(x)$ is arbitrary in $a < x < b$ and $\eta(a)$ is arbitrary, then the statement

$$\int_a^b G\eta\,dx + B(a)\eta(a) = 0 \tag{2.3.11a}$$

implies that

$$G = 0 \text{ in } a < x < b \text{ and } B(a) = 0 \tag{2.3.11b}$$

because $\eta(x)$ is independent of $\eta(a)$.

2.3.4 The Euler Equations

As stated in the introduction, certain problems are formulated as one of seeking the extremum of functionals (i.e., functions of dependent unknowns of the problem). For example, problems of solid mechanics can be formulated as one of minimizing the total potential energy of the system (Reddy, 2002). Typically, the total potential energy (functional) is written in terms of the displacement field and applied loads. Then it is useful to derive the differential equations that govern the displacement field from this minimum principle. Here we outline the steps in obtaining such equations.

Consider, for example, the problem of finding a function $u = u(x)$ such that

$$u(a) = u_a, \quad u(b) = u_b \tag{2.3.12a}$$

and

$$I(u) = \int_a^b F(x, u(x), u'(x))\,dx \tag{2.3.12b}$$

is a minimum. In analyzing the problem, we are not interested in all functions u but only in those functions that satisfy the stated boundary (or end) conditions. The set of all such functions is called, for obvious reasons, the *set of competing functions* (or set of admissible functions). We shall denote the set by $\mathcal{C}$. The problem is to seek an element u from $\mathcal{C}$ that renders I a minimum. If $u \in \mathcal{C}$ (the symbol $\in$ means 'an element of'), then $(u + \epsilon v) \in \mathcal{C}$ for every v satisfying the conditions $v(a) = v(b) = 0$. The space of all such elements is called the space of admissible variations, as already mentioned. Figure 2.3.1 shows a typical competing function $\bar{u}(x) = u(x) + \epsilon v(x)$ and a typical admissible variation $v(x)$.

Let $I(u)$ be a differentiable functional in the sense that

$$\frac{dI(u+\epsilon v, u'+\epsilon v')}{d\epsilon}$$

exists, and let $\mathcal{C}$ denote the space of competing functions. Then, an element u in $\mathcal{C}$ is said to yield a *relative minimum (maximum)* for $I(\bar{u})$ in $\mathcal{C}$ if

$$I(\bar{u}) - I(u) \geq 0 \qquad (\leq 0) \tag{2.3.13}$$

If $I(\bar{u})$ assumes a relative minimum (maximum) at $u \in \mathcal{C}$ relative to elements $\bar{u} \in \mathcal{C}$, then it follows from the definition of the space of admissible variations $\mathcal{H}$ and Eq. (2.3.13) that

$$I(u+\epsilon v) - I(u) \geq 0 \qquad (\leq 0) \tag{2.3.14}$$

for all $v \in \mathcal{H}$, $||v|| < \varepsilon$, and ϵ a real number. Since u is the minimizer, any other function $u \in \mathcal{C}$ is of the form $\bar{u} = u + \epsilon v$, and the actual minimizer is determined by setting $\epsilon = 0$. Once $u(x)$ and $v(x)$ are assigned, $I(\bar{u})$ is a function of ϵ alone, say $\bar{I}(\epsilon)$. Now a necessary condition for $I(\bar{u}) = \bar{I}(\epsilon)$ to attain a minimum is that

$$\frac{d\bar{I}(\epsilon)}{d\epsilon} = \frac{d}{d\epsilon}[I(u+\epsilon v)] = 0 \tag{2.3.15}$$

On the other hand, $I(\bar{u})$ attains its minimum at u, i.e., $\epsilon = 0$. These two conditions together imply $(d\bar{I}(\epsilon)/d\epsilon)|_{\epsilon=0} = 0$, which is nothing but

$$\delta I(u) = 0 \tag{2.3.16}$$

Analogous to the sufficient condition for ordinary functions, the sufficient condition for a functional to assume a relative minimum (maximum) is that the second variation $\delta^2 I(u)$ is greater (less) than zero. The second variation $\delta^2 I(u)$ of a functional $I(u)$ is given by

$$\delta^2 I(u) \equiv \frac{\epsilon^2}{2}\left[\frac{d^2}{d\epsilon^2} I(u+\epsilon v)\right]_{\epsilon=0} \tag{2.3.17}$$

for all $v \in \mathcal{C}$ and real number ϵ.

It is clear that any candidate for the minimizing functional should satisfy the end conditions in Eq. (2.3.12*a*) and be sufficiently differentiable (twice in the present case, as we shall see shortly). The set of all such functions is the set of admissible functions or competing functions for the present case. Functions from the admissible set can be viewed as smooth (i.e., differentiable twice) functions passing through points (a, u_a) and (b, u_b), as shown in Fig. 2.3.1. Clearly, any element $\bar{u}$ in $\mathcal{C}$ (the set of competing functions) has the form

$$\bar{u} = u + \epsilon v \tag{2.3.18}$$

where ϵ is a small number and v is a sufficiently differentiable function that satisfies the homogeneous form of the end conditions (because $\bar{u}$ must satisfy the specified end conditions) in Eq. (2.3.12*a*)

$$v(a) = v(b) = 0 \tag{2.3.19}$$

and u is the function that minimizes the functional in Eq. (2.3.12*b*). The set of all functions v is the set of admissible variations, $\mathcal{H}$. Now assuming that, for each admissible function $\bar{u}$, $F(x, \bar{u}, \bar{u}')$ exists and is continuously differentiable with respect to its arguments, and $I(\bar{u})$ takes one and only one real value, we seek the particular function $u(x)$ that makes the integral a minimum.

The necessary condition (2.3.15) for I to attain a minimum gives

$$0 = \frac{dI(u+\epsilon v)}{d\epsilon}\bigg|_{\epsilon=0} = \left[\frac{d}{d\epsilon}\int_a^b F(x,\bar{u},\bar{u}')dx\right]_{\epsilon=0}$$

$$= \int_a^b \left(\frac{\partial F}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial \epsilon} + \frac{\partial F}{\partial \bar{u}'}\frac{\partial \bar{u}'}{\partial \epsilon}\right)\bigg|_{\epsilon=0} dx = \int_a^b \left(\frac{\partial F}{\partial u}v + \frac{\partial F}{\partial u'}v'\right)dx \qquad (2.3.20)$$

where $\bar{u} = u + \epsilon v$. Integrating the second term in the last equation by parts to transfer differentiation from v to u, we obtain

$$0 = \int_a^b v\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]dx + \left(\frac{\partial F}{\partial u'}v\right)\bigg|_a^b \qquad (2.3.21)$$

The boundary term vanishes because v is zero at $x = a$ and $x = b$ [see Eq. (2.3.19)]. The fact that v is arbitrary inside the interval (a, b) and yet the equation should hold implies, by the fundamental lemma of the calculus of variations, that the expression in the square brackets is zero identically:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0 \text{ in } a < x < b \qquad (2.3.22)$$

Equation (2.3.22) is called the *Euler equation* of the functional in Eq. (2.3.12*b*). Of all the admissible functions, the one that satisfies (i.e., the solution of) Eq. (2.3.22) is the true minimizer of the functional I.

Next consider the problem of finding (u, v), defined on a two-dimensional region Ω, such that the following functional is to be minimized:

$$I(u,v) = \int_\Omega F(x, y, u, v, u_x, v_x, u_y, v_y)dxdy \qquad (2.3.23)$$

where $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, and so on. For the moment we assume that u and v are specified on the boundary Γ of Ω. The vanishing of the first variation of $I(u, v)$ is written as

$$\delta I(u,v) = \delta_u I(u,v) + \delta_v I(u,v) = 0$$

Here δ_u and δ_v denote (partial) variations with respect to u and v, respectively. We have

$$\delta I = \int_\Omega \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v_x}\delta v_x + \frac{\partial F}{\partial v_y}\delta v_y\right)dxdy \qquad (2.3.24)$$

The next step in the development involves the use of integration by parts, or the gradient theorem on the second, third, fifth, and sixth terms in Eq. (2.3.24). Consider the second term. We have

$$\int_\Omega \frac{\partial F}{\partial u_x}\frac{\partial \delta u}{\partial x}\,dxdy = \int_\Omega \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u\right]dxdy$$

$$= \oint_\Gamma \frac{\partial F}{\partial u_x}\delta u\, n_x\, ds - \int_\Omega \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u\, dxdy \qquad (2.3.25)$$

Using a similar procedure on the other terms and collecting the coefficients of δu and δv separately, we obtain

$$0 = \int_\Omega \left\{\left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]\delta u\right.$$

$$+\left[\frac{\partial F}{\partial v}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right)\right]\delta v\Bigg\}\,dxdy$$

$$+\oint_\Gamma\left[\left(\frac{\partial F}{\partial u_x}n_x+\frac{\partial F}{\partial u_y}n_y\right)\delta u+\left(\frac{\partial F}{\partial v_x}n_x+\frac{\partial F}{\partial v_y}n_y\right)\delta v\right]ds \qquad (2.3.26)$$

Since (u, v) are specified on Γ, $\delta u=\delta v=0$ and the boundary expressions vanish. Then, since δu and δv are arbitrary and independent of each other in Ω, the fundamental lemma yields the Euler equations

$$\delta u:\quad \frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)=0 \qquad (2.3.27a)$$

$$\delta v:\quad \frac{\partial F}{\partial v}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right)=0 \qquad (2.3.27b)$$

2.3.5 Natural and Essential Boundary Conditions

First, consider the problem of minimizing the functional in Eq. (2.3.12*b*) subject to *no* end conditions [hence, an element v of the set of admissible variation is arbitrary even at the end points, i.e., $v(a)\neq 0$ and $v(b)\neq 0$]. Then the functional $I(u)$ has the form

$$I(u)=\int_a^b F(x, u(x), u'(x))\,dx-Q_a\,u(a)-Q_b\,u(b) \qquad (2.3.28)$$

where Q_a and Q_b are known values. The necessary condition for I to attain a minimum yields [cf. Eq. (2.3.21)]

$$0=\int_a^b v\left[\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]dx+\left(\frac{\partial F}{\partial u'}v\right)\Big|_a^b-Q_a\,v(a)-Q_b\,v(b) \qquad (2.3.29)$$

Now suppose that $\frac{\partial F}{\partial u'}$ and v are selected such that

$$\left(-\frac{\partial F}{\partial u'}-Q_a\right)v=0 \text{ for } x=a,\quad \left(\frac{\partial F}{\partial u'}-Q_b\right)v=0 \text{ for } x=b \qquad (2.3.30)$$

Then using the fundamental lemma of the calculus of variations, we obtain the same Euler equation as in Eq. (2.3.22).

Equations in (2.3.30) are satisfied identically for any of the following combinations:

$$\begin{aligned}
&(1)\ \ v(a)=0,\ \ v(b)=0\\
&(2)\ \ v(a)=0,\ \ \left.\frac{\partial F}{\partial u'}\right|_b-Q_b=0\\
&(3)\ \ -\left.\frac{\partial F}{\partial u'}\right|_a-Q_a=0,\ \ v(b)=0\\
&(4)\ \ -\left.\frac{\partial F}{\partial u'}\right|_a-Q_a=0,\ \ \left.\frac{\partial F}{\partial u'}\right|_b-Q_b=0
\end{aligned} \qquad (2.3.31)$$

The requirement that $v=0$ at an end point is equivalent to the requirement that u is specified (to be some value) at that point. The end conditions in (2.3.31) are classified into two types:

essential boundary conditions, which require v (and possibly its derivatives) to vanish at the boundary, and *natural boundary conditions*, which require the specification of *the coefficient of* v (and possibly its derivatives). Thus, we have

Essential Boundary Conditions:

$$\text{specify } v = 0 \text{ or } u = \hat{u} \text{ on the boundary} \tag{2.3.32a}$$

Natural Boundary Conditions:

$$\text{specify } \frac{\partial F}{\partial u'} = Q \text{ on the boundary} \tag{2.3.32b}$$

where $Q = -Q_a$ at $x = a$ and $Q = Q_b$ at $x = b$. In a given problem, only one of the four combinations given in Eq. (2.3.31) can be specified. Problems in which all of the boundary conditions are of the essential type are called *Dirichlet boundary-value problems*, and those in which all of the boundary conditions are of the natural type are called *Neumann boundary-value problems*. *Mixed boundary-value problems* are those in which both essential and natural boundary conditions are specified. Essential boundary conditions are also known as *Dirichlet* or *geometric* boundary conditions, and natural boundary conditions are known as *Neumann* or *dynamic* boundary conditions.

As a general rule, the vanishing of the variation v (or δu) at a point implies that u is specified there (in general, to be nonzero). The specification of u on the boundary constitutes the essential boundary condition. Vanishing of the coefficient of the variation v in the boundary expression constitutes the natural boundary condition. This rule applies to any functional in one, two, and three dimensions, and to integrands that are functions of one or more dependent variables and their derivatives of any order.

Next consider the functional in Eq. (2.3.23), and suppose that (u, v) are arbitrary on Γ for the moment. It is easy to identify the natural and essential boundary conditions of the problem from Eq. (2.3.26): In each of the pairings on boundary Γ, specifying the first element (which contains no variations of the dependent variables) constitutes the natural boundary condition, and vanishing of the second element (or, equivalently, specifying the quantity in front of the variational operator) constitutes the essential boundary condition. Thus, we have either

$$u = \hat{u} \text{ (specified) so that } \delta u = 0 \quad \text{on} \quad \Gamma \tag{2.3.33a}$$

$$v = \hat{v} \text{ (specified) so that } \delta v = 0 \quad \text{on} \quad \Gamma \tag{2.3.33b}$$

or

$$\frac{\partial F}{\partial u_x} n_x + \frac{\partial F}{\partial u_y} n_y = 0 \quad \text{on} \quad \Gamma \tag{2.3.34a}$$

$$\frac{\partial F}{\partial v_x} n_x + \frac{\partial F}{\partial v_y} n_y = 0 \quad \text{on} \quad \Gamma \tag{2.3.34b}$$

Equations (2.3.33a) and (2.3.33b) represent the essential boundary conditions and Eqs. (2.3.34a) and (2.3.34b) the natural boundary conditions. The pair of elements (u, v) are called the *primary variables* and

$$Q_x \equiv \frac{\partial F}{\partial u_x} n_x + \frac{\partial F}{\partial u_y} n_y \text{ and } Q_y \equiv \frac{\partial F}{\partial v_x} n_x + \frac{\partial F}{\partial v_y} n_y$$

are called the *secondary variables*. Thus, specification of the primary variables constitute essential boundary conditions and specification of the secondary variables constitute natural boundary conditions. In general, one element of each pair (u, Q_x) and (v, Q_y) (but not both elements of the same pair) may be specified at any point of the boundary. Thus, there are four possible combinations of natural and essential boundary conditions for the problem under discussion.

Example 2.3.2

Consider an elastic bar of length L, modulus of elasticity E, and area of cross section A. Assume that it is fixed at the left end, spring supported at the right end, and subjected to distributed axial load $f(x)$ (see Fig. 2.3.2). The total potential energy functional $\Pi = U + V$ of the bar is (here we assumed that one knows how to write the total potential energy of the bar)

$$\Pi(u) = \int_0^L \left[\frac{EA}{2} \left(\frac{du}{dx} \right)^2 - fu \right] dx + \frac{k}{2}[u(L)]^2 \tag{2.3.35}$$

where u denotes the axial displacement of the bar. The first term in $\Pi(u)$ represents the strain energy U stored in the bar, the second term denotes the work done V on the bar by the distributed load f, and the last term denotes the strain energy stored in the linear elastic spring. We wish to determine the Euler equation of the bar by requiring that $\Pi(u)$ be a minimum subject to the geometric boundary condition $u(0) = 0$. The statement $\delta\Pi = 0$ is known as *the principle of minimum total potential energy*.

The first variation of Π is given by

$$\delta\Pi(u) = \int_0^L \left(EA \frac{du}{dx} \frac{d\delta u}{dx} - f\delta u \right) dx + ku(L)\,\delta u(L)$$

where δu is arbitrary in $0 < x < L$ and at $x = L$ but satisfies the condition $\delta u(0) = 0$. To use the fundamental lemma of variational calculus, we must relieve δu of any differentiation.

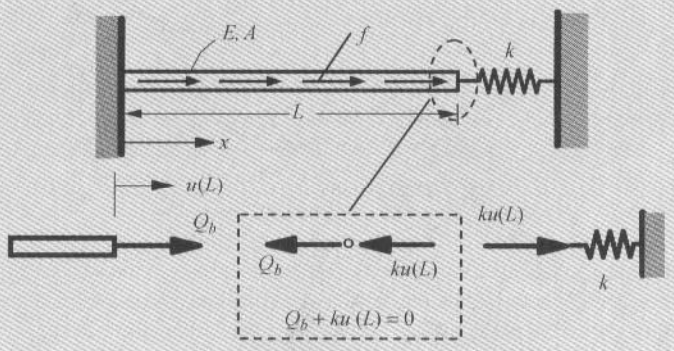


Figure 2.3.2 Elastic bar problem discussed in Example 2.3.2.

Integrating the first term by parts, we get

$$\delta\Pi(u) = \int_0^L \left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f \right] \delta u \, dx + \left[EA\frac{du}{dx}\,\delta u \right]_0^L + ku(L)\,\delta u(L)$$

$$= \int_0^L \delta u \left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f \right] dx + \delta u(L)\left[EA\frac{du}{dx} + ku(L) \right]_{x=L}$$

$$- \delta u(0)\left[EA\frac{du}{dx} \right]_{x=0}$$

The last term is zero because $\delta u(0) = 0$. Setting the coefficients of δu in $(0, L)$ and δu at $x = L$ to zero separately, we obtain the Euler equation and the natural (or force) boundary condition of the problem:

Euler Equation:

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f = 0, \quad 0 < x < L \tag{2.3.36a}$$

Natural Boundary Condition:

$$EA\frac{du}{dx} + ku(L) = 0 \text{ at } x = L \tag{2.3.36b}$$

Thus, the solution u of Eqs. (2.3.36*a*) and (2.3.36*b*) that satisfies $u(0) = 0$ is the minimizer of the energy functional $\Pi(u)$ in Eq. (2.3.35).

Equations (2.3.36*a*) and (2.3.36*b*) can be obtained directly from Eq. (2.3.22) and (2) of Eq. (2.3.31) (with $Q_b = -ku(L)$ as shown in Fig. 2.3.2) by substituting

$$F(x, u, u') = \frac{EA}{2}\left(\frac{du}{dx}\right)^2 - f(x)u(x), \quad \frac{\partial F}{\partial u} = -f, \quad \frac{\partial F}{\partial u'} = EA\frac{du}{dx} \tag{2.3.37}$$

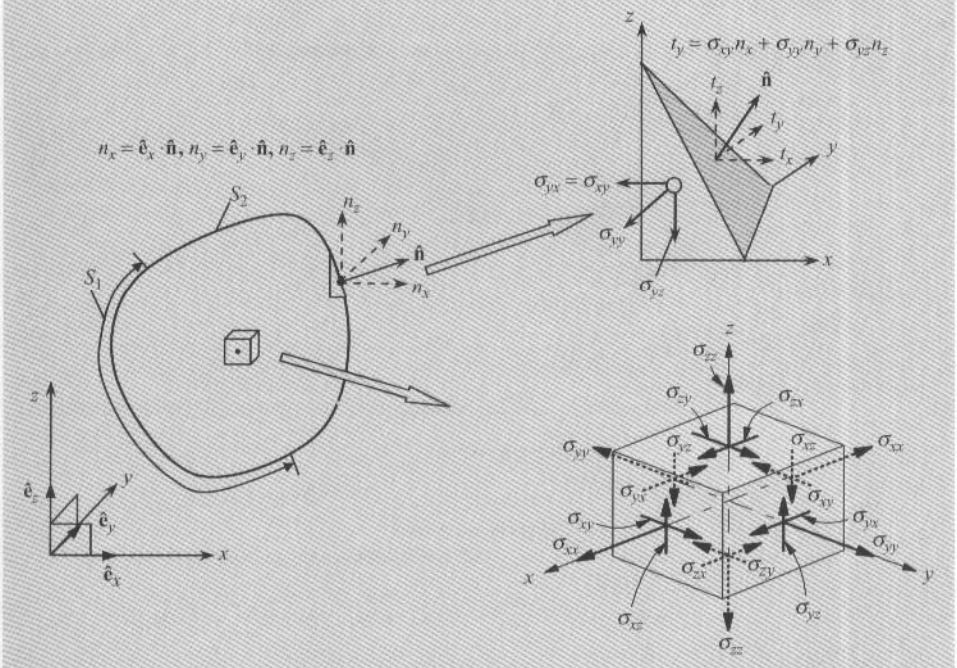
Example 2.3.3

The total potential energy of a linear elastic body in three dimensions, subjected to body force $\mathbf{f}$ (measured per unit volume) and surface traction $\hat{\mathbf{t}}$ (measured per unit area) on portion S_2 of the surface [see Fig. 2.3.3(*a*)] is given by (summation on repeated indices is implied)

$$\Pi(\mathbf{u}) = \int_V \left(\frac{1}{2}\sigma_{ij}\varepsilon_{ij} - f_i u_i \right) dV - \int_{S_2} \hat{t}_i u_i \, dS \tag{2.3.38}$$

where u_i denote the displacement components, σ_{ij} are the stress components [see Fig. 2.3.3(*b*)], and ε_{ij} are strain components. It is assumed that the body is subjected to specified displacements on the remaining portion S_1 of the surface, and therefore the virtual displacements vanish there:

$$u_i = \hat{u}_i \quad \text{and} \quad \delta u_i = 0 \quad \text{on} \quad S_1 \tag{2.3.39}$$



(*a*) A 3-D domain (*b*) Stress components

Figure 2.3.3 (*a*) An elastic body with specified displacements and tractions. (*b*) The stress components at a typical point inside the body and on the boundary.

In Eq. (2.3.38), and in the following discussion, summation on repeated indices is assumed. Use of the index notation and the summation convention in the 3-D case saves considerable space (i.e., the explicit form of Eq. (2.3.38) and subsequent manipulations would take considerably more space). The first term under the volume integral represents the strain energy density of the elastic body and the second term represents the work done by the body force $\mathbf{f}$; and the surface integral denotes the work done by the specified traction $\hat{\mathbf{t}}$.

For an isotropic body the stress-strain relations are given by (the generalized Hooke's law)

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} \tag{2.3.40}$$

where μ and λ are the Lamé (material) constants. Hence,

$$\sigma_{ij}\varepsilon_{ij} = 2\mu\varepsilon_{ij}\varepsilon_{ij} + \lambda\varepsilon_{ii}\varepsilon_{kk} \tag{2.3.41}$$

The linear strain-displacement relations of the linear theory are given by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.3.42}$$

where $u_{i,j} = (\partial u_i / \partial x_j)$.

Substituting Eqs. (2.3.41) and (2.3.42) into Eq. (2.3.38), we obtain

$$\Pi(\mathbf{u}) = \int_V \left[\frac{\mu}{4}\left(u_{i,j}+u_{j,i}\right)\left(u_{i,j}+u_{j,i}\right)+\frac{\lambda}{2}u_{i,i}u_{k,k}-f_i u_i\right]dV-\int_{S_2}\hat{t}_i u_i\, dS \tag{2.3.43}$$

Now we wish to derive the Euler equations associated with the functional in Eq. (2.3.43) using the principle of minimum total potential energy, $\delta\Pi = 0$ (i.e., minimize Π) and $\delta u_i = 0$ on S_1.

Setting the first variation of Π to zero, we obtain

$$0 = \int_V \left[\frac{\mu}{2}\left(\delta u_{i,j}+\delta u_{j,i}\right)\left(u_{i,j}+u_{j,i}\right)+\lambda\delta u_{i,i}u_{k,k}-f_i\,\delta u_i\right]dV-\int_{S_2}\hat{t}_i\,\delta u_i\, dS \tag{2.3.44}$$

wherein the product rule of variation is used and similar terms are combined. Using integration by parts

$$\int_V \delta u_{i,j}(u_{i,j}+u_{j,i})dV = -\int_V \delta u_i(u_{i,j}+u_{j,i})_{,j}\, dV+\oint_S \delta u_i(u_{i,j}+u_{j,i})n_j\, dS$$

where n_j denotes the jth direction cosine of the unit normal to the surface, we obtain

$$\begin{aligned}0 &= \int_V\left[-\frac{\mu}{2}(u_{i,j}+u_{j,i})_{,j}\delta u_i-\frac{\mu}{2}(u_{i,j}+u_{j,i})_{,i}\delta u_j-\lambda u_{k,ki}\delta u_i-f_i\,\delta u_i\right]dV\\ &\quad+\oint_S\left[\frac{\mu}{2}(u_{i,j}+u_{j,i})(n_j\delta u_i+n_i\delta u_j)+\lambda u_{k,k}n_i\,\delta u_i\right]dS-\int_{S_2}\delta u_i\,\hat{t}_i\, dS\\ &= \int_V\left[-\mu(u_{i,j}+u_{j,i})_{,j}-\lambda u_{k,ki}-f_i\right]\delta u_i\, dV\\ &\quad+\oint_S\left[\mu(u_{i,j}+u_{j,i})+\lambda u_{k,k}\delta_{ij}\right]n_j\delta u_i\, dS-\int_{S_2}\delta u_i\hat{t}_i\, dS\end{aligned} \tag{2.3.45}$$

In arriving at the last step, a change of dummy indices is made to combine terms. Recognizing that the expression inside the square brackets of the closed surface integral is nothing but σ_{ij}, and $\sigma_{ij}n_j = t_i$ by Cauchy's formula [which is nothing but a statement of the equilibrium of forces on the tetrahedral element; see Fig. 2.3.3(b)], we can write

$$\oint_S t_i\,\delta u_i\, dS = \int_{S_1} t_i\,\delta u_i\, dS+\int_{S_2} t_i\,\delta u_i\, dS = \int_{S_2} t_i\,\delta u_i\, dS$$

The integral over S_1 is zero by virtue of Eq. (2.3.39). Hence, we have

$$0 = \int_V\left[-\mu\left(u_{i,j}+u_{j,i}\right)_{,j}-\lambda u_{k,ki}-f_i\right]\delta u_i dV+\int_{S_2}\delta u_i\left(t_i-\hat{t}_i\right)dS$$

Using the fundamental lemma of calculus of variations, we set the coefficients of δu_i in V and δu_i on S_2 to zero separately and obtain

$$\mu(u_{i,jj}+u_{j,ij})+\lambda u_{k,ki}+f_i = 0 \text{ in } V \tag{2.3.46}$$

$$\sigma_{ij}n_j-\hat{t}_i = 0 \text{ on } S_2 \tag{2.3.47}$$

for $i = 1, 2, 3$.

Equations (2.3.46) are the well-known Navier's equations of equilibrium of three-dimensional elasticity, and (2.3.47) are the traction boundary conditions. The explicit forms of Eqs. (2.3.40), (2.3.42), (2.3.46), and (2.3.47) in a rectangular Cartesian coordinate system (x, y, z) are given below ($x_1 = x$, $x_2 = y$, $x_3 = z$, $f_1 = f_x$, $f_2 = f_y$, $f_3 = f_z$, $\sigma_{11} = \sigma_{xx}$, $\sigma_{13} = \sigma_{xz} = \sigma_{zx}$, and so on).

Stress-Strain Relations (Hooke's Law for an Orthotropic Material):

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{2.3.48a}$$

where $\gamma_{xy} = 2\varepsilon_{xy}$, $\gamma_{xz} = 2\varepsilon_{xz}$ and $\gamma_{yz} = 2\varepsilon_{yz}$ are the engineering shear strains. The elastic material stiffness coefficients C_{ij} are known in terms of the engineering material constants by

$$\begin{aligned} C_{11} &= \frac{1 - \nu_{23}\nu_{32}}{E_2 E_3 \Delta}, \quad C_{12} = \frac{\nu_{21} + \nu_{31}\nu_{23}}{E_2 E_3 \Delta} = \frac{\nu_{12} + \nu_{32}\nu_{13}}{E_1 E_3 \Delta} \\ C_{13} &= \frac{\nu_{31} + \nu_{21}\nu_{32}}{E_2 E_3 \Delta} = \frac{\nu_{13} + \nu_{12}\nu_{23}}{E_1 E_2 \Delta} \\ C_{22} &= \frac{1 - \nu_{13}\nu_{31}}{E_1 E_3 \Delta}, \quad C_{23} = \frac{\nu_{32} + \nu_{12}\nu_{31}}{E_1 E_3 \Delta} = \frac{\nu_{23} + \nu_{21}\nu_{13}}{E_1 E_3 \Delta} \\ C_{33} &= \frac{1 - \nu_{12}\nu_{21}}{E_1 E_2 \Delta}, \quad C_{44} = G_{23} \quad C_{55} = G_{31} \quad C_{66} = G_{12} \\ \Delta &= \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}}{E_1 E_2 E_3} \end{aligned} \tag{2.3.48b}$$

Here E_1, E_2, and E_3 are Young's moduli in 1, 2, and 3 material directions, respectively; ν_{ij} are Poisson's ratios, defined as the ratio of transverse strain in the jth direction to the axial strain in the ith direction when stressed in the ith direction; and G_{23}, G_{13}, and G_{12} are shear moduli in the 2-3, 1-3, and 1-2 planes, respectively. In addition, the following reciprocity relations among the engineering constants hold:

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}; \quad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2} \tag{2.3.49}$$

For the isotropic case, we have $E_1 = E_2 = E_3 = E$, $G_{12} = G_{13} = G_{23} = G$, and $\nu_{12} = \nu_{13} = \nu_{23} = \nu$ with $G = 0.5E/(1 + \nu)$. The Lamé constants are related to the engineering constants by

$$\lambda = K - \frac{2}{3}G, \quad \mu = G \tag{2.3.50}$$

where K is the bulk modulus.

Strain-Displacement Relations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}; \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}; \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right); \quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right); \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{2.3.51}$$

where (u, v, w) are the displacements of a point along the (x, y, z) coordinates, respectively.

Stress Equilibrium Equations:

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y &= 0 \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z &= 0 \end{aligned} \tag{2.3.52}$$

Stress-Traction Relations (Cauchy's formula; see Fig. 2.3.3):

$$\begin{aligned} \sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z &= t_x \\ \sigma_{xy} n_x + \sigma_{yy} n_y + \sigma_{yz} n_z &= t_y \\ \sigma_{xz} n_x + \sigma_{yz} n_y + \sigma_{zz} n_z &= t_z \end{aligned} \tag{2.3.53}$$

2.3.6 Hamilton's Principle

The principle of minimum total potential energy is limited to static equilibrium of solids. Hamilton's principle is a generalization of the principle of virtual displacements (see Reddy, 2002) to dynamics (i.e., time-dependent response) of solids. The principle assumes that the system under consideration is characterized by two energy functions: *kinetic energy* K and *potential energy* Π. For *continuous* systems (i.e., systems that cannot be described by a finite number of generalized coordinates), the energies can be expressed in terms of the dependent variables (which are functions of position) of the problem.

Newton's second law of motion for a continuous body can be written in general terms as

$$\mathbf{F} - m\mathbf{a} = \mathbf{0} \tag{2.3.54}$$

where m is the mass, $\mathbf{a}$ the acceleration vector, and $\mathbf{F}$ is the resultant of *all* forces acting on the body. The actual path $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ followed by a material particle in position $\mathbf{x}$ in the body is varied, consistent with kinematic (essential) boundary conditions, to $\mathbf{u} + \delta\mathbf{u}$, where $\delta\mathbf{u}$ is the admissible variation (or virtual displacement) of the path. We suppose that the varied path differs from the actual path except at initial and final times, t_1 and t_2, respectively. Thus, an admissible variation $\delta\mathbf{u}$ satisfies the conditions,

$$\delta\mathbf{u} = \mathbf{0} \text{ on } \Gamma_u \text{ for all } t \tag{2.3.55a}$$

$$\delta\mathbf{u}(\mathbf{x}, t_1) = \delta\mathbf{u}(\mathbf{x}, t_2) = \mathbf{0} \text{ for all } \mathbf{x} \tag{2.3.55b}$$

where Γ_u denotes the portion of the boundary of the body where the displacement vector $\mathbf{u}$ is specified. Note that the scalar product of Eq. (2.3.54) with $\delta\mathbf{u}$ gives work done at point $\mathbf{x}$, because $\mathbf{F}$, $\mathbf{a}$, and $\mathbf{u}$ are vector functions of position (whereas the work is a scalar). Integration of the product over the volume (and surface) of the body gives the total work done by all points in moving through their respective displacements.

The *work done on the body* at time t by the resultant force in moving through the virtual displacement $\delta\mathbf{u}$ is given by

$$\int_V \mathbf{f}\cdot\delta\mathbf{u}\,dV + \int_{\Gamma_\sigma} \hat{\mathbf{t}}\cdot\delta\mathbf{u}\,dS - \int_V \overleftrightarrow{\sigma} : \delta\overleftrightarrow{\varepsilon}\,dV \tag{2.3.56}$$

where $\mathbf{f}$ is the body force vector, $\hat{\mathbf{t}}$ the specified surface traction vector, and $\overleftrightarrow{\sigma}$ and $\overleftrightarrow{\varepsilon}$ are the stress and strain tensors. The "double-dot product" has the meaning $\overleftrightarrow{\sigma} : \delta\overleftrightarrow{\varepsilon} = \sigma_{ij}\varepsilon_{ji}$. The last term in Eq. (2.3.56) represents the *virtual work* of internal forces *stored in the body*. The strains $\delta\overleftrightarrow{\varepsilon}$ are assumed to be compatible in the sense that the strain-displacement relations (2.3.42) are satisfied. The work done by the inertia force $m\mathbf{a}$ in moving through the virtual displacement $\delta\mathbf{u}$ is given by

$$\int_V \rho\frac{\partial^2\mathbf{u}}{\partial t^2}\cdot\delta\mathbf{u}\,dV \tag{2.3.57}$$

where ρ is the mass density (can be a function of position) of the medium. We have the result

$$\int_{t_1}^{t_2}\left\{\int_V \rho\frac{\partial^2\mathbf{u}}{\partial t^2}\cdot\delta\mathbf{u}\,dV - \left[\int_V\left(\mathbf{f}\cdot\delta\mathbf{u} - \overleftrightarrow{\sigma} : \delta\overleftrightarrow{\varepsilon}\right)dV + \int_{\Gamma_\sigma}\hat{\mathbf{t}}\cdot\delta\mathbf{u}\,dS\right]\right\}dt = 0$$

or

$$\int_{t_1}^{t_2}\left[\int_V \rho\frac{\partial\mathbf{u}}{\partial t}\cdot\frac{\partial\delta\mathbf{u}}{\partial t}\,dV + \int_V\left(\mathbf{f}\cdot\delta\mathbf{u} - \overleftrightarrow{\sigma} : \delta\overleftrightarrow{\varepsilon}\right)dV + \int_{\Gamma_\sigma}\hat{\mathbf{t}}\cdot\delta\mathbf{u}\,dS\right]dt = 0 \tag{2.3.58}$$

In arriving at the expression in Eq. (2.3.58), integration by parts is used on the first term; the integrated terms vanish because of the initial and final conditions in Eq. (2.3.55*b*). Equation (2.3.58) is known as the general form of Hamilton's principle for a continuous medium (conservative or not, and elastic or not).

For an ideal elastic body, we recall from the previous discussions that the forces $\mathbf{f}$ and $\mathbf{t}$ are conservative,

$$\delta V = -\left(\int_V \mathbf{f}\cdot\delta\mathbf{u}\,dV + \int_{\Gamma_\sigma}\hat{\mathbf{t}}\cdot\delta\mathbf{u}\,dS\right) \tag{2.3.59a}$$

and that there exists a strain energy density function $U_0 = U_0(\varepsilon_{ij})$ such that

$$\delta U_0(\varepsilon_{ij}) = \frac{\partial U_0}{\partial\varepsilon_{ij}}\delta\varepsilon_{ij} = \sigma_{ij}\delta\varepsilon_{ij} \tag{2.3.59b}$$

Substituting Eqs. (2.3.59*a*) and (2.3.59*b*) into Eq. (2.3.58), we obtain

$$\delta\int_{t_1}^{t_2}[K - (V+U)]dt = 0 \tag{2.3.60}$$

where K and U are the kinetic and strain energies:

$$K = \int_V \frac{\rho}{2}\frac{\partial\mathbf{u}}{\partial t}\cdot\frac{\partial\mathbf{u}}{\partial t}\,dV, \quad U = \int_V U_0\,dV \tag{2.3.61}$$

Equation (2.3.60) represents Hamilton's principle for an elastic body. Recall that the sum of the strain energy and potential energy of external forces, $U + V$, is called the total potential energy, Π, of the body. For bodies involving no motion (i.e., forces are applied sufficiently slowly such that the motion is independent of time and the inertia forces are negligible), Hamilton's principle (2.3.60) reduces to the principle of minimum total potential energy:

$$\delta(U + V) \equiv \delta\Pi = 0 \tag{2.3.62}$$

The Euler equations, known as the *Euler–Lagrange equations*, associated with the Lagrangian $L = K - \Pi$, can be obtained from Eq. (2.3.60):

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} L(\mathbf{u}, \nabla\mathbf{u}, \dot{\mathbf{u}})\, dt \\ &= \int_{t_1}^{t_2} \left[\int_V \left(\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div} \overleftrightarrow{\sigma} - \mathbf{f} \right) \cdot \delta\mathbf{u}\, dV + \int_{\Gamma_\sigma} (\mathbf{t} - \hat{\mathbf{t}}) \cdot \delta\mathbf{u}\, dS \right] dt \end{aligned} \tag{2.3.63}$$

where integration by parts, gradient theorems, and Eqs. (2.3.55*a*) and (2.3.55*b*) were used in arriving at Eq. (2.3.63) from Eq. (2.3.60). Because $\delta\mathbf{u}$ is arbitrary for t, $t_1 < t < t_2$, for $\mathbf{x}$ in V, and also on Γ_σ, it follows that

$$\begin{aligned} \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div} \overleftrightarrow{\sigma} - \mathbf{f} = \mathbf{0} \quad &\text{in } V \\ \mathbf{t} - \hat{\mathbf{t}} = \mathbf{0} \quad &\text{on } \Gamma_\sigma \end{aligned} \tag{2.3.64}$$

Equations (2.3.64) are the Euler-Lagrange equations for an elastic body [cf. Eq. (2.3.46)]. Next, we consider a specific example of application of Hamilton's principle [see Reddy (2002)].

Example 2.3.4

Consider the axial motion of an elastic bar of length L, area of cross section A, modulus of elasticity E, and mass density ρ, and subjected to distributed force f per unit length and an end load P. We wish to determine the equations of motion for the bar. The kinetic and total potential energies of the system are

$$K = \int_V \frac{\rho}{2} \left(\frac{\partial u}{\partial t} \right)^2 dV = \int_0^L \frac{\rho A}{2} \left(\frac{\partial u}{\partial t} \right)^2 dx \tag{2.3.65a}$$

$$\begin{aligned} \Pi &= \int_V \frac{1}{2} \sigma_{ij} \varepsilon_{ij} dV - \int_0^L f u\, dx - P u(L) \\ &= \int_0^L \frac{A}{2} \sigma_{xx} \varepsilon_{xx}\, dx - \int_0^L f u\, dx - P u(L) \end{aligned} \tag{2.3.65b}$$

wherein u, σ_{xx}, and ε_{xx} are assumed to be functions of x only, and

$$\begin{aligned} u(0, t) &= 0 \quad \text{(bar is fixed at } x = 0) \\ \varepsilon_{xx} &= \frac{\partial u}{\partial x} \quad \text{(strain-displacement relation)} \end{aligned} \tag{2.3.65c}$$

Substituting for K and Π from Eqs. (2.3.65a) and (2.3.65b) into Eq. (2.3.60), we obtain

$$
\begin{aligned}
0 &= \int_{t_1}^{t_2} \left\{ \int_0^L \left[A\rho \frac{\partial u}{\partial t} \frac{\partial \delta u}{\partial t} - A\sigma_{xx} \delta \left(\frac{\partial u}{\partial x} \right) + f \delta u \right] dx + P \delta u(L) \right\} dt \\
&= \int_0^L \left[\int_{t_1}^{t_2} -\frac{\partial}{\partial t} \left(\rho A \frac{\partial u}{\partial t} \right) \delta u \, dt + \rho A \frac{\partial u}{\partial t} \delta u \Big|_{t_1}^{t_2} \right] dx \\
&\quad + \int_{t_1}^{t_2} \left\{ \int_0^L \left[\frac{\partial}{\partial x} (A\sigma_{xx}) + f \right] \delta u \, dx - (A\sigma_{xx} \delta u) \Big|_0^L + P \delta u(L) \right\} dt \\
&= -\int_{t_1}^{t_2} \left\{ \int_0^L \left[\frac{\partial}{\partial t} \left(\rho A \frac{\partial u}{\partial t} \right) - \frac{\partial}{\partial x} (A\sigma_{xx}) - f \right] \delta u \, dx \right. \\
&\quad \left. - (A\sigma_{xx} - P)\big|_{x=L} \delta u(L) \right\} dt
\end{aligned}
\tag{2.3.66}
$$

where $\delta u(0, t) = 0$ and $\delta u(x, t_1) = \delta u(x, t_2) = 0$ are used to simplify the expression. The Euler-Lagrange equations are obtained by setting the coefficients of δu in $(0, L)$ and at $x = L$ to zero separately:

$$
\frac{\partial}{\partial t} \left(\rho A \frac{\partial u}{\partial t} \right) - \frac{\partial}{\partial x} (A\sigma_{xx}) - f = 0, \quad 0 < x < L \tag{2.3.67a}
$$

$$
(A\sigma_{xx})\big|_{x=L} - P = 0 \tag{2.3.67b}
$$

for all $t, t_1 < t < t_2$. For linear elastic materials, we have $\sigma_{xx} = E\varepsilon_{xx} = E(\partial u/\partial x)$, and Eqs. (2.3.67$a$) and (2.3.67$b$) become

$$
\frac{\partial}{\partial t} \left(\rho A \frac{\partial u}{\partial t} \right) - \frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) - f = 0, \quad 0 < x < L \tag{2.3.68a}
$$

$$
\left(AE \frac{\partial u}{\partial x} \right)\Big|_{x=L} - P = 0 \tag{2.3.68b}
$$

Now suppose that the bar also experiences a nonconservative (viscous damping) force proportional to the velocity,

$$
F^* = -\mu \frac{\partial u}{\partial t} \tag{2.3.69}
$$

where μ is the damping coefficient (a constant). Then the Euler-Lagrange equations from Eq. (2.3.58) are given by

$$
\frac{\partial}{\partial t} \left(\rho A \frac{\partial u}{\partial t} \right) - \frac{\partial}{\partial x} \left(EA \frac{\partial}{\partial x} \right) - f + \mu \frac{\partial u}{\partial t} = 0, \quad 0 < x < L \tag{2.3.70a}
$$

$$
\left(AE \frac{\partial u}{\partial x} \right)_{x=L} - P = 0 \tag{2.3.70b}
$$

2.4 INTEGRAL FORMULATIONS

2.4.1 Introduction

Recall from Section 2.1.3 that the motivation for the use of weighted-integral statements of differential equations comes from the fact that we wish to have a means to determine the unknown parameters c_j in the approximate solution $U_N = \sum_j c_j \phi_j$. The variational methods of approximation, e.g., the Ritz, Galerkin, least-squares, collocation, or, in general, weighted-residual methods to be discussed in Section 2.5 are based on weighted-integral statements of the governing equations. Since the finite element method is a technique for constructing approximation functions required in an elementwise application of a variational method, it is necessary to study the weighted-integral formulation and the so-called weak formulation of differential equations. The weak formulations also facilitate, in a natural way, the classification of boundary conditions into natural and essential types. As we shall see shortly, this classification plays a crucial role in the derivation of the approximation functions and the selection of the nodal degrees of freedom of the finite element model.

In this section, our primary objective will be to construct the weak form of a given differential equation and to classify the boundary conditions associated with the equation. A *weak form* is defined to be a weighted-integral statement of a differential equation in which the differentiation is transferred from the dependent variable to the weight function such that all natural boundary conditions of the problem are also included in the integral statement. These ideas will be more clear in the sequel.

2.4.2 Weighted-Integral and Weak Formulations

Consider the problem of solving the differential equation

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] = f(x) \quad \text{for} \quad 0 < x < L \tag{2.4.1a}$$

for $u(x)$, subject to the boundary conditions

$$u(0) = u_0, \quad \left(a\frac{du}{dx}\right)\bigg|_{x=L} = Q_L \tag{2.4.1b}$$

Here $a(x)$ and $f(x)$ are known functions of the coordinate x; u_0 and Q_L are known values; and L is the size of the one-dimensional domain. When the specified values are nonzero ($u_0 \neq 0$ or $Q_L \neq 0$), the boundary conditions are said to be nonhomogeneous; when the specified values are zero, the boundary conditions are said to be homogeneous. The homogeneous form of the boundary condition $u(0) = u_0$ is $u(0) = 0$, and the homogeneous form of the boundary condition $(adu/dx)|_{x=L} = Q_L$ is $(adu/dx)|_{x=L} = 0$.

Equations of the type (2.4.1a) arise, for example, in the study of axial heat conduction in a rod (e.g., heat exchanger fin) or radial heat conduction in a long axisymmetric cylinder. In the former case, $a = kA$, with k being the thermal conductivity and the A the cross-sectional area, the L being the length of the rod. For the axisymmetric case, $a = 2\pi Lkx$, x being the radial coordinate r and L the length of the cylinder. In both cases, f denotes the heat generation term, u_0 is the specified temperature, and Q_L is the specified heat. Other physical problems are also described by the same equation but with different meanings for the variables. Typical examples of field problems with a description of the variables are presented in Table 2.4.1.

Table 2.4.1 Some examples of engineering problems in which the second-order equation (2.4.1*a*) and its boundary conditions (2.4.1*b*) arise.

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)=f \quad \text{for } 0<x<L; \quad u(0)=u_0; \quad \left(a\frac{du}{dx}\right)_{x=L}=Q_L$$

Field	Primary variable u	Coefficient* a	Source term f	Secondary variable Q_0
1. Cables	Transverse deflection	T	Distributed vertical force	Axial force
2. Bars	Longitudinal displacement	EA	Distributed axial force	Axial load
3. Heat transfer	Temperature	k	Internal heat generation	Heat flux
4. Pipe flow	Hydrostatic pressure	$\frac{\pi D^4}{128\mu}$	Flow source	Flow rate
5. Viscous flows	Velocity	μ	Pressure gradient	Stress
6. Seepage	Fluid head	ε	Fluid flux	Flow
7. Electrostatics	Electrical potential	ϵ	Charge density	Electric flux

*E = Young's modulus; A = area of cross section; D = diameter of the pipe; k = thermal conductivity; μ = viscosity; T = tension; ε = permeability; and ϵ = dielectric constant.

Residual Function. Suppose that we seek an approximation of $u(x)$ in the form

$$u(x)\approx U_N(x)=\sum_{j=1}^{N} c_j\phi_j(x)+\phi_0(x) \tag{2.4.2}$$

and determine c_j such that $U_N(x)$ satisfies the differential equation (2.4.1*a*). Substitution of U_N into Eq. (2.4.1*a*) yields

$$-\frac{d}{dx}\left[a(x)\frac{dU_N}{dx}\right]=f(x) \quad \text{for} \quad 0<x<L \tag{2.4.3a}$$

Since the left side of the equality is now an approximate value, we cannot expect it to be equal, in general, to the right side of the equality. The difference

$$R(x,c_j)\equiv -\frac{d}{dx}\left[a(x)\frac{dU_N}{dx}\right]-f(x)\neq 0 \quad \text{for} \quad 0<x<L \tag{2.4.3b}$$

is called the *residual* of approximation in the differential equation. It is a function of x and c_j. Any approximate method (especially a variational method) seeks a set of N equations among c_j by making R equal to zero. Since it cannot be made zero identically at every point of the domain (as explained in Section 2.1.3), we must find an alternate way to find the necessary relations among c_j such that R is zero. If we require R to be zero at N selected points of the domain, we have

$$R(x,c_j)=0 \text{ for } x=x_i,\ i=1,2,\cdots,N \tag{2.4.4}$$

which is known as the *collocation method.* Another way to make R zero is to minimize the integral of the square of the residual (R is squared to make it positive) with respect to c_j:

$$\delta I \equiv \delta \int_0^L R^2 \, dx = 0 \quad \text{or} \quad \frac{\partial}{\partial c_j} \int_0^L R^2 \, dx = 0 \tag{2.4.5}$$

The method based on (2.4.5) is called the *least-squares method.* Equations (2.4.4) and (2.4.5), each, give N equations that can be solved for parameters c_j.

Weighted-Residual Method. Yet another way to determine the c_j is to require R to vanish in a "weighted-residual" sense:

$$\int_0^L w_i(x) \, R(x, c_j) \, dx = 0 \quad (i = 1, 2, \cdots, N) \tag{2.4.6}$$

where $w_i(x)$ are a set of linearly independent functions, called *weight functions,* which in general can be different from the approximation functions $\phi_i(x)$. This method is known as the *weighted-residual method.* Indeed, the statement in (2.4.6) includes (2.4.4) as well as (2.4.5) as special cases. When $w_i = \phi_i$, Eq. (2.4.6) is known as the *Galerkin method.* Thus, we have the following special cases of (2.4.6):

Petrov–Galerkin method:	$w_i = \psi_i \neq \phi_i$	
Galerkin's method:	$w_i = \phi_i$	
Least squares method:	$w_i = \dfrac{d}{dx}\left(a(x)\dfrac{d\phi_i}{dx}\right)$	(2.4.7)
Collocation method:	$w_i = \delta(x - x_i)$	

Here x_i is the ith collocation point of the domain of the problem and $\delta(\cdot)$ is the Dirac delta function defined such that its value is zero for all nonzero values of its arguments:

$$\delta(x - x_0) = 0 \ \text{ when } \ x \neq x_0, \qquad \int_{-\infty}^{\infty} f(x)\delta(x - x_0) \, dx = f(x_0) \tag{2.4.8}$$

Due to the different choices of w_i—even when the ϕ_i used in (2.4.4), (2.4.5) and (2.4.6) are the same—the system of algebraic equations will have different characteristics in different method. For linear differential equations of any order, only the least-squares method yields a system of matrix equations whose coefficient matrix is symmetric. One other method that has the symmetry property is the *Ritz method,* which uses the weak form of even-order (second, fourth, and so on; called self-adjoint) differential equations with $w_i = \phi_i$; the Ritz method is *not* a special case of the weighted-residual method. As we shall see shortly, the weak form of a self-adjoint differential equation always contains the same order derivatives of both the weight function w and the dependent unknown u, and the order is equal to half that of the original differential equation. In the following paragraphs, we discuss the weak form development.

Development of Weak Forms. There are three steps in the development of the weak form of any differential equation. These steps are illustrated by means of the model differential equation (2.4.1a) and boundary conditions (2.4.1b).

Step 1. (Weighted-integral statement.) This step is the same as in a weighted-residual method. Move all terms of the differential equation to one side (so that it reads $\ldots = 0$), multiply the entire equation with a function $w(x)$, and integrate over the domain $\Omega = (0, L)$

of the problem:

$$0 = \int_0^L w \left[-\frac{d}{dx}\left(a\frac{du}{dx}\right) - f \right] dx \tag{2.4.9}$$

Recall that the expression in the square brackets is not identically zero since u is replaced by its approximation, U_N. Mathematically, in (2.4.9) the error in the differential equation (due to the approximation of the solution) is made zero in the weighted-integral sense. The integral statement (2.4.9) allows us to choose N linearly independent functions for w and obtain N equations for $c_1, c_2, \cdots, c_N$ of (2.4.2).

Note that the weighted-integral statement of any differential equation can be readily written. The weighted-integral statement is equivalent only to the differential equation and does not include any boundary conditions. The weight function w in (2.4.9) can be any nonzero integrable function and has no differentiability requirements.

Step 2. While the weighted-integral statement (2.4.9) allows us to obtain the necessary number (N) of algebraic relations among c_j for N different choices of the weight function w, it requires that the approximation functions ϕ_j be such that U_N [see (2.4.2)] is differentiable as many times as called for in the original differential equation (2.4.1a) and satisfies the specified boundary conditions. If this is not a concern, one can proceed with the integral statement (2.4.9) and obtain the necessary algebraic equations for c_j (using any one of the choices listed in Eq. (2.4.7) for $w \sim w_i$).

If we plan to use the approximation functions ϕ_i for $w \sim w_i$, it makes sense to shift half of the derivatives from u to w so that both are differentiated equally, and we have fewer (or weaker) continuity requirements on ϕ_j. The resulting integral form is known as the weak form. Of course, weakening the differentiability of u (and hence ϕ_i) is purely a mathematical (and perhaps computational) consideration. As will be seen shortly, the weak formulation has two desirable characteristics. First, it requires weaker, as already indicated, continuity of the dependent variable, and for self-adjoint equations (as is the case with problems studied in this book) it always results in a symmetric coefficient matrix. Second, the natural boundary conditions of the problem are included in the weak form, and therefore the approximate solution U_N is required to satisfy only the essential boundary conditions of the problem. These two features of a weak form play an important role in the development of finite element models of a problem.

A word of caution is in order. Differentiating the weight function instead of the dependent variable (in addition to the weakening the continuity requirements on ϕ_i) is also dictated by the need to include physically meaningful boundary terms into the weak form, regardless of the effect on the continuity requirements. Therefore, this trade-off should not be performed if it results in boundary terms that are not physically meaningful.

Returning to the integral statement (2.4.9), we integrate the first term of the expression by parts to obtain

$$\begin{aligned} 0 &= \int_0^L \left\{ w \left[-\frac{d}{dx}\left(a\frac{du}{dx}\right) \right] - w f \right\} dx \\ &= \int_0^L \left(a\frac{dw}{dx}\frac{du}{dx} - w f \right) dx - \left[w\, a\frac{du}{dx} \right]_0^L \end{aligned} \tag{2.4.10}$$

where the integration-by-parts formula [see Eq. (2.2.22)]

$$\int_0^L w\, dv = -\int_0^L v\, dw + [wv]_0^L$$

is used with $v = -a du/dx$ on the first term to arrive at the second line of (2.4.10). Note that now the weight function w is required to be differentiable at least once.

An important part of Step 2 is to identify the two types of boundary conditions associated with *any* differential equation: *natural* and *essential*. The classification is important for both the variational methods of approximation considered in this chapter and the finite element formulations presented in the subsequent chapters. The following rule is used to identify the natural boundary conditions and their form. After trading between differentiating the weight function w and the variable u of the problem, examine all boundary terms of the integral statement. The boundary terms will involve both the weight function and the dependent variable. Coefficients of the weight function (and possibly its derivatives for higher-order equations) in the boundary expression(s) are termed the *secondary variables* (SV). For example, for the problem at hand the coefficient of w in the boundary term is $a(du/dx)$, which is the secondary variable. Specification of a secondary variable on the boundary constitutes the *natural boundary condition* (NBC).

The dependent variable of the problem (u), expressed in the *same form* as the weight function (w) appearing in the boundary term, is called the *primary variable* (PV), and its specification on the boundary constitutes the *essential boundary condition* (EBC). For the case under consideration, the weight function appears in the boundary expression [see (2.4.10)] as w (in higher-order equations, it may appear as w in one boundary term and as dw/dx in other). Therefore, the primary variable is u (for higher-order equations, the primary variables may include u as well as du/dx), and the essential boundary condition involves specifying u at the boundary points.

The secondary variables always have physical meaning and are often quantities of interest. In the case of heat transfer problems, the secondary variable represents heat, Q. We shall denote the secondary variable by

$$Q \equiv \left(a\frac{du}{dx}\right) n_x \tag{2.4.11}$$

where n_x denotes the direction cosine, i.e., n_x = cosine of the angle between the positive x axis and the normal to the boundary. For one-dimensional problems, the normal at the boundary points is always along the length of the domain. Thus, we have $n_x = -1$ at the left end and $n_x = 1$ at the right end of the domain.

It should be noted that the number and form of the primary and secondary variables depend on the order of the differential equation. The number of primary and secondary variables is always the same, and with each primary variable there is an associated secondary variable, i.e., they always appear in pairs (e.g., displacement and force, temperature and heat, and so on). Only one of the pair, either the primary or the secondary variable, may be specified at a point of the boundary. Thus, a given problem can have its specified boundary conditions in one of three categories: (1) all specified boundary conditions are EBC; (2) some of the specified boundary conditions are EBC and the remaining are NBC; or (3) all specified boundary conditions are NBC. For a single second-order equation, as in the present case, there is one primary variable u and one secondary variable Q. At a boundary

point, only one of the pair (u, Q) can be specified. For a fourth-order equation, such as that for the classical (i.e., Euler-Bernoulli) theory of beams, there are two of each kind (i.e., two PVs and two SVs), as will be illustrated later (see Example 2.4.2). In general, a $2m$th-order differential equation require m integration by parts to transfer m derivatives from u to w and therefore there will be m boundary terms involving m primary variables and m secondary variables, i.e., m pairs of primary and secondary variables.

Returning to Eq. (2.4.10), we rewrite it using the notation of (2.4.11):

$$\begin{aligned} 0 &= \int_0^L \left(a \frac{dw}{dx} \frac{du}{dx} - w f \right) dx - \left[w a \frac{du}{dx} \right]_0^L \\ &= \int_0^L \left(a \frac{dw}{dx} \frac{du}{dx} - w f \right) dx - \left(w a \frac{du}{dx} n_x \right) \Big|_{x=0} - \left(w a \frac{du}{dx} n_x \right) \Big|_{x=L} \\ &= \int_0^L \left(a \frac{dw}{dx} \frac{du}{dx} - w f \right) dx - (w Q)_0 - (w Q)_L \end{aligned} \tag{2.4.12}$$

Equation (2.4.12) is called the *weak form* of the differential equation (2.4.1a). The word "weak" refers to the reduced (i.e., weakened) continuity of u, which is required to be twice-differentiable in the weighted-integral statement (2.4.9) but only once-differentiable in (2.4.12).

Step 3. The third and last step of the weak formulation is to impose the actual boundary conditions of the problem under consideration. It is here that we require the weight function w to vanish at boundary points where the essential boundary conditions are specified, i.e., w is required to satisfy the *homogeneous form* of the specified essential boundary conditions of the problem (recall Section 2.3). In weak formulations, the weight function has the meaning of a *virtual* change (or variation) of the primary variable $w \sim \delta u$. If a primary variable is specified at a point, the virtual change there must be zero. For the problem at hand, the boundary conditions are given in (2.4.1b). By the rules of classification of the boundary conditions, $u = u_0$ is the essential boundary condition and $(adu/dx)|_{x=L} = Q_L$ is the natural boundary condition. Thus, the weight function w is required to satisfy $w(0) = 0$ because $u(0) = u_0$. Since $w(0) = 0$ and

$$Q(L) = \left(a \frac{du}{dx} n_x \right) \Big|_{x=L} = \left(a \frac{du}{dx} \right) \Big|_{x=L} = Q_L$$

Eq. (2.4.12) reduces to the expression

$$0 = \int_0^L \left(a \frac{dw}{dx} \frac{du}{dx} - w f \right) dx - w(L)\, Q_L \tag{2.4.13}$$

which is the weak form equivalent to the original differential equation (2.4.1a) and the natural boundary condition (2.4.1b). This completes the steps involved in the development of the weak form of a differential equation.

The terms "variational form" and "weak form" will be used interchangeably. The weak form of a differential equation is a weighted-integral statement equivalent to the differential equation *and* the specified natural boundary condition of the problem. Note that the weak form exists for all problems—linear or nonlinear—that are described by second- and higher-order differential equations. When the differential equation is linear and of even order, the

resulting weak form will have a *symmetric* bilinear form in the dependent variable u and weight function w, as we shall see shortly.

In summary, there are three steps in the development of a weak form. In the first step, we put all expressions of the differential equation on one side (so that the other side is equal to zero), then multiply the entire equation by a weight function and integrate over the domain of the problem. The resulting expression is called the weighted-integral form of the equation. In the second step, we use integration by parts to distribute differentiation evenly between the dependent variable and the weight function, and use the boundary terms to identify the form of the primary and secondary variables. In the third step, we modify the boundary terms by restricting the weight function to satisfy the homogeneous form of the specified essential boundary conditions and replacing the secondary variables by their specified values.

It should be recalled that a weighted-integral statement or the weak form of a differential equation is needed to obtain as many algebraic equations as there are unknown coefficients in the approximation of the dependent variables of the equation. For different choices of the weight function, different algebraic equations can be obtained. Because of the restrictions placed on the weight function in Step 3 ($w \sim \delta u$) of the variational formulation, it must belong to the same space of functions as the approximation functions (i.e., $w \sim \phi_i$).

2.4.3 Linear and Bilinear Forms and Quadratic Functionals

It is informative, although not necessary for the use of variational methods or the finite element method, to see the relation between the weak form and the minimum of a quadratic functional associated with the differential equation. The weak form (2.4.13) contains two types of expressions: those involving both the dependent variable u and the weight function w, and those involving only the latter. We shall denote these two types of expressions by $B(w, u)$ and $l(w)$, respectively:

$$B(w,u) = \int_0^L a \frac{dw}{dx}\frac{du}{dx}\,dx, \quad l(w) = \int_0^L w\, f\, dx + w(L)\, Q_L \tag{2.4.14}$$

Hence, the weak form (2.4.13) can be expressed in the form

$$0 = B(w,u) - l(w) \quad \text{or} \quad B(w,u) = l(w) \tag{2.4.15}$$

which is termed the *variational problem* associated with Eqs. (2.4.1a) and (2.4.1b). Using the definitions of linear and bilinear forms from Section 2.2.4, it can be verified that $B(w, u)$ is bilinear and symmetric in w and u and that $l(w)$ is linear. The variational problem associated with (2.4.1a) and (2.4.1b) can be stated as one of finding the solution u (from a suitable vector space, H) such that

$$B(w,u) = l(w) \tag{2.4.16}$$

holds for any w (in H) that satisfies the homogeneous form of the specified essential boundary conditions and continuity conditions implied by the weak form. The function w can be viewed as a variation of the actual solution $w = \delta u$, and Eq. (2.4.15) can be written as

$$0 = B(\delta u, u) - l(\delta u) \tag{2.4.17}$$

If $B(\cdot, \cdot)$ is bilinear and symmetric and $l(\cdot)$ is linear, we have

$$B(\delta u, u) = \frac{1}{2}\delta[B(u,u)], \quad l(\delta u) = \delta[l(u)] \tag{2.4.18a}$$

so that (2.4.17) can be expressed as

$$0 = B(\delta u, u) - l(\delta u) = \frac{1}{2}\delta[B(u,u)] - \delta[l(u)] \equiv \delta I(u) \tag{2.4.18b}$$

where

$$I(u) = \frac{1}{2}B(u,u) - l(u) \tag{2.4.19}$$

The result in (2.4.18a) can be verified for the problem at hand:

$$\begin{aligned} B(\delta u, u) &= \int_0^L a\frac{d\delta u}{dx}\frac{du}{dx}\,dx = \delta\int_0^L \frac{a}{2}\left(\frac{du}{dx}\right)^2 dx \\ &= \frac{1}{2}\delta\int_0^L a\frac{du}{dx}\frac{du}{dx}\,dx = \frac{1}{2}\delta[B(u,u)] \\ l(\delta u) &= \int_0^L \delta u\, f\, dx + \delta u(L)Q_L \\ &= \delta\left[\int_0^L u\, f\, dx + u(L)Q_L\right] = \delta[l(u)] \end{aligned}$$

Now we can restate the variational problem (2.4.16) as one of minimizing the functional $I(u)$:

$$\delta I = 0 = B(\delta u, u) - l(\delta u), \quad \delta^2 I = B(\delta u, \delta u) > 0 \text{ for } \delta u \neq 0 \tag{2.4.20}$$

Thus, the function u that minimizes $I(u)$ is the solution of (2.4.16); conversely, the solution of (2.4.16) minimizes the functional. Mathematical proof of these assertions can be found in Reddy (1986, 2002).

Note that the key step in the derivation of the functional $I(u)$ from the weak form is the bilinearity and symmetry of the bilinear form $B(w,u)$. Thus, whenever $B(w,u)$ is bilinear and symmetric, and $l(w)$ is linear, the functional associated with the variational problem (2.4.16) is given by (2.4.19). When $B(w,u)$ is not linear in w and u, but is symmetric, the functional $I(u)$ can be derived, but not from (2.4.19). The interested reader may consult the books by Oden and Reddy (1983) and Reddy (1986).

For solid mechanics problems, $I(u)$ represents the total potential energy functional, and $\delta I = 0$ is the statement of the principle of the minimum total potential energy: Of all admissible functions u, the one that makes the total potential energy $I(u)$ a minimum also satisfies the differential equation(s) and natural boundary condition(s). In other words, the weak form of a differential equation is the same as the statement of the principle of minimum total potential energy. For problems outside solid mechanics, the functional $I(u)$, if it exists, may not have any physical meaning, but it is still useful for mathematical analysis (e.g., in proving the existence and uniqueness of solutions).

As noted earlier, every differential equation admits a weighted-integral statement, and a weak form exists provided the equation is of order two or higher. When the bilinear form is symmetric, we will also have a functional whose first variation set equal to zero is equivalent to the governing equations. Recall that we can always construct the least-squares functional associated with any set of governing equations. However, the traditional variational methods and the finite element method use only an integral statement or a weak form of the equation(s) to be solved.

2.4.4 Examples

Here, we consider some representative examples of differential equations in one and two dimensions, and develop their weak forms. These examples are of primary interest in the study of the finite element method in the coming chapters. All problems considered here correspond to one or more physical problems of science and engineering.

Example 2.4.1 (Model Second-Order Equation in One Dimension)

Consider the differential equation

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu=f \quad \text{for} \quad 0<x<L \tag{2.4.21a}$$

subject to the boundary conditions

$$u(0)=u_0, \quad \left(a\frac{du}{dx}\right)+\beta\,(u-u_\infty)=Q_0 \ \text{ at } x=L \tag{2.4.21b}$$

The data of the problem consists of specifying $a(x)$, $c(x)$, $f(x)$, u_0, β, u_∞, and Q_0.

Following the three steps outlined above for the construction of variational statements, we obtain

$$\begin{aligned} &\textit{Step 1:} && 0=\int_0^L w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f\right]dx \\ &\textit{Step 2:} && 0=\int_0^L\left(a\frac{dw}{dx}\frac{du}{dx}+cwu-wf\right)-\left[w\,a\frac{du}{dx}\right]_0^L \end{aligned} \tag{2.4.22}$$

From the boundary term, it is clear that the specification of u is an essential boundary condition, and the specification of $a\,du/dx$ is a natural boundary condition. Since $w=0$ at $x=0$ (because u is specified there) and

$$a\frac{du}{dx}=Q_0-\beta\,(u-u_\infty) \ \text{ at } x=L$$

we obtain

$$\textit{Step 3:} \quad 0=\int_0^L\left(a\frac{dw}{dx}\frac{du}{dx}+cwu\right)dx-\int_0^L wf\,dx+w(L)\beta(u(L)-u_\infty)-w(L)Q_0 \tag{2.4.23a}$$

Variational Problem and Quadratic Functional: Equation (2.4.23*a*) can be expressed in the form

$$B(w,u)=l(w) \tag{2.4.23b}$$

where

$$\begin{aligned} B(w,u)&=\int_0^L\left(a\frac{dw}{dx}\frac{du}{dx}+cwu\right)dx+\beta w(L)u(L) \\ l(w)&=\int_0^L wf\,dx+\beta w(L)u_\infty+Q_0u(L) \end{aligned} \tag{2.4.23c}$$

It is clear that since $B(w, u)$ is linear and symmetric in w and u, and $l(w)$ is linear in w, we can compute the quadratic functional from (2.4.19):

$$I(u)=\frac{1}{2}\int_0^L\left[a\left(\frac{du}{dx}\right)^2+cu^2\right]dx+\frac{1}{2}\beta[u(L)]^2-\int_0^L fu\,dx-\beta u(L)u_\infty-Q_0u(L) \tag{2.4.24}$$

Equations of the type of (2.4.21*a*) arise, for example, in the study of the axial deformation of a bar or flow of heat in a bar. In the former case, u denotes the axial displacement, a the product of modulus of elasticity (E) and area of cross section (A), and c the resistance offered by an external medium to the axial deformation (often, $c=0$). In the case of heat flow, u denotes the temperature (T), a the product of thermal conductivity (k) times the area of cross section (A) of the bar, and c is the product of film conductance (β) times the perimeter (P).

The next example illustrates the variational formulation of a fourth-order differential equation governing bending of elastic beams according to the Euler-Bernoulli beam theory [see Reddy (2002)].

Example 2.4.2 (Euler-Bernoulli Beam)

Consider the problem of finding the function $w(x)$ that satisfies the differential equation

$$\frac{d^2}{dx^2}\left[b(x)\frac{d^2w}{dx^2}\right]+c_fw-q(x)=0 \quad \text{for} \quad 0<x<L \tag{2.4.25}$$

This equation arises in the study of the elastic bending of beams (under the Euler-Bernoulli hypothesis that plane sections perpendicular to the axis of the beam before deformation remain plane after deformation), where w denotes the transverse deflection of the beam, L is the length of the beam, $b(x)>0$ is the flexural rigidity of the beam (i.e., the product of modulus of elasticity E and second moment of inertia I), c_f is the foundation modulus, and $q(x)$ is the transverse distributed load, as shown in Fig. 2.4.1. At the moment, we do not have to consider any specific boundary conditions of the problem.

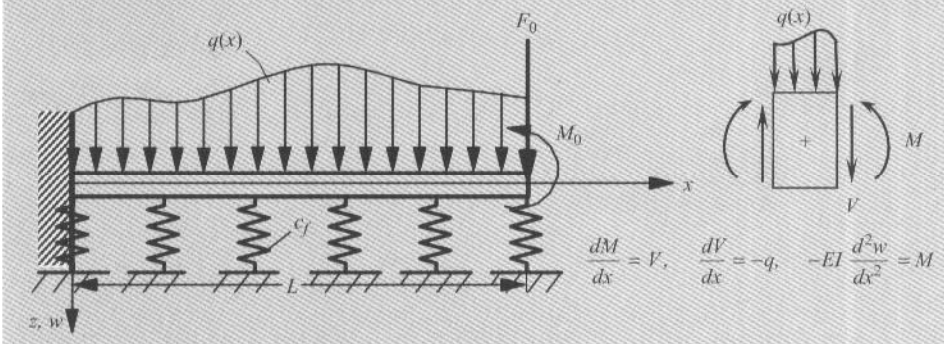


Figure 2.4.1 Cantilever beam with a set of loads.

Since the equation contains a fourth-order derivative, we should integrate it twice by parts to distribute the derivatives equally between the dependent variable w and the weight function v. In this case, v must be twice differentiable and satisfy the homogeneous form of an EBC. Multiplying (2.4.25) by v, and integrating the first term by parts twice with respect to x, we obtain

$$\textit{Step 1:}\quad 0=\int_0^L v\left[\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right)+c_f w-q\right]dx \tag{2.4.26}$$

$$\begin{aligned}\textit{Step 2:}\quad 0&=\int_0^L\left[\left(-\frac{dv}{dx}\right)\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)+c_f vw-vq\right]dx+\left[v\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_0^L\\&=\int_0^L\left(\frac{d^2v}{dx^2}b\frac{d^2w}{dx^2}+c_f vw-vq\right)dx+\left[v\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)-\frac{dv}{dx}b\frac{d^2w}{dx^2}\right]_0^L\end{aligned} \tag{2.4.27}$$

From the last line, it follows that the specification of w and dw/dx constitutes the essential (geometric or static) boundary conditions, and the specification of

$$\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\ \text{(shear force)}\quad\text{and}\quad b\left(\frac{d^2w}{dx^2}\right)\ \text{(bending moment)} \tag{2.4.28a}$$

constitutes the natural boundary conditions for the Euler-Bernoulli beam theory.

Now consider a specific beam problem. Let us consider a beam fixed at the left end and subjected to transverse force and bending moment at $x=L$, as shown in Fig. 2.4.1:

$$w(0)=0,\quad \left(\frac{dw}{dx}\right)\bigg|_{x=0}=0,\quad \left(b\frac{d^2w}{dx^2}\right)\bigg|_{x=L}=-M_0,\quad \left[\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_{x=L}=-F_0 \tag{2.4.28b}$$

where M_0 is the bending moment and F_0 is the transverse load. Since w and dw/dx (both are primary variables) are specified at $x=0$, we require the weight function v and its derivative dv/dx to be zero there

$$v(0)=\left(\frac{dv}{dx}\right)\bigg|_{x=0}=0$$

The remaining two boundary conditions in (2.4.28b) are natural boundary conditions, which place no restrictions on v and its derivatives. Thus, Eq. (2.4.27) becomes

$$\textit{Step 3:}\quad 0=\int_0^L\left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}+c_f vw-vq\right)dx-v(L)F_0+\left(\frac{dv}{dx}\right)\bigg|_{x=L}M_0 \tag{2.4.29}$$

Variational Problem and Quadratic Functional: Equation (2.4.30a) can be written in the form

$$B(v,w)=l(v) \tag{2.4.30a}$$

where

$$\begin{aligned}B(v,w)&=\int_0^L\left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}+c_f vw\right)dx\\ l(v)&=\int_0^L vq\,dx+v(L)F_0-\left(\frac{dv}{dx}\right)\bigg|_{x=L}M_0\end{aligned} \tag{2.4.30b}$$

The functional, known as the total potential energy of the beam, is obtained using (2.4.19):

$$I(w) = \int_0^L \left[\frac{b}{2}\left(\frac{d^2w}{dx^2}\right)^2 + \frac{c_f}{2}w^2 - wq \right] dx - w(L)F_0 + \left(\frac{dw}{dx}\right)\Big|_{x=L} M_0 \tag{2.4.31}$$

Note that for the fourth-order equation, the essential boundary conditions involve not only the dependent variable but also its first derivative. As pointed out earlier, at any boundary point, only one of the two boundary conditions (essential or natural) can be specified. For example, if the transverse deflection is specified at a boundary point, then one cannot specify the shear force V at the same point, and vice versa. Similar comments apply to the slope dw/dx and the bending moment M. Note that in the present case, w and dw/dx are the primary variables, and V and M are the secondary variables.

The next example is concerned with the weak-form development of a pair of second-order differential equations in one dimension. The approach used for a single equation is followed for each equation of the pair. However, writing the variational problem and the associated functional is a bit tricky.

Example 2.4.3 (Timoshenko Beam)

Consider the following pair of coupled differential equations governing bending of the Timoshenko beam:

$$-\frac{d}{dx}\left[S\left(\frac{dw}{dx}+\phi_x\right)\right] + c_f w = q \tag{2.4.32a}$$

$$-\frac{d}{dx}\left(D\frac{d\phi_x}{dx}\right) + S\left(\frac{dw}{dx}+\phi_x\right) = 0 \tag{2.4.32b}$$

where S is the shear stiffness ($S = K_s GA$; K_s is the shear correction coefficient, G is the shear modulus, and A is the area of the cross section), $D = EI$ is the bending stiffness, w is the transverse deflection, ϕ_x is the rotation, k is the foundation modulus, and q is the distributed transverse load. We shall develop the weak form of the above equations using the three-step procedure.

Steps 1 and 2: Multiply the first equation with weight function v_1 and the second one with weight function v_2 and integrate over the length of the beam:

$$\textit{Step 1a:}\quad 0 = \int_0^L v_1 \left\{ -\frac{d}{dx}\left[S\left(\frac{dw}{dx}+\phi_x\right)\right] + c_f w - q \right\} dx$$

$$\begin{aligned}\textit{Step 2a:}\quad 0 &= \int_0^L \left[\frac{dv_1}{dx} S\left(\frac{dw}{dx}+\phi_x\right) + c_f v_1 w - v_1 q\right] dx - \left[v_1 S\left(\frac{dw}{dx}+\phi_x\right)\right]_0^L \\ &= \int_0^L \left[\frac{dv_1}{dx} S\left(\frac{dw}{dx}+\phi_x\right) + c_f v_1 w - v_1 q\right] dx - v_1(L)\left[S\left(\frac{dw}{dx}+\phi_x\right)\right]_{x=L} \\ &\quad + v_1(0)\left[S\left(\frac{dw}{dx}+\phi_x\right)\right]_{x=0}\end{aligned} \tag{2.4.33a}$$

$$\textit{Step 1}b\text{:}\quad 0=\int_0^L v_2\left[-\frac{d}{dx}\left(D\frac{d\phi_x}{dx}\right)+S\left(\frac{dw}{dx}+\phi_x\right)\right]dx$$

$$\textit{Step 2}b\text{:}\quad 0=\int_0^L\left[D\frac{dv_2}{dx}\frac{d\phi_x}{dx}+v_2\,S\left(\frac{dw}{dx}+\phi_x\right)\right]dx-\left[v_2\,D\frac{d\phi_x}{dx}\right]_0^L$$

$$=\int_0^L\left[D\frac{dv_2}{dx}\frac{d\phi_x}{dx}+v_2\,S\left(\frac{dw}{dx}+\phi_x\right)\right]dx$$

$$-v_2(L)\left[D\frac{d\phi_x}{dx}\right]_{x=L}+v_2(0)\left[D\frac{d\phi_x}{dx}\right]_{x=0}\qquad(2.4.33b)$$

Note that integration-by-parts was used such that the expression $w_{,x}+\phi_x$ is preserved, as it enters the boundary term representing the shear force. Such considerations can only be used by knowing the mechanics of the problem at hand. Also, note that the pair of weight functions (v_1, v_2) satisfy the homogeneous form of specified essential boundary conditions on the pair (w, ϕ_x) (with the correspondence $v_1 \sim w$ and $v_2 \sim \phi_x$).

Steps 3: An examination of the boundary terms shows that $w \sim v_1$ and $\phi_x \sim v_2$ are the primary variables, and the secondary variables are given by [cf. Eq. (2.4.28*a*)]

$$S\left(\frac{dw}{dx}+\phi_x\right)\quad\text{(shear force)}\qquad(2.4.34a)$$

$$D\frac{d\phi_x}{dx}\quad\text{(bending moment)}\qquad(2.4.34b)$$

To finalize the weak forms, we must take care of the boundary terms by considering a specific beam problem. Using the beam of Fig. 2.4.1, we see that

$$w(0)=0,\quad \phi_x(0)=0,\quad \left[S\left(\frac{dw}{dx}+\phi_x\right)\right]_{x=L}=F_0,\quad \left[D\frac{d\phi_x}{dx}\right]_{x=L}=M_0\qquad(2.4.35)$$

and hence, $v_1(0)=0$ and $v_2(0)=0$. Consequently, the weak forms (2.4.33*a*) and (2.4.33*b*) become

$$0=\int_0^L\left[\frac{dv_1}{dx}\,S\left(\frac{dw}{dx}+\phi_x\right)+c_f v_1 w-v_1 q\right]dx-v_1(L)\,F_0\qquad(2.4.36a)$$

$$0=\int_0^L\left[\frac{dv_2}{dx}\,D\frac{d\phi_x}{dx}+v_2\,S\left(\frac{dw}{dx}+\phi_x\right)\right]dx-v_2(L)\,M_0\qquad(2.4.36b)$$

Variational Problem and Quadratic Functional: To write the variational problem of finding (w, ϕ_x) such that

$$B((v_1, v_2), (w, \phi_x)) = l((v_1, v_2))\qquad(2.4.37)$$

holds for all (v_1, v_2), we must combine the two weak forms into a single expression

$$0=\int_0^L\left[\left(\frac{dv_1}{dx}+v_2\right)S\left(\frac{dw}{dx}+\phi_x\right)+\frac{dv_2}{dx}\,D\frac{d\phi_x}{dx}+c_f v_1 w-v_1 q\right]dx-v_1(L)\,F_0-v_2(L)\,M_0$$

(2.4.38)

Thus, the bilinear and linear forms of the problem are given by

$$B((v_1, v_2), (w, \phi_x)) = \int_0^L \left[S\left(\frac{dv_1}{dx} + v_2\right)\left(\frac{dw}{dx} + \phi_x\right) + D\frac{dv_2}{dx}\frac{d\phi_x}{dx} + c_f v_1 w \right] dx \quad (2.4.39a)$$

$$l((v_1, v_2)) = \int_0^L v_1 q \, dx + v_1(L) F_0 + v_2(L) M_0 \quad (2.4.39b)$$

Clearly, $B((v_1, v_2), (w, \phi_x))$ is symmetric in its arguments (i.e. interchange of v_1 with w *and* v_2 with ϕ_x yields the same expression). Hence the functional is given by

$$I((w, \phi_x)) = \frac{1}{2}\int_0^L \left[S\left(\frac{dw}{dx} + \phi_x\right)^2 + D\left(\frac{d\phi_x}{dx}\right)^2 + \frac{c_f}{2} w^2 \right] dx - \left(\int_0^L w q \, dx + w(L) F_0 + \phi_x(L) M_0 \right) \quad (2.4.40)$$

The last example of this section is concerned with a second-order differential equation in two dimensions. The equation arises in a number of fields, including heat transfer, stream function or velocity potential formulation of inviscid flows, transverse deflections of a membrane, torsion of a cylindrical member, and others.

Example 2.4.4 (Poisson's Equation in Two Dimensions)

Consider the problem of determining the solution $u(x, y)$ to the partial differential equation,

$$-\frac{\partial}{\partial x}\left(a_1 \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_2 \frac{\partial u}{\partial y}\right) + a_0 u = f \quad \text{in } \Omega \quad (2.4.41)$$

in a closed two-dimensional domain Ω with boundary Γ, as shown in Fig. 2.4.2. Here a_0, a_1, a_2, and f are known functions of position (x, y) in Ω. The function u is required to satisfy, in addition to the differential equation (2.4.41), certain boundary conditions on the boundary Γ of Ω. The weak formulation to be discussed next will reveal the precise form of the essential and natural boundary conditions of the equation.

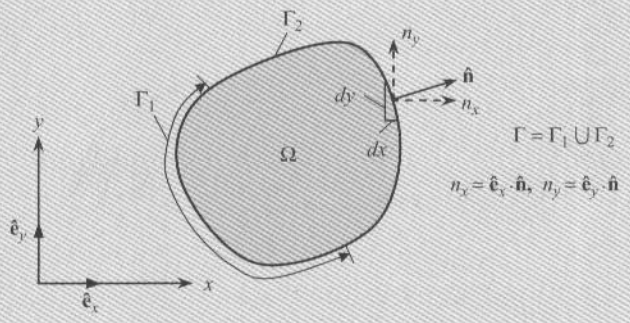


Figure 2.4.2 A two-dimensional domain Ω with boundary Γ.

The three step procedure applied to Eq. (2.4.41) results in the following equations:

$$\textit{Step 1:} \qquad 0 = \int_\Omega w \left[-\frac{\partial}{\partial x}\left(a_1 \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_2 \frac{\partial u}{\partial y}\right) + a_0 u - f \right] dx dy \tag{2.4.42}$$

$$\begin{aligned} \textit{Step 2:} \qquad 0 = & \int_\Omega \left(a_1 \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + a_2 \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} + a_0 w u - w f \right) dx dy \\ & - \oint_\Gamma w \left(a_1 \frac{\partial u}{\partial x} n_x + a_2 \frac{\partial u}{\partial y} n_y \right) ds \end{aligned} \tag{2.4.43}$$

where we used integration by parts [or the gradient and divergence theorems, Eqs. (2.2.27*b*) and (2.2.28*b*)] to transfer differentiation from u to w so that both u and w have the same order derivatives in Ω. The boundary term shows that u is the primary variable while

$$a_1 \frac{\partial u}{\partial x} n_x + a_2 \frac{\partial u}{\partial y} n_y$$

is the secondary variable.

Step 3: The last step in the procedure is to impose the specified boundary conditions. Suppose that u is specified on portion Γ_1 and the natural boundary condition is specified on the remaining portion Γ_2 of the boundary as (see Fig. 2.4.2):

$$u = \hat{u} \text{ on } \Gamma_1, \quad a_1 \frac{\partial u}{\partial x} n_x + a_2 \frac{\partial u}{\partial y} n_y = \hat{g} \text{ on } \Gamma_2 \tag{2.4.44}$$

Then w is arbitrary on Γ_2 and equal to zero on Γ_1. Consequently, Eq. (2.4.43) simplifies to

$$0 = \int_\Omega \left(a_1 \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + a_2 \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} + a_0 w u - w f \right) dx dy - \int_{\Gamma_2} w \hat{g}\, ds \tag{2.4.45}$$

Variational Problem and Quadratic Functional: The weak statement (2.4.45) can be expressed as $B(w, u) = \ell(w)$, where the bilinear form and linear form are

$$B(w,u) = \int_\Omega \left(a_1 \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + a_2 \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} + a_0 w u \right) dx dy, \tag{2.4.46a}$$

$$\ell(w) = \int_\Omega w f\, dx dy + \int_{\Gamma_2} w \hat{g}\, ds, \tag{2.4.46b}$$

The associated quadratic functional is

$$\begin{aligned} I(u) = & \frac{1}{2}\int_\Omega \left[a_1 \left(\frac{\partial u}{\partial x}\right)^2 + a_2 \left(\frac{\partial u}{\partial y}\right)^2 + a_0 u^2 \right] dx dy \\ & - \int_\Omega u f\, dx dy - \int_{\Gamma_2} \hat{g} u\, ds. \end{aligned} \tag{2.4.46c}$$

In the case of transverse deflections of a membrane, $I(u)$ represents the total potential energy.

As a specific example of the Poisson equation, consider steady heat conduction in a two-dimensional domain Ω, enclosed by lines AB, BC, CD, DE, EF, FG, GH, and HA, as indicated

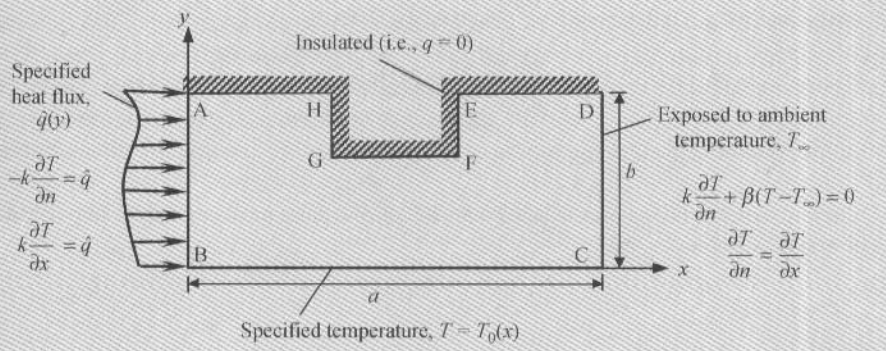


Figure 2.4.3 A two-dimensional heat transfer problem with a variety of boundary conditions.

in Fig. 2.4.3. The governing equation is

$$-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=g_0 \quad \text{in} \quad \Omega \tag{2.4.47}$$

where g_0 is the uniform heat generation, k is the conductivity of the isotropic material of the domain, and T is the temperature. At the second step of the weak-form development, we have

$$0=\int_\Omega \left[k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right)-wg_0\right]dx\,dy-\oint_\Gamma wk\left(\frac{\partial T}{\partial x}n_x+\frac{\partial T}{\partial y}n_y\right)ds \tag{2.4.48}$$

In order to simplify the boundary expression, we first recognize the fact that the boundary Γ of the domain consists of several line segments, and they are subject to different types of boundary conditions as shown in Fig. 2.4.3:

$$\begin{aligned}
&\Gamma_1 = \text{AB}: n_x=-1,\ n_y=0; && \text{heat flux is specified } q=\hat{q}(y)\\
&\Gamma_2 = \text{BC}: n_x=0,\ \ n_y=-1; && \text{temperature is specified } T=T_0(x)\\
&\Gamma_3 = \text{CD}: n_x=1,\ \ n_y=0; && \text{boundary exposed to ambient temperature } T_\infty\\
& && \text{(convection) } k\frac{\partial T}{\partial n}+\beta\,(T-T_\infty)=0 \qquad (2.4.49)\\
&\Gamma_4 = \text{DEFGHA}: && \text{insulated boundary } \frac{\partial T}{\partial n}=0
\end{aligned}$$

Using the boundary information, the boundary integral in (2.4.48) can be simplified as follows (note that $w=0$ on portion of the boundary wherever T is specified, Γ_2):

$$\begin{aligned}
\oint_\Gamma w\left(k\frac{\partial T}{\partial n}\right)ds &= \int_{\Gamma_1} wq_n\,ds+\int_{\Gamma_2} 0\left(k\frac{\partial T}{\partial n}\right)ds\\
&\quad -\int_{\Gamma_3} w[\beta(T-T_\infty)]\,ds+\int_{\Gamma_4} w\,0\,ds\\
&= -\int_0^b w(0,y)\hat{q}(y)\,dy-\beta\int_0^b w(a,y)[T(a,y)-T_\infty]\,dy
\end{aligned} \tag{2.4.50}$$

Substituting (2.4.50) into (2.4.48), we obtain the weak form

$$0 = \int_\Omega \left[k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial T}{\partial y} \right) - w g_0 \right] dx\, dy + \int_0^b w(0, y)\hat{q}(y) dy + \beta \int_0^b w(a, y)[T(a, y) - T_\infty]\, dy \tag{2.4.51}$$

Collecting terms involving both w and T into $B(\cdot, \cdot)$, and those involving only w into $l(\cdot)$, we can write (2.4.51) as

$$B(w, T) = l(w) \tag{2.4.52a}$$

where

$$B(w, T) = \int_\Omega k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial T}{\partial y} \right) dx\, dy + \beta \int_0^b w(a, y) T(a, y) dy$$

$$l(w) = \int_\Omega w g_0\, dx\, dy - \int_0^b w(0, y)\hat{q}(y)\, dy + \beta \int_0^b w(a, y) T_\infty\, dy \tag{2.4.52b}$$

The quadratic functional is given by

$$I(T) = \frac{k}{2}\int_\Omega \left[\left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right] dx\, dy + \frac{\beta}{2}\int_0^b T^2(a, y)\, dy - \int_\Omega T g_0\, dx\, dy + \int_0^b T(0, y)\hat{q}(y)\, dy - \beta \int_0^b T(a, y) T_\infty\, dy \tag{2.4.53}$$

Note that the boundary integrals in this example are defined along the y or x axes, because the boundaries are parallel to either the x- or y-axis.

2.5 VARIATIONAL METHODS

2.5.1 Introduction

Our objective in this section is to study the variational methods of approximation because they provide a background for the development of finite element models. The methods to be discussed include the Ritz and weighted-residual (e.g., Galerkin, Petrov–Galerkin, least-squares, and collocation) methods. In all these methods, we seek an approximate solution in the form of a linear combination of suitable approximation functions ϕ_j and undetermined parameters c_j: $\sum_j c_j \phi_j$. The Ritz method uses the weak form, whereas the weighted-residual methods use the weighted-integral form (Step 1 of the weak-form development) to determine the parameters c_j. Various methods differ from each other in the choice of weight function w and approximation functions ϕ_j. As we shall see in the coming chapters, the finite element method is a elementwise application of the classical variational methods.

2.5.2 The Ritz Method

In the Ritz method, the coefficients c_j of the approximation are determined using the weak form of the problem, and hence, the choice of weight functions is restricted to the

approximation functions, $w = \phi_j$ ($w \sim u$). Recall that the weak form contains both the governing differential equation and the natural boundary conditions of the problem, and hence it places weaker continuity requirements on the approximate solution than the original differential equation or its weighted-integral form. The method is described here for a linear variational problem (which is the same as the weak form).

Consider the variational problem resulting from the weak form: Find the solution u such that

$$B(w, u) = l(w) \tag{2.5.1}$$

for all sufficiently differentiable functions w that satisfy the homogeneous form of specified essential boundary conditions on u. In general, $B(\cdot, \cdot)$ can be unsymmetric in w and u, and it can be even nonlinear in u [$B(\cdot, \cdot)$ is always linear in w]. When $B(\cdot, \cdot)$ is bilinear and symmetric in w and u and l is linear, the problem in (2.5.1) is equivalent to minimization of the quadratic functional [see Eq. (2.4.19)]

$$I(u) = \frac{1}{2} B(u, u) - l(u) \tag{2.5.2}$$

In the Ritz method, we seek an approximate solution to (2.5.1) in the form of a finite series [see Eq. (2.4.2)]

$$U_N(x) = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x) \tag{2.5.3}$$

where the constants c_j, called the *Ritz coefficients*, are determined such that (2.5.1) holds for each $w = \phi_i$ ($i = 1, 2, \ldots, N$), i.e., (2.5.1) holds for N different choices of w, so that N independent algebraic relations among c_j are obtained. The functions ϕ_j and ϕ_0, called *approximation functions*, are chosen such that U_N satisfies the specified essential boundary conditions [recall that the specified natural boundary conditions are already included in the variational problem (2.5.1) and hence in the functional $I(u)$]. More details on this will be discussed shortly. The ith algebraic equation is obtained from Eq. (2.5.1) by substituting $w = \phi_i$ and U_N from (2.5.2) for u:

$$B\left(\phi_i, \sum_{j=1}^{N} c_j \phi_j + \phi_0\right) = l(\phi_i) \quad (i = 1, 2, \cdots, N)$$

Since $B(\cdot, \cdot)$ is linear in u, we have

$$\sum_{j=1}^{N} B(\phi_i, \phi_j) c_j = l(\phi_i) - B(\phi_i, \phi_0)$$

or

$$\sum_{j=1}^{N} K_{ij} c_j = F_i, \quad i = 1, 2, \ldots, N \tag{2.5.4a}$$

$$K_{ij} = B(\phi_i, \phi_j), \quad F_i = l(\phi_i) - B(\phi_i, \phi_0) \tag{2.5.4b}$$

The algebraic equations in (2.5.4a) can be expressed in matrix form as

$$[K]\{c\} = \{F\} \quad \text{or} \quad \mathbf{Kc} = \mathbf{F} \tag{2.5.5}$$

As stated earlier, for symmetric bilinear forms the Ritz method can also be viewed as seeking a solution of the form in (2.5.3) in which the parameters c_j are determined by minimizing the quadratic functional $I(u)$ in (2.5.2). After substituting U_N from (2.5.3) for u into (2.5.2) and integrating, the functional $I(u)$ becomes an ordinary function of the parameters $c_1, c_2, \ldots c_N$. Then the necessary condition for the minimization of $I(c_1, c_2, \ldots, c_N)$ is that its partial derivatives with respect to each of the parameters be zero:

$$\frac{\partial I}{\partial c_1} = 0, \quad \frac{\partial I}{\partial c_2} = 0, \quad \cdots, \quad \frac{\partial I}{\partial c_N} = 0 \tag{2.5.6}$$

Thus, there are N linear algebraic equations in N unknowns, c_j $(j = 1, 2, \cdots, N)$. These equations are exactly the same as those in (2.5.4) for all problems for which the variational problem (2.5.1) is equivalent to $\delta I = 0$. Of course, when $B(\cdot, \cdot)$ is not symmetric, we do not have a quadratic functional. In other words, (2.5.4a) is more general than (2.5.6), and they are the same when $B(\cdot, \cdot)$ is bilinear and symmetric. In all problems of interest in the present study, we shall have a symmetric bilinear form.

2.5.3 Approximation Functions

Returning to the approximation U_N in (2.5.3), we wish to discuss the selection of the approximation functions ϕ_j and ϕ_0 for the Ritz method. First, we note that U_N must satisfy only the specified essential boundary conditions of the problem, since the specified natural boundary conditions are included in the variational problem (2.5.1). The particular form of U_N in (2.5.3) facilitates satisfaction of specified boundary conditions. To see this, suppose that the approximate solution is sought in the form

$$U_N(x) = \sum_{j=1}^{N} c_j \phi_j(x)$$

and suppose that the specified essential boundary condition is $u(x_0) = u_0$. Then U_N must also satisfy the condition $U_N(x_0) = u_0$ at a boundary point $x = x_0$:

$$\sum_{j=1}^{N} c_j \phi_j(x_0) = u_0$$

Since c_j are unknown parameters to be determined, it is not easy to choose $\phi_j(x)$ such that the above relation holds. If $u_0 = 0$, then we can select all ϕ_j such that $\phi_j(x_0) = 0$ and satisfy the condition $U_N(x_0) = 0$. By writing the approximate solution U_N in the form (2.5.3), a sum of a homogeneous part $\sum c_j \phi_j(x)$ and a nonhomogeneous part $\phi_0(x)$, we require $\phi_0(x)$ to satisfy the specified essential boundary conditions while the homogeneous part vanishes at the same boundary point where the essential boundary condition is specified. This follows from

$$U_N(x_0) = \sum_{j=1}^{N} c_j \phi_j(x_0) + \phi_0(x_0)$$

$$u_0 = \sum_{j=1}^{N} c_j \phi_j(x_0) + u_0 \rightarrow \sum_{j=1}^{N} c_j \phi_j(x_0) = 0$$

which is satisfied, for arbitrary c_j, by choosing $\phi_j(x_0) = 0$.

If all specified essential boundary conditions are homogeneous (i.e., the specified value u_0 is zero), then ϕ_0 is taken to be zero and ϕ_j must still satisfy the same conditions, $\phi_j(x_0) = 0$, $j = 1, 2, \ldots, N$. Note that the requirement that w be zero at the boundary points where the essential boundary conditions are specified is satisfied by the choice $w = \phi_j(x)$.

In summary, the approximation functions $\phi_i(x)$ and $\phi_0(x)$ are required to satisfy the following conditions:

1. (a) ϕ_i must be such that $B(\phi_i, \phi_j)$ is defined and nonzero, i.e., ϕ_i are sufficiently differentiable and integrable as required in the evaluation of $B(\phi_i, \phi_j)$.
   (b) ϕ_i must satisfy the homogeneous form of the specified essential boundary conditions of the problem.
2. For any N, the set $\{\phi_i\}_{i=1}^N$ along with the columns (and rows) of $B(\phi_i, \phi_j)$ must be *linearly independent*.
3. The set $\{\phi_i\}$ must be *complete*. For example, when ϕ_i are algebraic polynomials, completeness requires that the set $\{\phi_i\}$ contains all terms of the lowest order admissible, up to the highest order desired.
4. The only requirement on ϕ_0 is that it satisfy the specified essential boundary conditions. When the specified essential boundary conditions are zero, then ϕ_0 is identically zero. Also, for completeness reasons, ϕ_0 must be the lowest-order function that satisfies the specified essential boundary conditions.

2.5.4 Examples

Here, we consider a few examples of the application of the Ritz method to equilibrium, eigenvalue, and time-dependent problems.

Example 2.5.1

Consider the differential equation (see Example 2.4.1; set $a = 1$, $c = -1$, $L = 1$, and $f = -x^2$)

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \quad \text{for} \quad 0 < x < 1 \tag{2.5.7}$$

We consider two sets of boundary conditions:

$$\textit{Set 1:}\ u(0) = 0, \quad u(1) = 0 \tag{2.5.8}$$

$$\textit{Set 2:}\ u(0) = 0, \quad \left.\frac{du}{dx}\right|_{x=1} = 1 \tag{2.5.9}$$

Boundary Conditions Set 1. The bilinear form and the linear functional associated with Eqs. (2.5.7) and (2.5.8) are [see (2.4.23c)]

$$B(w, u) = \int_0^1 \left(\frac{dw}{dx}\frac{du}{dx} - wu\right) dx, \quad l(w) = -\int_0^1 w x^2 \, dx \tag{2.5.10}$$

and K_{ij} and F_i are given by

$$K_{ij} = B(\phi_i, \phi_j) = \int_0^1 \left(\frac{d\phi_i}{dx} \frac{d\phi_j}{dx} - \phi_i \phi_j \right) dx, \quad l(\phi_i) = -\int_0^1 \phi_i \, x^2 \, dx \tag{2.5.11}$$

The same result as in (2.5.11) can be obtained by minimizing the quadratic functional [set $a = 1$, $c = -1$, $f = -x^2$, $L = 1$, and $u(1) = 0$ in Eq. (2.4.24)]

$$I(u) = \frac{1}{2} \int_0^1 \left[\left(\frac{du}{dx} \right)^2 - u^2 + 2x^2 u \right] dx$$

Substituting for $u \approx U_N$ from (2.5.3), with $\phi_0 = 0$, into the above functional, we obtain

$$I(c_1, c_2, \ldots, c_N) = \frac{1}{2} \int_0^1 \left[\left(\sum_{j=1}^N c_j \frac{d\phi_j}{dx} \right)^2 - \left(\sum_{j=1}^N c_j \phi_j \right)^2 + 2x^2 \left(\sum_{j=1}^N c_j \phi_j \right) \right] dx$$

The necessary condition for the minimum of I is that

$$\begin{aligned} \frac{\partial I}{\partial c_i} = 0 &= \int_0^1 \left[\frac{d\phi_i}{dx} \left(\sum_{j=1}^N c_j \frac{d\phi_j}{dx} \right) - \phi_i \left(\sum_{j=1}^N c_j \phi_j \right) + \phi_i x^2 \right] dx \\ &= \sum_{j=1}^N \left[\int_0^1 \left(\frac{d\phi_i}{dx} \frac{d\phi_j}{dx} - \phi_i \phi_j \right) dx \right] c_j + \int_0^1 \phi_i \, x^2 \, dx \\ &= \sum_{j=1}^N K_{ij} c_j - F_i \quad \text{for } i = 1, 2, \ldots, N \end{aligned} \tag{2.5.12}$$

Clearly, K_{ij} and F_i are the same as those defined in Eq. (2.5.11). Equations (2.5.12) hold for any choice of admissible approximation functions ϕ_i.

Next, we discuss the choice of ϕ_i and ϕ_0. Since both boundary conditions $u(0) = u(1) = 0$ are homogeneous and of the essential type, we take $\phi_0 = 0$ and select ϕ_i in the N-parameter Ritz approximation to satisfy the conditions $\phi_i(0) = \phi_i(1) = 0$. Clearly, the function $(0 - x)(1 - x)$ vanishes at $x = 0$ and $x = 1$, and its first derivative is nonzero. Hence, we take $\phi_1 = x(1 - x)$. The next function in the sequence is obviously $\phi_2 = x^2(1 - x)$(or $\phi_2 = x(1 - x)^2$). Thus, the following set of functions are admissible:

$$\phi_1 = x(1 - x), \; \phi_2 = x^2(1 - x), \ldots, \; \phi_i = x^i(1 - x), \ldots, \; \phi_N = x^N(1 - x) \tag{2.5.13}$$

The approximation

$$U_N = c_1 x(1 - x) + c_2 x^2(1 - x) + \ldots + c_N x^N(1 - x) \tag{2.5.14a}$$

is equivalent to

$$U_N = \hat{c}_1 x(1 - x) + \hat{c}_2 x(1 - x)^2 + \ldots + \hat{c}_N x(1 - x)^N \tag{2.5.14b}$$

It should be noted that if we select, for example, the functions $\phi_1 = x^2(1 - x)$, $\phi_2 = x^3(1 - x)$, and so on [not including $x(1 - x)$], the completeness requirement is violated, because the set cannot be used to generate the linear term x of the exact solution, if the solution has such a term. As a rule, we must start with the lowest-order admissible function and include all admissible, higher-order functions up to the desired degree.

For the choice of approximation functions in (2.5.13), the matrix coefficients $K_{ij} = B(\phi_i, \phi_j)$ and vector coefficients $F_i = l(\phi_i)$ can be computed as follows:

$$K_{ij} = \int_0^1 \{[ix^{i-1} - (i+1)x^i][jx^{j-1} - (j+1)x^j] - (x^i - x^{i+1})(x^j - x^{j-1})\}\, dx$$

$$= \frac{2ij}{(i+j)[(i+j)^2 - 1]} - \frac{2}{(i+j+1)(i+j+2)(i+j+3)} \tag{2.5.15a}$$

$$F_i = -\int_0^1 x^2(x^i - x^{i+1})dx = -\frac{1}{(i+3)(i+4)} \tag{2.5.15b}$$

for $i, j = 1, 2, \ldots, N$. We consider the one-, two- and three-parameter approximations to illustrate how the Ritz solution converges to the exact solution of the problem

$$u(x) = \frac{\sin x + 2\sin(1-x)}{\sin 1} + x^2 - 2 \tag{2.5.16}$$

For $N = 1$, we have

$$K_{11} = \frac{3}{10}, \quad F_1 = -\frac{1}{20} \quad \rightarrow \quad c_1 = -\frac{1}{6} = -0.1667$$

The one-parameter Ritz solution is given by

$$U_1 = c_1\phi_1 = -\frac{1}{6}(x - x^2)$$

For $N = 2$, we have

$$\frac{1}{420}\begin{bmatrix} 126 & 63 \\ 63 & 52 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = -\frac{1}{60}\begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$$

Solving the linear equations using Cramer's rule, we obtain

$$c_1 = -\frac{10}{123} = -0.0813, \quad c_2 = -\frac{21}{123} = -0.1707$$

The two-parameter Ritz solution is given by

$$U_2 = c_1\phi_1 + c_2\phi_2 = -\frac{10}{123}(x - x^2) - \frac{21}{123}(x^2 - x^3)$$
$$= -0.0813x - 0.0894x^2 + 0.1707x^3$$

For $N = 3$, we have

$$\frac{1}{2520}\begin{bmatrix} 756 & 378 & 228 \\ 378 & 312 & 237 \\ 228 & 237 & 206 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = -\frac{1}{420}\begin{Bmatrix} 21 \\ 14 \\ 10 \end{Bmatrix}$$

Note that the previously computed coefficients K_{ij} and F_i for $i, j = 1, 2$ remain unchanged and we only need to compute K_{3i}, $i = 1, 2, 3$, and F_3. The solution of the above equations is

$$c_1 = -0.0952, \quad c_2 = -0.1005, \quad c_3 = -0.0702$$

Table 2.5.1 Comparison of the Ritz and exact solutions of the equation

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \text{ for } 0 < x < 1; \quad u(0) = u(1) = 0$$

Ritz coefficients[†]		x	Ritz solution, $-10u$			Exact solution
			$N=1$	$N=2$	$N=3$	
$N=1$:	$c_1=-0.1667$	0.0	0.0	0.0	0.0	0.0
		0.1	0.1500	0.0885	0.0954	0.0955
		0.2	0.2667	0.1847	0.1890	0.1890
$N=2$:	$c_1=-0.0813$	0.3	0.3500	0.2783	0.2766	0.2764
	$c_2=-0.1707$	0.4	0.4000	0.3590	0.3520	0.3518
		0.5	0.4167	0.4167	0.4076	0.4076
$N=3$:	$c_1=-0.0952$	0.6	0.4000	0.4410	0.4340	0.4342
	$c_2=-0.1005$	0.7	0.3500	0.4217	0.4200	0.4203
	$c_3=-0.0702$	0.8	0.2667	0.3486	0.3529	0.3530
		0.9	0.1500	0.2115	0.2183	0.2182
		1.0	0.0	0.0	0.0	0.0

[†]The four-parameter Ritz solution coincides with the exact solution up to four decimal places.

The three-parameter Ritz solution is given by

$$\begin{aligned} U_3 &= c_1\phi_1 + c_2\phi_2 + c_3\phi_3 = -0.0952(x - x^2) - 0.1005(x^2 - x^3) - 0.0702(x^3 - x^4) \\ &= -0.0952x - 0.0053x^2 + 0.0303x^3 + 0.0702x^4 \end{aligned}$$

The values of the Ritz coefficients c_i, $i = 1, 2, \ldots, N$ for various values of N can be obtained by solving the matrix equation $[K]\{c\} = \{F\}$ with the coefficients of K_{ij} and F_i given by (2.5.15*a*) and (2.5.15*b*). The Ritz coefficients and a comparison of the Ritz solution with the exact solution (2.5.16) is presented in Table 2.5.1 and Fig. 2.5.1. If the exact solution (2.5.16) is expanded in a series in terms of powers of x, we note that it is an infinite series. However, the three-parameter Ritz solution is already a good approximation of the exact solution, as can be seen from Fig. 2.5.1 and Table 2.5.1.

Boundary Conditions Set 2. For the second set of boundary conditions (2.5.9), the bilinear form is the same as that given in (2.5.10). The linear form is given by

$$l(w) = -\int_0^1 w\, x^2\, dx + w(1) \tag{2.5.17}$$

Therefore, we have

$$F_i = -\int_0^1 x^2 \phi_i\, dx + \phi_i(1) \tag{2.5.18}$$

As for the approximation functions, ϕ_0 is still zero and the ϕ_i must be selected to satisfy the condition $\phi_i(0) = 0$. Clearly, $\phi_1(x) = x$, $\phi_2 = x^2$, and so on meet the requirement. Thus, we have

$$U_N = c_1 x + c_2 x^2 + \ldots + c_i x^i + \ldots + c_N x^N \tag{2.5.19}$$

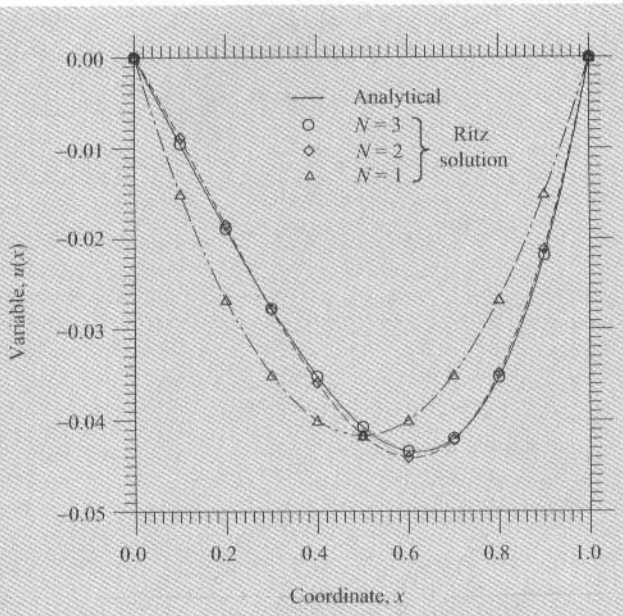


Figure 2.5.1 Comparison of the Ritz solution with the exact solution of Eqs. (2.5.7) and (2.5.8). The three-parameter Ritz solution and the exact solution do not differ on the scale of the plot.

The coefficients K_{ij} and F_i can be computed using

$$K_{ij} = \int_0^1 (ijx^{i+j-2} - x^{(i+j)})\,dx = \frac{ij}{i+j-1} - \frac{1}{i+j+1}$$

$$F_i = -\int_0^1 x^{(i+2}\,dx + 1 = -\frac{1}{i+3} + 1 \tag{2.5.20}$$

For example, for $N = 2$ we have

$$\frac{1}{60}\begin{bmatrix} 40 & 45 \\ 45 & 68 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{1}{20}\begin{Bmatrix} 15 \\ 16 \end{Bmatrix}$$

Solving the linear equations using Cramer's rule, we obtain

$$c_1 = \frac{180}{139} = 1.2950, \quad c_2 = -\frac{21}{139} = -0.1511$$

The two-parameter Ritz solution is given by

$$U_2 = c_1\phi_1 + c_2\phi_2 = 1.2950x - 0.1511x^2$$

The exact solution for this case is

$$u(x) = \frac{2\cos(1-x) - \sin x}{\cos 1} + x^2 - 2 \tag{2.5.21}$$

A comparison of the Ritz solutions with the exact solution is presented in Table 2.5.2.

Table 2.5.2 Comparison of the Ritz and exact solutions of the equation

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \text{ for } 0 < x < 1; \quad u(0) = 0, \quad \left(\frac{du}{dx}\right)\Big|_{x=1} = 1$$

Ritz coefficients†		x	Ritz solution, u			Exact solution
			$N=1$	$N=2$	$N=3$	
$N=1$	$c_1 = 1.1250$	0.0	0.0	0.0	0.0	0.0
		0.1	0.1125	0.1280	0.1271	0.1262
		0.2	0.2250	0.2530	0.2519	0.2513
$N=2$	$c_1 = 1.2950$	0.3	0.3375	0.3749	0.3740	0.3742
	$c_2 = -0.1511$	0.4	0.4500	0.4938	0.4934	0.4944
		0.5	0.5625	0.6097	0.6099	0.6112
$N=3$	$c_1 = 1.2831$	0.6	0.6750	0.7226	0.7234	0.7244
	$c_2 = -0.1142$	0.7	0.7875	0.8325	0.8337	0.8340
	$c_3 = -0.0246$	0.8	0.9000	0.9393	0.9407	0.9402
		0.9	1.0125	1.0431	1.0443	1.0433
		1.0	1.1250	1.1439	1.1442	1.1442

†The four-parameter Ritz solution coincides with the exact solution up to four decimal places.

Example 2.5.2

Consider the problem of finding the transverse deflection of a cantilever beam under a uniform transverse load of intensity q_0 per unit length and subjected to point load F_0 and bending moment M_0 at the free end (see Example 2.4.2). The governing equations according to the Euler-Bernoulli beam theory are

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - q_0 = 0 \quad \text{for} \quad \begin{cases} 0 < x < L \\ EI > 0 \end{cases} \tag{2.5.22}$$

$$w(0) = \left(\frac{dw}{dx}\right)\Big|_{x=0} = 0, \quad \left(EI\frac{d^2w}{dx^2}\right)\Big|_{x=L} = -M_0, \quad \left[\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]\Big|_{x=L} = F_0 \tag{2.5.23}$$

The weak form of (2.5.22) and (2.5.23) (which includes the specified NBC) was derived in Example 2.4.2, and is given by (2.4.30*a*).

We now construct an N-parameter Ritz solution using the variational form, (2.4.30*b*). Since the specified EBCs $w(0) = 0$ and $(dw/dx)|_{x=0}$ are homogeneous, we set $\phi_0 = 0$. Next, we select algebraic approximation functions ϕ_i that satisfy the continuity conditions and homogeneous form of the specified EBCs. The lowest-order algebraic function that meets these conditions is $\phi_1 = x^2$. The next function in the sequence is $\phi_2 = x^3$. Thus, we have

$$\phi_1 = x^2, \ \phi_2 = x^3, \ \ldots, \ \phi_i = x^{i+1}, \ \ldots, \ \phi_N = x^{N+1} \tag{2.5.24}$$

The N-parameter Ritz approximation is

$$W_N(x) = \sum_{j=1}^{N} c_j \phi_j, \quad \phi_j = x^{j+1} \tag{2.5.25}$$

Substituting (2.5.25) for w and $v = \phi_i$ into (2.4.30a) with $c_f = 0$, we obtain

$$[K]\{c\} = \{F\} \tag{2.5.26a}$$

$$K_{ij} = B(\phi_i, \phi_j) = \int_0^L EI(i+1)ix^{i-1}(j+1)jx^{j-1}\,dx$$

$$= EI(L)^{i+j-1}\frac{ij(i+1)(j+1)}{(i+j-1)} \tag{2.5.26b}$$

$$F_i = l(\phi_i) = \int_0^L q_0 x^{i+1}\,dx + (L)^{i+1}F_0 - (i+1)(L)^i M_0$$

$$= q_0(L)^{i+2}\frac{1}{i+2} + (L)^{i+1}F_0 - (i+1)(L)^i M_0 \tag{2.5.26c}$$

For $N = 1$, Eq. (2.5.26a) gives

$$c_1 = \left(\frac{q_0 L^4}{12EI} + \frac{F_0 L^3}{4EI} - \frac{M_0 L}{2EI}\right)$$

and the one-parameter Ritz solution is

$$W_1(x) = \left(\frac{q_0 L^4}{12EI} + \frac{F_0 L^3}{4EI} - \frac{M_0 L}{2EI}\right)\frac{x^2}{L^2}$$

For $N = 2$, we have

$$EI\begin{bmatrix} 4L & 6L^2 \\ 6L^2 & 12L^3 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{q_0 L^3}{12}\begin{Bmatrix} 4 \\ 3L \end{Bmatrix} + F_0 L^2\begin{Bmatrix} 1 \\ L \end{Bmatrix} - M_0 L\begin{Bmatrix} 2 \\ 3L \end{Bmatrix} \tag{2.5.27}$$

Solving for c_1 and c_2, we obtain

$$c_1 = \frac{1}{24EI}\left(5q_0 L^2 + 12F_0 L - 12M_0\right), \quad c_2 = -\frac{1}{12EI}(q_0 L + 2F_0)$$

and solution (2.5.25) becomes

$$W_2(x) = \frac{q_0 L^4}{24EI}\left(5\frac{x^2}{L^2} - 2\frac{x^3}{L^3}\right) + \frac{F_0 L^3}{6EI}\left(3\frac{x^2}{L^2} - \frac{x^3}{L^3}\right) - \frac{M_0 L^2}{2EI}\frac{x^2}{L^2} \tag{2.5.28}$$

For $N = 3$, we obtain the matrix equation

$$EI\begin{bmatrix} 4 & 6L & 8L^2 \\ 6L & 12L^2 & 18L^3 \\ 8L^2 & 18L^2 & \frac{144}{5}L^4 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \frac{q_0 L^3}{60}\begin{Bmatrix} 20 \\ 15L \\ 12L^2 \end{Bmatrix} + F_0 L^2\begin{Bmatrix} 1 \\ L \\ L^2 \end{Bmatrix} - M_0 L\begin{Bmatrix} 2 \\ 3L \\ 4L^2 \end{Bmatrix} \tag{2.5.29}$$

The solution of these equations when substituted into (2.5.25) for $N = 3$, gives

$$W_3(x) = \frac{q_0 L^4}{24EI}\frac{x^2}{L^2}\left(6 - 4\frac{x}{L} + \frac{x^2}{L^2}\right) + \frac{FL^3}{6EI}\left(3\frac{x^2}{L^2} - \frac{x^3}{L^3}\right) - \frac{M_0 L^2}{2EI}\frac{x^2}{L^2} \tag{2.5.30}$$

which coincides with the exact solution of Eqs. (2.5.22) and (2.5.23). Note that the one-parameter solution is exact when the beam is subjected to end moment M_0 only; the two-parameter solution is exact when the beam is subjected to both F_0 and M_0. If we try to compute the four-parameter solution without knowing that the three-parameter solution is exact for a beam subjected to distributed load q_0, point load F_0, and bending moment M_0, the parameters c_j $(j > 3)$ will be zero.

The next example deals with two-dimensional heat conduction in a square region. Note that the dependent variable, namely the temperature, is denoted by T, consistent with the standard notation used in heat transfer books.

Example 2.5.3

Consider the Poisson equation governing two-dimensional heat transfer in a square region

$$-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=g_0 \quad \text{in} \quad \Omega=\{(x, y): 0<(x, y)<1\} \tag{2.5.31}$$

with the following boundary conditions:

$$T=0 \quad \text{on sides } x=1 \quad \text{and} \quad y=1 \tag{2.5.32a}$$

$$\frac{\partial T}{\partial n}=0 \quad \text{on sides } x=0 \quad \text{and} \quad y=0 \tag{2.5.32b}$$

where g_0 is the rate of uniform heat generation in the region. Equation (2.5.31) is called Poisson's equation $(-k\nabla^2 T=g_0)$.

The variational problem is of the form (see Example 2.4.4; set $u=T$, $a_1=a_2=k$, $a_0=0$, and $f=g_0$)

$$B(w, T)=l(w) \tag{2.5.33a}$$

where the bilinear and linear functionals are

$$\begin{aligned} B(w, T) &= \int_0^1 \int_0^1 k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right) dx\, dy \\ l(w) &= \int_0^1 \int_0^1 w g_0\, dx\, dy \end{aligned} \tag{2.5.33b}$$

We consider an N-parameter approximation of the form

$$T_N=\sum_{i,j=1}^{N} c_{ij} \cos\alpha_i x \cos\alpha_j y, \quad \alpha_i=\frac{1}{2}(2i-1)\pi \tag{2.5.34}$$

Note that (2.5.34) involves a double summation. Since the boundary conditions are homogeneous, we have $\phi_0=0$. Incidentally, ϕ_i also satisfies the natural boundary conditions of the

problem but that is not necessary to be admissible. While the choice $\hat{\phi}_{ij} = \sin i\pi x \sin j\pi y$ meets the essential boundary conditions, $\hat{\phi}_{ij}$ is not complete because it cannot be used to generate the solution that *does not* vanish on the sides $x = 0$ and $y = 0$. Hence, $\hat{\phi}_{ij}$ are not admissible.

The coefficients K_{ij} and F_i can be computed by substituting (2.5.34) into (2.5.33*b*). Since the double Fourier series has two summations, we introduce the notation

$$\begin{aligned} K_{(ij)(kl)} &= k\int_0^1\int_0^1 [(\alpha_i \sin\alpha_i x \cos\alpha_j y)(\alpha_k \sin\alpha_k x \cos\alpha_l y) \\ &\quad +(\alpha_j \cos\alpha_i x \sin\alpha_j y)(\alpha_l \cos\alpha_k x \sin\alpha_l y)]\, dx\, dy \\ &= \begin{cases} 0 & \text{if } i \neq k \text{ or } j \neq l \\ \frac{1}{4}k(\alpha_i^2 + \alpha_j^2) & \text{if } i = k \text{ and } j = l \end{cases} \end{aligned} \tag{2.5.35a}$$

$$F_{ij} = g_0 \int_0^1\int_0^1 \cos\alpha_1 x \cos\alpha_j y\, dx\, dy = \frac{g_0}{\alpha_i\alpha_j} \sin\alpha_i \sin\alpha_j \tag{2.5.35b}$$

In evaluating the integrals, the following orthogonality conditions were used

$$\int_0^1 \sin\alpha_i x \sin\alpha_j x\, dx = \begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{2} & \text{if } i = j \end{cases}$$

$$\int_0^1 \cos\alpha_i x \cos\alpha_j x\, dx = \begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{2} & \text{if } i = j \end{cases}$$

Owing to the diagonal form of the coefficient matrix (2.5.35*a*), we can readily solve for the coefficients c_{ij}:

$$c_{ij} = \frac{F_{ij}}{K_{(ij)(ij)}} = \frac{4g_0}{k} \frac{\sin\alpha_i \sin\alpha_j}{(\alpha_i^2 + \alpha_j^2)\alpha_i\alpha_j} \tag{2.5.36}$$

The one- and two-parameter Ritz solutions are (the one-parameter solution has one term but the two-parameter solution has four terms)

$$T_1 = \frac{32g_0}{k\pi^4} \cos\frac{1}{2}\pi x \cos\frac{1}{2}\pi y \tag{2.5.37}$$

$$\begin{aligned} T_2 = \frac{g_0}{k}[&0.3285 \cos\frac{1}{2}\pi x \cos\frac{1}{2}\pi y - 0.0219(\cos\frac{1}{2}\pi x \cos\frac{3}{2}\pi y \\ &+ \cos\frac{3}{2}\pi x \cos\frac{1}{2}\pi y) + 0.0041 \cos\frac{3}{2}\pi x \cos\frac{3}{2}\pi y] \end{aligned} \tag{2.5.38}$$

If algebraic polynomials are to be used in the approximation of T, we can choose $\phi_1 = (1-x)(1-y)$ or $\phi_1 = (1-x^2)(1-y^2)$, both of which satisfy the (homogeneous) essential boundary conditions. However, the choice $\phi_1 = (1-x^2)(1-y^2)$ also meets the natural boundary conditions of the problem. The one-parameter Ritz solution for the choice $\phi_1 = (1-x^2)(1-y^2)$ is

$$T_1(x, y) = \frac{5g_0}{16k}(1-x^2)(1-y^2) \tag{2.5.39}$$

The exact solution of Eqs. (2.5.31), (2.5.32*a*), and (2.5.32*b*) is

$$T(x, y) = \frac{g_0}{2k}\left[(1 - y^2) + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos \alpha_n y \cosh \alpha_n x}{\alpha_n^3 \cosh \alpha_n}\right] \tag{2.5.40}$$

where $\alpha_n = \frac{1}{2}(2n - 1)\pi$. The Ritz solutions (2.5.37), (2.5.38), and (2.5.39) are compared with the exact solution (2.5.40) in Fig. 2.5.2. The analytical solution is evaluated using 50 terms of the series in Eq. (2.5.40).

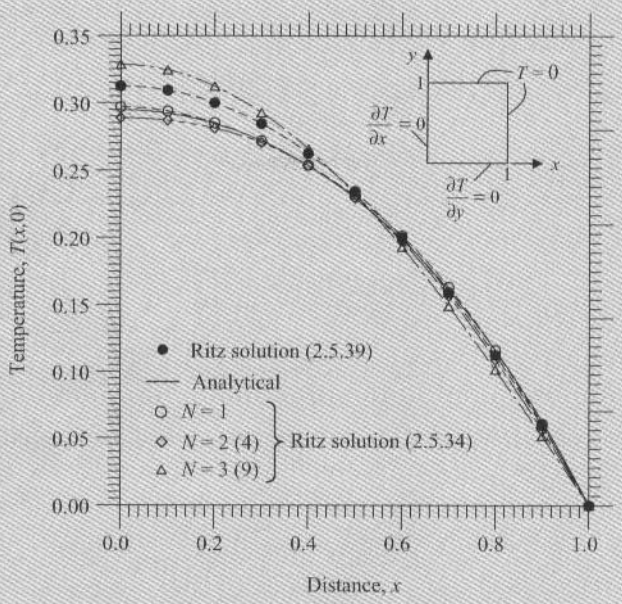


Figure 2.5.2 Comparison of the Ritz solutions with the analytical solution of the Poisson equations (2.5.31), (2.5.32*a*) and (2.5.32*b*) in two dimensions.

The next example deals with an eigenvalue problem, namely axial vibration of a bar [see Example 2.3.4 and Reddy (2002)].

Example 2.5.4

Consider a uniform cross-section bar of length L, with the left end fixed and the right end connected to a rigid support via a linear elastic spring (with spring constant k), as shown in Fig. 2.5.3. We wish to determine the first two natural axial frequencies of the bar using the Ritz method.

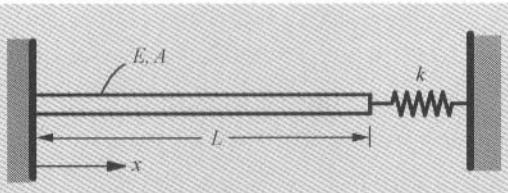


Figure 2.5.3 A uniform bar with an end spring.

The starting point is to construct the Lagrangian function $L = K - (U + V)$ (see Section 2.3.6). The kinetic energy K and the strain energy U associated with the axial motion of a bar are

$$K = \int_0^L \frac{\rho A}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx, \quad U = \int_0^L \frac{EA}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{k}{2}[u(L,t)]^2 \tag{2.5.41}$$

For potential energy due to applied loads, $V = 0$.

Substituting for K and U from Eq. (2.5.41) and $V = 0$ in Hamilton's principle (2.3.60), we obtain [$\delta u(x, t_1) = \delta u(x, t_2) = 0$ and $\delta u(0, t) = 0$]

$$\begin{aligned}
0 &= \int_{t_1}^{t_2} \delta(K - U)dt \\
&= \int_{t_1}^{t_2} \delta\left\{\frac{1}{2}\int_0^L \left[\rho A\left(\frac{\partial u}{\partial t}\right)^2 - EA\left(\frac{\partial u}{\partial x}\right)^2\right]dx - \frac{k}{2}[u(L,t)]^2\right\}dt \\
&= \int_{t_1}^{t_2}\left[\int_0^L \left(\rho A\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} - EA\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right)dx - ku(L,t)\delta u(L,t)\right]dt \\
&= \int_{t_1}^{t_2}\left[\int_0^L \left(-\rho A\frac{\partial^2 u}{\partial t^2}\delta u - EA\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right)dx - ku(L,t)\delta u(L,t)\right]dt
\end{aligned} \tag{2.5.42}$$

The Euler–Lagrange equations associated with (2.5.42) are given by

$$\begin{aligned}
&-\rho A\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = 0, \quad 0 < x < L \\
&\left[\left(EA\frac{\partial u}{\partial x}\right) + ku(x,t)\right]_{x=L} = 0
\end{aligned} \tag{2.5.43}$$

Natural vibration is nothing but the periodic motion of the form

$$u(x,t) = u_0(x)e^{i\omega t}, \quad i = \sqrt{-1} \tag{2.5.44}$$

where ω is the frequency of natural vibration and $u_0(x)$ is the amplitude. Substituting Eq. (2.5.44) into Eq. (2.5.42), we obtain

$$0 = \int_0^L \left(\rho A\omega^2 u_0 \delta u_0 - EA\frac{du_0}{dx}\frac{d\delta u_0}{dx}\right)dx - ku_0(L)\delta u_0(L) \tag{2.5.45}$$

where $(i\omega)^2 = -\omega^2$, and $\int_{t_1}^{t_2} e^{2i\omega t}\,dt$, being nonzero, is factored out. Equation (2.5.45) is the weak form of the equations governing natural vibration of a bar with left end fixed and the right end spring supported. The Euler equation and natural boundary condition associated with Eq. (2.5.45) are

$$-\frac{d}{dx}\left(EA\frac{du_0}{dx}\right) - \rho A\omega^2 u_0 = 0, \quad 0 < x < L \tag{2.5.46a}$$

$$EA\frac{du_0}{dx} + ku_0 = 0 \quad \text{at } x = L \tag{2.5.46b}$$

The essential boundary condition is $u_0(0) = 0$. Conversely, we can use the three step procedure to obtain the weak form (2.5.45) from Eqs. (2.5.46*a*) and (2.5.46*b*).

Next, we seek a $N-$parameter Ritz approximation (obviously, $\phi_0 = 0$)

$$u_0(x) \approx U_N(x) = \sum_{i=1}^{N} c_i\phi_i(x)$$

Substituting into Eq. (2.5.45), we obtain

$$0 = \sum_{i=1}^{N}\left\{\sum_{j=1}^{N}\left[\lambda\int_0^L \rho A\phi_i\phi_j\,dx - \left(\int_0^L EA\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx + k\phi_i(L)\phi_j(L)\right)\right]c_j\right\}\delta c_i$$

where $\lambda = \omega^2$. Because of the independent nature of δc_i, we obtain

$$0 = \sum_{j=1}^{N}\left[\lambda\int_0^L \rho A\phi_i\phi_j\,dx - \left(\int_0^L EA\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx + k\phi_i(L)\phi_j(L)\right)\right]c_j \tag{2.5.47a}$$

and in matrix form

$$([A] - \lambda[M])\{c\} = \{0\} \quad \text{or} \quad (\mathbf{A} - \lambda\mathbf{M})\mathbf{c} = 0 \tag{2.5.47b}$$

where

$$a_{ij} = \int_0^L EA\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx + k\phi_i(L)\phi_j(L), \quad m_{ij} = \int_0^L \rho A\phi_i\phi_j\,dx \tag{2.5.47c}$$

Equation (2.5.47*b*) represents a matrix eigenvalue problem, and we obtain N eigenvalues, λ_i, $i = 1, 2, \cdots, N$.

For the problem at hand, the approximation functions $\phi_i(x)$ are required to be differentiable once with respect to x and vanish at $x = 0$. Hence, we choose

$$\phi_i(x) = \left(\frac{x}{L}\right)^i \tag{2.5.48a}$$

Substituting ϕ_i from (2.5.48*a*) into (2.5.47*c*), we obtain

$$m_{ij} = \int_0^L \rho A\phi_i\phi_j\,dx = \rho AL\frac{1}{i+j+1}$$

$$a_{ij} = \int_0^L EA\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx + k\phi_i(L)\phi_j(L) = \frac{EA}{L}\frac{ij}{i+j-1} + k \tag{2.5.48b}$$

Since we wish to determine two eigenvalues, we take $N=2$ and obtain

$$m_{11}=\frac{\rho AL}{3},\quad m_{12}=\frac{\rho AL}{4},\quad m_{22}=\frac{\rho AL}{5},\quad a_{11}=\frac{EA}{L}+k,\quad a_{12}=\frac{EA}{L}+k,\quad a_{22}=\frac{4EA}{3L}+k$$

and the matrix eigenvalue problem (2.5.47*b*) becomes

$$\left(\frac{EA}{3L}\begin{bmatrix}3+3\alpha & 3+3\alpha\\ 3+3\alpha & 4+3\alpha\end{bmatrix}-\lambda\frac{\rho AL}{60}\begin{bmatrix}20 & 15\\ 15 & 12\end{bmatrix}\right)\begin{Bmatrix}c_1\\ c_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{2.5.49}$$

where $\alpha=kL/EA$. The algebraic eigenvalue problem (2.5.49) must be solved for $\lambda=\omega^2$ and c_i [hence, for the mode shape $u_0(x)$].

To carry out the remaining steps to obtain the numerical values for the natural frequencies, we take $\alpha=kL/EA=1$. Then, for a nontrivial solution (i.e., $c_1\neq 0,\quad c_2\neq 0$), we set the determinant of the coefficient matrix in Eq. (2.5.49) to zero:

$$\begin{vmatrix}2-\dfrac{\bar\lambda}{3} & 2-\dfrac{\bar\lambda}{4}\\ 2-\dfrac{\bar\lambda}{4} & \dfrac{7}{3}-\dfrac{\bar\lambda}{5}\end{vmatrix}=0,\quad \bar\lambda=\frac{\lambda\rho L^2}{E}=\frac{\omega^2\rho L^2}{E} \tag{2.5.50a}$$

or

$$15\bar\lambda^2-640\bar\lambda+2400=0$$

The quadratic equation has two roots

$$\bar\lambda_1=4.1545,\quad \bar\lambda_2=38.512\ \rightarrow\ \omega_1=\frac{2.038}{L}\sqrt{\frac{E}{\rho}},\quad \omega_2=\frac{6.206}{L}\sqrt{\frac{E}{\rho}} \tag{2.5.50b}$$

The eigenvectors (or mode shapes) are given by

$$U_2^{(i)}=c_1^{(i)}\frac{x}{L}+c_2^{(i)}\frac{x^2}{L^2}$$

where $c_1^{(i)}$ and $c_2^{(i)}$ are calculated from the equations [see Eq. (2.5.49)]

$$\begin{bmatrix}2-\dfrac{\bar\lambda_i}{3} & 2-\dfrac{\bar\lambda_i}{4}\\ 2-\dfrac{\bar\lambda_i}{4} & \dfrac{7}{3}-\dfrac{\bar\lambda_i}{5}\end{bmatrix}\begin{Bmatrix}c_1^{(i)}\\ c_2^{(i)}\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

The above pair of equations is linearly dependent. Hence, one of the two equations can be used to determine c_2 in terms of c_1 (or vice versa) for each value of λ. We obtain

$$\bar\lambda_1=4.1545:\quad c_1^{(1)}=1.0000,\quad c_2^{(1)}=-0.6399\ \rightarrow\ U_2^{(1)}(x)=\frac{x}{L}-0.6399\frac{x^2}{L^2}$$

$$\bar\lambda_2=38.512:\quad c_1^{(2)}=1.0000,\quad c_2^{(2)}=-1.4207\ \rightarrow\ U_2^{(2)}(x)=\frac{x}{L}-1.4207\frac{x^2}{L^2}$$

Plots of the two mode shapes are shown in Fig. 2.5.4.

The exact values of λ are the roots of the transcendental equation (the reader may verify this)

$$\lambda+\tan\lambda=0 \tag{2.5.51a}$$

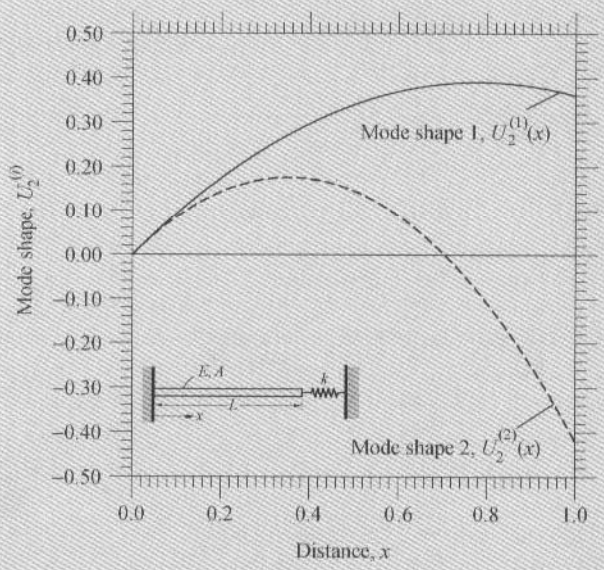


Figure 2.5.4 First two mode shapes obtained by the Ritz method for the natural longitudinal vibrations of a spring-supported bar.

whose first two roots are ($\omega^2 = \lambda$)

$$\omega_1 = \frac{2.02875}{L}\sqrt{\frac{E}{\rho}}, \quad \omega_2 = \frac{4.91318}{L}\sqrt{\frac{E}{\rho}} \tag{2.5.51b}$$

Note that the first approximate frequency is closer to the exact than the second.

If we select ϕ_0 and ϕ_i to satisfy the natural boundary condition also, the degree of polynomials will inevitably go up. For example, the lowest-order function that satisfies the homogeneous form (we still have $\phi_0 = 0$) of the natural boundary condition $u'(1) + u(1) = 0$ is

$$\hat{\phi}_1 = 3x - 2x^2$$

The one-parameter solution with the choice of $\hat{\phi}_1 = 3x - 2x^2$ gives $\lambda_1 = 50/12 = 4.1667$, which is no better than the two-parameter solution computed using $\phi_1 = x$ and $\phi_2 = x^2$. Of course, solution $c_1\hat{\phi}_1$ would yield a more accurate value for λ_1 than the solution $c_1\phi_1$. Although $c_1\hat{\phi}_1$ and $c_1\phi_1 + c_2\phi_2$ are of the same degree (polynomials), the latter gives better accuracy for λ_1 because of the number of parameters is greater, which provides greater freedom to adjust the parameters c_i in satisfying the weak form.

The eigenvalues and mode shapes determined through eigenvalue analysis are also useful in determining the transient response. The general homogeneous solution to the transient problem is [see Eq. (2.5.44)]

$$U_2(x, t) = (c_{11}\cos\omega_1 t + c_{12}\sin\omega_1 t)\,x + (c_{21}\cos\omega_2 t + c_{22}\sin\omega_2 t)\,x^2$$

where c_{1j} and c_{2j} are constants to be determined using the initial conditions.

2.5.5 The Method of Weighted Residuals

As noted in Section 2.4.2, one can always write the weighted-integral form of a differential equation, whether the equation is linear or nonlinear (in the dependent variables). The weak form can be developed if the equations are second-order or higher, even if they are nonlinear. However, it is not always possible to construct a functional $I(u)$ whose first variation is equal to the variational form, $\delta I = B(\delta u, u) - l(\delta u) = 0$. The Ritz method can be applied to all problems, including nonlinear problems, that have weak forms.

The weighted-residual method is a generalization of the Ritz method in that the weight functions can be chosen from an independent set of functions, and it requires only the weighted-integral form to determine the parameters. Since the latter form does not include any of the specified boundary conditions of the problem, the approximation functions must be selected such that the approximate solution satisfies both the natural and essential boundary conditions. In addition, the weight functions can be selected independently of the approximation functions, but are required to be linearly independent (so that the resulting algebraic equations are linearly independent).

We discuss the general method of weighted residuals first, and then consider certain special cases that are known by specific names (e.g., the Galerkin method, the collocation method, the least-squares method, and so on). Although a limited use of the weighted-residual method is made in this book (see Chapter 14), it is informative to have a knowledge of this class of methods for use in the formulation of certain nonlinear problems and non-self-adjoint problems (i.e., which do not admit a functional formulation).

The method of weighted residuals can be described in its generality by considering the operator equation

$$A(u) = f \quad \text{in} \quad \Omega \tag{2.5.52}$$

where A is an operator (linear or nonlinear), often a differential operator, acting on the dependent variable u, and f is a known function of the independent variables. Some examples of such operators are given below.

$$\begin{aligned}
&(1)\quad A(u) = -\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu \\
&(2)\quad A(u) = \frac{d^2}{dx^2}\left(b\frac{d^2u}{dx^2}\right) \\
&(3)\quad A(u) = -\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y\frac{\partial u}{\partial y}\right)\right] \\
&(4)\quad A(u) = -\frac{d}{dx}\left(u\frac{du}{dx}\right) \\
&(5)\quad A(u,v) = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned} \tag{2.5.53}$$

For an operator A to be *linear* in its arguments, it must satisfy the relation

$$A(\alpha u + \beta v) = \alpha A(u) + \beta A(v) \tag{2.5.54}$$

for any scalars α and β and dependent variables u and v. It can be easily verified that all operators in (2.5.53), except for 4 and 5, are linear. When an operator does not satisfy the condition (2.5.54), it is said to be *nonlinear*.

The function u is not only required to satisfy the operator equation (2.5.52), it is also required to satisfy the boundary conditions associated with the operator equation. From the examples considered so far, the boundary conditions associated with the operators defined in 1, 2, and 3 of (2.5.53) are obvious (see Examples 2.4.1–2.4.3).

In the weighted-residual method, the solution u is approximated, in much the same way as in the Ritz method, by the expression

$$U_N(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi_j(\mathbf{x}) + \phi_0(\mathbf{x}) \tag{2.5.55}$$

except that the requirements on ϕ_0 and ϕ_j for the weighted-residual method are more stringent than those for the Ritz method. Substitution of the approximate solution U_N into the left-hand side of (2.5.52) gives a function $A(U_N)$ that, in general, is not equal to the specified function f. The difference $A(U_N) - f$, called the *residual* of the approximation, is nonzero:

$$R \equiv A(U_N) - f = A\left(\sum_{j=1}^{N} c_j \phi_j + \phi_0\right) - f \neq 0 \tag{2.5.56a}$$

Note that the residual R is a function of position as well as of the parameters c_j. In the weighted-residual method, as the name suggests, the parameters c_j are determined by requiring the residual R to vanish in the weighted-integral sense:

$$\int_{\Omega} \psi_i(\mathbf{x}) R(\mathbf{x}, c_j)\, dx\, dy = 0 \quad (i = 1, 2, \cdots, N) \tag{2.5.56b}$$

where Ω is a two-dimensional domain and ψ_i are *weight functions,* which, in general, are not the same as the approximation functions ϕ_i. The set $\{\psi_i\}$ must be a linearly independent set; otherwise, the equations provided by (2.5.56*b*) will not be linearly independent and hence will not be solvable.

The requirements on ϕ_0 and ϕ_j for the weighted-residual method are different from those for the Ritz method, which is based on the weak (integral) form of the differential equation. The differentiability requirement on ϕ_j in the weighted-residual method is dictated by the integral statement (2.5.56*b*), as opposed to the weak form in the Ritz method. Thus, ϕ_j must have nonzero derivatives up to the order appearing in the operator equation (2.5.52). Since the weighted-integral form (2.5.56*b*) does not include any of the specified (either essential or natural) boundary conditions, we must also require U_N in (2.5.55) to satisfy all specified boundary conditions of the problem. Consequently, ϕ_0 is required to satisfy the homogeneous form of all specified boundary conditions of the problem. These requirements on ϕ_0 and ϕ_j will increase the order of the polynomial expressions used for the weighted-residual method. In general, the ϕ_j used in this method are higher-order functions than those used in the Ritz method, and the functions used in the latter may not satisfy the continuity (i.e., differentiability) requirements of the weighted-residual method. Various special cases of the weighted-residual method are discussed in the following paragraphs.

The Petrov–Galerkin Method. The weighted-residual method is referred to as the *Petrov–Galerkin method* when $\psi_i \neq \phi_i$. When the operator A is linear, (2.5.56*b*) can be simplified to the form

$$\sum_{j=1}^{N} \left[\int_\Omega \psi_i A(\phi_j)\, d\mathbf{x} \right] c_j = \int_\Omega \psi_i [f - A(\phi_0)]\, d\mathbf{x}$$

or

$$\sum_{j=1}^{N} A_{ij} c_j = F_i \quad (\mathbf{Ac} = \mathbf{F}) \tag{2.5.57}$$

Note that the coefficient matrix $[A]$ is not symmetric:

$$A_{ij} = \int_\Omega \psi_i A(\phi_j)\, d\mathbf{x} \neq A_{ji} \tag{2.5.58}$$

The Galerkin Method. If the weight function ψ_i is chosen to be equal to the approximation function ϕ_i, the weighted-residual method is better known as the Galerkin method. The algebraic equations of the Galerkin approximation are

$$\mathbf{Ac} = \mathbf{F} \tag{2.5.59a}$$

where

$$A_{ij} = \int_\Omega \phi_i A(\phi_j)\, d\mathbf{x} \quad F_i = \int_\Omega \phi_i [f - A(\phi_0)]\, d\mathbf{x} \tag{2.5.59b}$$

Once again, we note that A_{ij} is not symmetric.

In general, the Galerkin method is *not* the same as the Ritz method. This should be clear from the fact that the former uses the weighted-integral form whereas the latter uses the weak (or variational) form to determine the coefficients c_j. Consequently, the approximation functions used in the Galerkin method are required to be of higher order than those in the Ritz method.

If the equation permits, and one wishes, differentiation from the dependent variable(s) can be transferred to the weight function $w = \phi_i$, thereby obtaining the weak form. Then there is no difference between the Galerkin method and the Ritz method. Thus, Ritz and Galerkin methods yield the same solutions in two cases: (*a*) when the specified boundary conditions of the problem are all of the essential type, and therefore the requirements on ϕ_i in the two methods become the same and the weighted-integral form reduces to the weak form; and (*b*) when the approximation functions of the Galerkin method are used in the Ritz method.

The Least-Squares Method. In least-squares method, we determine the parameters c_j by minimizing the integral of the square of the residual (2.5.56*a*):

$$\frac{\partial}{\partial c_i} \int_\Omega R^2(\mathbf{x}, c_j)\, d\mathbf{x} = 0$$

or

$$\int_\Omega \frac{\partial R}{\partial c_i} R\, d\mathbf{x} = 0 \tag{2.5.60a}$$

Comparison of (2.5.60a) with (2.5.56b) shows that $\psi_i = \partial R/\partial c_i$. If A is a linear operator, $\psi_i = A(\phi_i)$, and (2.5.60a) becomes

$$\sum_{j=1}^{N}\left[\int_\Omega A(\phi_i)A(\phi_j)\,d\mathbf{x}\right]c_j = \int_\Omega A(\phi_i)[f - A(\phi_0)]\,d\mathbf{x}$$

or

$$\mathbf{Ac} = \mathbf{F} \tag{2.5.60b}$$

where

$$A_{ij} = \int_\Omega A(\phi_i)A(\phi_j)\,d\mathbf{x}, \quad F_i = A(\phi_i)[f - A(\phi_0)]\,d\mathbf{x} \tag{2.5.60c}$$

Note that the coefficient matrix A_{ij} is symmetric, but it involves the same order of differentiation as in the governing differential equation $A(u) - f = 0$.

The Collocation Method. In the collocation method, we seek an approximate solution U_N to (2.5.52) in the form of (2.5.55) by requiring the residual to vanish identically at N selected points $\mathbf{x}^i \equiv (x^i, y^i, z^i)$ $(i = 1, 2, \ldots, N)$ in the domain Ω

$$R(\mathbf{x}^i, c_j) = 0 \quad (i = 1, 2, \ldots, N) \tag{2.5.61}$$

The selection of the points $\mathbf{x}^i$ is crucial in obtaining a well-conditioned system of equations and ultimately in obtaining an accurate solution. The collocation method can be shown to be a special case of (2.5.56b) with $\psi_i = \delta(\mathbf{x} - \mathbf{x}^i)$, where $\delta(\mathbf{x})$ is the Dirac delta function, which is defined by

$$\int_\Omega f(\mathbf{x})\delta(\mathbf{x} - \xi)\,d\mathbf{x} = f(\xi) \tag{2.5.62}$$

With this choice of weight functions, the weighted-residual statement (2.5.56b) becomes

$$\int_\Omega \delta(\mathbf{x} - \mathbf{x}^i)R(\mathbf{x}, c_j)\,d\mathbf{x} = 0$$

or

$$R(\mathbf{x}^i, c_j) = 0 \tag{2.5.63}$$

We consider an example to illustrate the use of various types of weighted-residual methods.

Example 2.5.5

Consider the differential equation (see Example 2.5.1 with Set 2 boundary conditions):

$$-\frac{d^2u}{dx^2} - u + x^2 = 0, \quad u(0) = 0, \quad u'(1) = 1 \tag{2.5.64}$$

For a weighted-residual method, ϕ_0 and ϕ_i should satisfy the following conditions:

$\phi_0(0) = 0$, $\phi_0'(1) = 1$ (satisfy actual boundary conditions)
$\phi_i(0) = 0$, $\phi_i'(1) = 0$ (satisfy homogeneous form of the specified boundary conditions)

For a choice of algebraic polynomials, we assume $\phi_0(x) = a + bx$ and use the two conditions on ϕ_0 to determine the constants a and b. We obtain

$$\phi_0(x) = x$$

Since there are two homogeneous conditions, we must assume at least a three-parameter polynomial to obtain a nonzero function, $\phi_1 = a + bx + cx^2$. Using the conditions on ϕ_i, we obtain

$$\phi_1 = cx(2 - x)$$

The constant c can be set equal to unity because it will be absorbed into the parameter c_1. For ϕ_2, we can assume one of the forms

$$\phi_2 = a + bx + dx^3 \quad \text{or} \quad \phi_2 = a + cx^2 + dx^3$$

with $d \neq 0$; ϕ_2 does not contain all-order terms in either case, but the approximate solution is complete because $\{\phi_1, \phi_2\}$ contains all terms up to degree three. For the second choice of ϕ_2, we obtain

$$\phi_2 = x^2\left(1 - \frac{2}{3}x\right)$$

The residual in the approximation of the equation is

$$\begin{aligned} R &= -\left(0 + \sum_{i=1}^{N} c_i \frac{d^2\phi_i}{dx^2}\right) - \left(\phi_0 + \sum_{i=1}^{N} c_i\phi_i\right) + x^2 \\ &= c_1(2 - 2x + x^2) + c_2\left(-2 + 4x - x^2 + \frac{2}{3}x^3\right) - x + x^2 \end{aligned} \tag{2.5.65}$$

We next consider various methods.

The Petrov–Galerkin Method. Let the weight functions be

$$\psi_1 = x, \quad \psi_2 = x^2 \tag{2.5.66}$$

Then

$$\int_0^1 xR\,dx = 0, \quad \int_0^1 x^2R\,dx = 0$$

or

$$\frac{7}{12}c_1 + \frac{13}{60}c_2 - \frac{1}{12} = 0, \quad \frac{11}{30}c_1 + \frac{11}{45}c_2 - \frac{1}{20} = 0 \tag{2.5.67}$$

Solving for c_i, we obtain $c_1 = \frac{103}{682}$ and $c_2 = -\frac{15}{682}$; the solution becomes

$$U_{PG} = 1.302053x - 0.173021x^2 - 0.014663x^3 \tag{2.5.68}$$

The Galerkin Method. Taking $\psi_i = \phi_i$, we have

$$\int_0^1 x(2 - x)R\,dx = 0, \quad \int_0^1 x^2\left(1 - \frac{2}{3}x\right)R\,dx = 0$$

or

$$\frac{4}{5}c_1 + \frac{28}{45}c_2 - \frac{7}{60} = 0, \quad \frac{17}{90}c_1 + \frac{29}{315}c_2 - \frac{1}{36} = 0 \tag{2.5.69}$$

Hence, the solution becomes (with $c_1 = \frac{623}{4306}$, $c_2 = \frac{21}{4306}$),

$$U_G = 1.2894x - 0.1398x^2 - 0.00325x^3 \tag{2.5.70}$$

The Least-Squares Method. Taking $\psi_i = \frac{\partial R}{\partial c_i}$, we have

$$\int_0^1 (2 - 2x + x^2) R\, dx = 0, \quad \int_0^1 \left(-2 + 4x - x^2 + \frac{2}{3}x^3 \right) R\, dx = 0$$

or

$$\frac{28}{15}c_1 - \frac{47}{90}c_2 - \frac{13}{60} = 0, \quad -\frac{47}{90}c_1 + \frac{253}{315}c_2 - \frac{1}{36} = 0 \tag{2.5.71}$$

The least-squares solution is given by (with $c_1 = \frac{1292}{9935}$, $c_2 = \frac{991}{19870}$),

$$U_{LS} = 1.2601x - 0.08017x^2 - 0.03325x^3 \tag{2.5.72}$$

The Collocation Method. Choosing the points $x = \frac{1}{3}$ and $x = \frac{2}{3}$ as the collocation points, we evaluate the residuals at these points and set them equal to zero:

$$\begin{aligned} R\left(\frac{1}{3}\right) = 0: &\quad 117c_1 - 61c_2 = 18 \\ R\left(\frac{2}{3}\right) = 0: &\quad 90c_1 - 34c_2 = 18 \end{aligned} \tag{2.5.73}$$

The solution is given by ($c_1 = 1710/9468$, $c_2 = 486/9468$)

$$U_C = 1.3612x - 0.12927x^2 - 0.03422x^3 \tag{2.5.74}$$

The four approximate solutions are compared in Table 2.5.3 with the exact solution (2.5.21). For this problem, the Petrov–Galerkin method gives the most accurate solution.

Table 2.5.3 Comparison of the Ritz, weighted-residual, and exact solutions[1] of the boundary value problem in Example 2.5.5.

x	u_{exact}	U_R	U_{PG}	U_G	U_{LS}	U_C
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.1262	0.1280	0.1285	0.1275	0.1252	0.1348
0.2	0.2513	0.2529	0.2536	0.2523	0.2485	0.2668
0.3	0.3742	0.3749	0.3754	0.3741	0.3699	0.3958
0.4	0.4943	0.4938	0.4941	0.4932	0.4891	0.5216
0.5	0.6112	0.6097	0.6096	0.6093	0.6058	0.6440
0.6	0.7244	0.7226	0.7221	0.7226	0.7200	0.7628
0.7	0.8340	0.8324	0.8317	0.8329	0.8314	0.8778
0.8	0.9402	0.9393	0.9384	0.9404	0.9397	0.9887
0.9	1.0433	1.0431	1.0424	1.0448	1.0449	1.0954
1.0	1.1442	1.1439	1.1437	1.1463	1.1467	1.1977

[1] Subscripts are as follows: R, Ritz; PG, Petrov–Galerkin; G, Galerkin; LS, least-squares; C, collocation.

2.6 SUMMARY

In this chapter, we have studied two major topics that are of immediate interest in the study of finite element method in the forthcoming chapters:

1. Weighted-integral and weak formulations of differential equations
2. Solution of boundary value problems by the Ritz and weighted-residual (e.g., the Galerkin, least-squares, and collocation) methods

The weighted-integral statements are required in order to generate the necessary and sufficient number of algebraic equations to solve for the parameters c_i in the approximate solution. Thus, the algebraic equations are equivalent to minimizing, in a weighted-integral sense, the error introduced in the approximation of the differential equation.

In studying the two topics, a three-step procedure for developing the weak form of differential equations is presented, and methods for obtaining algebraic equations in terms of the unknown parameters in the approximate solution are developed. These topics are immediately applicable in the finite element method, which is an elementwise application of a variational method. Thus, the material covered in this chapter forms the core of the finite element method. A few remarks are in order on the classical variational methods studied here.

- The traditional variational methods (e.g., Ritz, Galerkin, and least-squares) presented in Section 2.5 provide a simple means of finding spatially continuous approximate solutions to physical problems. The approximate solutions obtained via these methods are continuous functions of position in the domain.
- The main limitation of classical variational methods that prevents them from being competitive with traditional finite difference methods is the difficulty encountered in constructing the approximation functions. The construction process becomes more difficult when the domain is geometrically complex. The so-called "meshless methods" are a return to the classical variational methods but with a procedure to construct the approximation functions.
- From the preceding discussion, it is apparent that the variational methods can provide a powerful means of finding approximate solutions, provided one can find a way to systematically construct approximation functions, for almost any geometry, that depend only on the differential equation being solved and not on the boundary conditions of the problem. This property enables one to develop a computer program for a particular class of problem (each problem in the class differs from the others only in the data), i.e., a *general-purpose* computer program. Since the functions must be constructed for a geometrically complex domain, it seems that (recall the discussion of the method of composites for the determination of the center of mass of an irregular shape from Chapter 1) the region must be represented (or approximated if required) as an assemblage of simple geometric shapes for which the construction of approximation functions becomes simpler. The finite element method to be discussed in the forthcoming chapters is based on these ideas.
- In the finite element method, a given domain is represented (discretized) by a collection of geometrically simple shapes (elements), and on each element of the collection, the governing equation is *formulated* using any one of the variational methods. The approximation functions are systematically generated for each (typical) element using the

essential boundary conditions. The elements are *connected* together by imposing the continuity of the dependent variables across the interelement boundaries.

The remaining chapters of this book are devoted to the introduction of the finite element method and its use in the analysis of several model differential equations representing mathematical models for many physical processes.

PROBLEMS

In Problems 2.1–2.5, construct the weak forms and, whenever possible, quadratic functionals.

2.1 A nonlinear equation:

$$-\frac{d}{dx}\left(u\frac{du}{dx}\right)+f=0 \quad \text{for} \quad 0<x<L$$
$$\left(u\frac{du}{dx}\right)\Big|_{x=0}=0 \quad u(1)=\sqrt{2}$$

2.2 The Euler-Bernoulli-von Kármán nonlinear theory of beams:

$$-\frac{d}{dx}\left\{EA\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\}=f \quad \text{for} \quad 0<x<L$$
$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)-\frac{d}{dx}\left\{a\frac{dw}{dx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\}=q$$
$$u=w=0 \quad \text{at} \quad x=0,\ L; \quad \left(\frac{dw}{dx}\right)\Big|_{x=0}=0; \quad \left(EI\frac{d^2w}{dx^2}\right)\Big|_{x=L}=M_0$$

where EA, EI, f, and q are functions of x, and M_0 is a constant. Here u denotes the axial displacement and w the transverse deflection of the beam.

2.3 A second-order equation:

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)+f=0 \quad \text{in} \quad \Omega$$
$$u=u_0 \ \text{ on } \ \Gamma_1, \quad \left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)n_x+\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)n_y=t_0 \ \text{ on } \ \Gamma_2$$

where $a_{ij}=a_{ji}$ $(i, j=1, 2)$ and f are given functions of position (x, y) in a two-dimensional domain Ω, and u_0 and t_0 are known functions on portions Γ_1 and Γ_2 of the boundary Γ: $\Gamma_1+\Gamma_2=\Gamma$.

2.4 Navier-Stokes equations for two-dimensional flow of viscous, incompressible fluids:

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}&=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}&=0 \end{aligned}\right\} \quad \text{in} \quad \Omega \qquad (1)$$

$$u = u_0, \quad v = v_0 \quad \text{on} \quad \Gamma_1 \tag{2}$$

$$\left.\begin{aligned} \nu\left(\frac{\partial u}{\partial x}n_x + \frac{\partial u}{\partial y}n_y\right) - \frac{1}{\rho}Pn_x = \hat{t}_x \\ \nu\left(\frac{\partial v}{\partial x}n_x + \frac{\partial v}{\partial y}n_y\right) - \frac{1}{\rho}Pn_y = \hat{t}_y \end{aligned}\right\} \quad \text{on} \quad \Gamma_2 \tag{3}$$

2.5 Two-dimensional flow of viscous, incompressible fluids (stream function–vorticity formulation):

$$\left.\begin{aligned} -\nabla^2\psi - \zeta = 0 \\ -\nabla^2\zeta + \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} = 0 \end{aligned}\right\} \quad \text{in} \quad \Omega$$

Assume that all essential boundary conditions are specified to be zero.

2.6 Compute the coefficient matrix and the right-hand side of the N-parameter Ritz approximation of the equation

$$-\frac{d}{dx}\left[(1+x)\frac{du}{dx}\right] = 0 \quad \text{for} \quad 0 < x < 1$$
$$u(0) = 0, \quad u(1) = 1$$

Use algebraic polynomials for the approximation functions. Specialize your result for $N = 2$ and compute the Ritz coefficients. *Answer:* $c_1 = \frac{55}{131}$ and $c_2 = -\frac{20}{131}$.

2.7 Use trigonometric functions for the two-parameter approximation of the equation in Problem 2.6 and obtain the Ritz coefficients.

2.8 A steel rod of diameter $d = 2$ cm, length $L = 25$ cm, and thermal conductivity $k = 50$ W/(m · °C) is exposed to ambient air $T_\infty = 20$°C with a heat-transfer coefficient $\beta = 64$ W/(m^2 · °C). Given that the left end of the rod is maintained at a temperature of $T_0 = 120$°C and the other end is exposed to the ambient temperature, determine the temperature distribution in the rod using a two-parameter Ritz approximation with polynomial approximation functions. The equation governing the problem is given by

$$-\frac{d^2\theta}{dx^2} + c\theta = 0 \quad \text{for} \quad 0 < x < 25 \text{ cm}$$

where $\theta = T - T_\infty$, T is the temperature, and c is given by

$$c = \frac{\beta P}{Ak} = \frac{\beta\pi D}{\frac{1}{4}\pi D^2 k} = \frac{4\beta}{kD} = 256/\text{m}^2$$

P being the perimeter and A the cross-sectional area of the rod. The boundary conditions are

$$\theta(0) = T(0) - T_\infty = 100°\text{C}, \quad \left(k\frac{d\theta}{dx} + \beta\theta\right)\Big|_{x=L} = 0$$

Answer: For $L = 0.25$ m, $\phi_0 = 100$, $\phi_i = x^i$, the Ritz coefficients are $c_1 = -1,033.385$ and $c_2 = 2,667.261$.

2.9 Set up the equations for the N-parameter Ritz approximation of the following equations associated with a simply supported beam and subjected to a uniform transverse load $q = q_0$:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q_0 \quad \text{for} \quad 0 < x < L$$

$$w = EI\frac{d^2w}{dx^2} = 0 \quad \text{at} \quad x = 0,\ L$$

(a) Use algebraic polynomials.

(b) Use trigonometric functions.

Compare the two-parameter Ritz solutions with the exact solution. *Answer:* (a) $c_1 = q_0L^2/(24EI)$ and $c_2 = 0$.

2.10 Repeat Problem 2.9 for $q = q_0 \sin(\pi x/L)$. *Answer:* $N = 2$: $c_1 = c_2L = 2q_0L^2/(3EI\pi^3)$.

2.11 Repeat Problem 2.9 for $q = Q_0\delta(x - \frac{1}{2}L)$, where $\delta(x)$ is the Dirac delta function (i.e., a point load Q_0 is applied at the center of the beam).

2.12 Develop the N-parameter Ritz solution for a simply supported beam under uniform transverse load using Timoshenko beam theory. The governing equations are given in Eqs. (2.4.32a) and (2.4.32b). Use Trigonometric functions to approximate w and Ψ.

2.13 Solve the Poisson equation governing heat conduction in a square region:

$$-k\nabla^2 T = g_0$$

$$T = 0 \quad \text{on sides} \quad x = 1 \quad \text{and} \quad y = 1$$

$$\frac{\partial T}{\partial n} = 0 \quad \text{(insulated) on sides} \quad x = 0 \quad \text{and} \quad y = 0$$

using a one-parameter Ritz approximation of the form

$$T_1(x, y) = c_1(1 - x^2)(1 - y^2)$$

Answer: $c_1 = \frac{5g_0}{16k}$.

2.14 Determine ϕ_i for a two-parameter Galerkin approximation with algebraic approximation functions for Problem 2.8.

2.15 Consider the (Neumann) boundary value problem

$$-\frac{d^2u}{dx^2} = f \quad \text{for} \quad 0 < x < L$$

$$\left(\frac{du}{dx}\right)\Big|_{x=0} = \left(\frac{du}{dx}\right)\Big|_{x=L} = 0$$

Find a two-parameter Galerkin approximation of the problem using trigonometric approximation functions, when (a) $f = f_0 \cos(\pi x/L)$ and (b) $f = f_0$. *Answer:* (a) $\phi_i = \cos(i\pi x/L)$, $c_1 = f_0L^2/\pi^2$, $c_i = 0$ for $i \neq 1$.

2.16 Find a one-parameter approximate solution of the nonlinear equation

$$-2u\frac{d^2u}{dx^2} + \left(\frac{du}{dx}\right)^2 = 4 \quad \text{for} \quad 0 < x < 1$$

subject to the boundary conditions $u(0) = 1$ and $u(1) = 0$, and compare it with the exact solution $u_0 = 1 - x^2$. Use (a) the Galerkin method, (b) the least-squares method, and (c) the Petrov–Galerkin method with weight function $w = 1$. *Answer:* (a) $(c_1)_1 = 1$, and $(c_1)_2 = -3$.

2.17 Give a one-parameter Galerkin solution of the equation

$$-\nabla^2 u = 1 \quad \text{in} \quad \Omega \quad (= \text{unit square})$$
$$u = 0 \quad \text{on} \quad \Gamma$$

Use (*a*) algebraic and (*b*) trigonometric approximation functions.
Answer: (*b*) $c_{ij} = \frac{16}{\pi^4}\frac{1}{ij(i^2+j^2)}$ $(i, j \quad \text{odd})$, $\phi_{ij} = \sin i\pi x \sin j\pi y$

2.18 Repeat Problem 2.17(*a*) for an equilateral triangular domain. *Hint:* Use the product of equations of the lines representing the sides of the triangle for the approximation function.
Answer: $c_1 = -\frac{1}{2}$.

2.19 Consider the differential equation

$$-\frac{d^2u}{dx^2} = \cos \pi x \quad \text{for} \quad 0 < x < 1$$

subject to the following three sets of boundary conditions:

(1) $u(0) = 0, \quad u(1) = 0$

(2) $u(0) = 0, \quad \left(\frac{du}{dx}\right)\Big|_{x=1} = 0$

(3) $\left(\frac{du}{dx}\right)\Big|_{x=0} = 0, \quad \left(\frac{du}{dx}\right)\Big|_{x=1} = 0$

Determine a three-parameter solution, with trigonometric functions, using (*a*) the Ritz method, (*b*) the least-squares method, and (*c*) collocation at $x = \frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}$, and compare with the exact solutions:

(1) $u_0 = \pi^{-2}(\cos \pi x + 2x - 1)$

(2) $u_0 = \pi^{-2}(\cos \pi x - 1)$

(3) $u_0 = \pi^{-2} \cos \pi x$

Answer: (1*a*) $c_i = \frac{4}{\pi^5 i(i^2-1)}$.

2.20 Consider a cantilever beam of variable flexural rigidity, $EI = a_0[2 - (x/L)^2]$ and carrying a distributed load, $q = q_0[1 - (x/L)]$. Find a three-parameter solution using $\phi_i = X^{(j+1)}$ and the collocation method. *Answer:* $c_1 = -\frac{q_0 L^2}{4a_0}$, $c_2 = \frac{q_0 L}{12a_0}$, and $c_3 = 0$.

2.21 Consider the problem of finding the fundamental frequency of a circular membrane of radius a, fixed at its edge. The governing equation for axisymmetric vibration is

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) - \lambda u = 0, \quad 0 < r < a$$

where λ is the frequency parameter and u is the deflection of the membrane. (*a*) Determine the trigonometric approximation functions for the Galerkin method, (*b*) use one-parameter Galerkin approximation to determine λ, and (*c*) use two-parameter Galerkin approximation to determine λ. *Answer:* $\lambda = 5.832/a^2$.

2.22 Find the first two eigenvalues associated with the differential equation

$$-\frac{d^2u}{dx^2} = \lambda u, \; 0 < x < 1$$
$$u(0) = 0, \; u(1) + u'(1) = 0$$

Use the least-squares method with algebraic polynomials. Use the operator definition to be $A = -(d^2/dx^2)$ to avoid increasing the degree of the characteristic polynomial for λ. *Answer:* $\lambda_1 = 4.212$ and $\lambda_2 = 34.188$.

2.23 Repeat Problem 2.22 using the Ritz method with algebraic polynomials. *Answer:* $\lambda_1 = 4.1545$ and $\lambda_2 = 38.512$.

2.24 Consider the Poisson equation

$$-\nabla^2 u = 0, \quad 0 < x < 1, \quad 0 < y < \infty$$
$$u(0, y) = u(1, y) = 0 \quad \text{for} \quad y > 0$$
$$u(x, 0) = x(1 - x), \quad u(x, \infty) = 0, \quad 0 \le x \le 1$$

Assuming an approximation of the form

$$u(x, y) = c_1(y)x(1 - x)$$

find the differential equation for $c_1(y)$ and solve it exactly. *Answer:* $U_1(x, y) = (x - x^2)e^{-\sqrt{10}y}$.

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