where

a_W	a_E	a_P
$\dfrac{\Gamma_w}{\delta x_{WP}} A_w$	$\dfrac{\Gamma_e}{\delta x_{PE}} A_e$	$a_W + a_E - S_P$

The values of S_u and S_p can be obtained from the source model (4.8): $\bar{S}\Delta V = S_u + S_p\phi_P$. Equations (4.11) and (4.8) represent the discretised form of equation (4.1). This type of discretised equation is central to all further developments.

Step 3: Solution of equations

Discretised equations of the form (4.11) must be set up at each of the nodal points in order to solve a problem. For control volumes that are adjacent to the domain boundaries the general discretised equation (4.11) is modified to incorporate boundary conditions. The resulting system of linear algebraic equations is then solved to obtain the distribution of the property ϕ at nodal points. Any suitable matrix solution technique may be enlisted for this task. In Chapter 7 we describe matrix solution methods that are specially designed for CFD procedures. The techniques of dealing with different types of boundary conditions will be examined in detail in Chapter 9.

4.3 Worked examples: one-dimensional steady state diffusion

The application of the finite volume method to the solution of simple diffusion problems involving conductive heat transfer is presented in this section. The equation governing one-dimensional steady state conductive heat transfer is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + S = 0 \tag{4.12}$$

where thermal conductivity k takes the place of Γ in equation (4.3) and the dependent variable is temperature T. The source term can, for example, be heat generation due to an electrical current passing through the rod. Incorporation of boundary conditions as well as the treatment of source terms will be introduced by means of three worked examples.

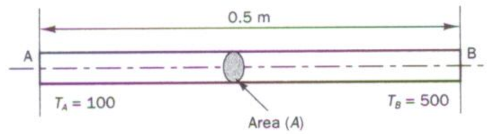
Example 4.1

Consider the problem of source-free heat conduction in an insulated rod whose ends are maintained at constant temperatures of 100°C and 500°C respectively. The one-dimensional problem sketched in Figure 4.3 is governed by

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \tag{4.13}$$

Calculate the steady state temperature distribution in the rod. Thermal conductivity k equals 1000 W/m.K, cross-sectional area A is 10×10^{-3} m^2.

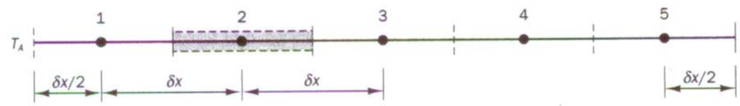
Figure 4.3



Solution

Let us divide the length of the rod into five equal control volumes as shown in Figure 4.4. This gives $\delta x = 0.1$ m.

Figure 4.4 The grid used



The grid consists of five nodes. For each one of nodes 2, 3 and 4 temperature values to the east and west are available as nodal values. Consequently, discretised equations of the form (4.10) can be readily written for control volumes surrounding these nodes:

$$\left(\frac{k_e}{\delta x_{PE}}A_e + \frac{k_w}{\delta x_{WP}}A_w\right)T_P = \left(\frac{k_w}{\delta x_{WP}}A_w\right)T_W + \left(\frac{k_e}{\delta x_{PE}}A_e\right)T_E \tag{4.14}$$

The thermal conductivity ($k_e = k_w = k$), node spacing (δx) and cross-sectional area ($A_e = A_w = A$) are constants. Therefore the **discretised equation for nodal points 2, 3 and 4** is

$$\boxed{a_P T_P = a_W T_W + a_E T_E} \tag{4.15}$$

with

a_W	a_E	a_P
$\frac{k}{\delta x}A$	$\frac{k}{\delta x}A$	$a_W + a_E$

S_u and S_p are zero in this case since there is no source term in the governing equation (4.13).

Nodes 1 and 5 are boundary nodes, and therefore require special attention. Integration of equation (4.13) over the control volume surrounding point 1 gives

$$kA\left(\frac{T_E - T_P}{\delta x}\right) - kA\left(\frac{T_P - T_A}{\delta x/2}\right) = 0 \tag{4.16}$$

This expression shows that the flux through control volume boundary A has been approximated by assuming a linear relationship between temperatures at boundary point A and node P. We can rearrange (4.16) as follows:

$$\left(\frac{k}{\delta x}A + \frac{2k}{\delta x}A\right)T_P = 0 \,.\, T_W + \left(\frac{k}{\delta x}A\right)T_E + \left(\frac{2k}{\delta x}A\right)T_A \tag{4.17}$$

Comparing equation (4.17) with equation (4.10), it can be easily identified that the fixed temperature boundary condition enters the calculation as a source term $(S_u + S_P T_P)$ with $S_u = (2kA/\delta x)T_A$ and $S_p = -2kA/\delta x$, and that the link to the (west) boundary side has been suppressed by setting coefficient a_W to zero.

Equation (4.17) may be cast in the same form as (4.11) to yield the **discretised equation for boundary node 1**:

$$\boxed{a_P T_P = a_W T_W + a_E T_E + S_u} \tag{4.18}$$

with

a_W	a_E	a_P	S_P	S_u
0	$\dfrac{kA}{\delta x}$	$a_W + a_E - S_p$	$-\dfrac{2kA}{\delta x}$	$\dfrac{2kA}{\delta x}T_A$

The control volume surrounding node 5 can be treated in a similar manner. Its discretised equation is given by

$$kA\left(\frac{T_B - T_P}{\delta x/2}\right) - kA\left(\frac{T_P - T_W}{\delta x}\right) = 0 \tag{4.19}$$

As before we assume a linear temperature distribution between node P and boundary point B to approximate the heat flux through the control volume boundary. Equation (4.19) can be rearranged as

$$\left(\frac{k}{\delta x}A + \frac{2k}{\delta x}A\right)T_P = \left(\frac{k}{\delta x}A\right)T_W + 0\,.\,T_E + \left(\frac{2k}{\delta x}A\right)T_B \tag{4.20}$$

The **discretised equation for boundary node 5** is

$$\boxed{a_P T_P = a_W T_W + a_E T_E + S_u} \tag{4.21}$$

where

a_W	a_E	a_P	S_P	S_u
$\dfrac{kA}{\delta x}$	0	$a_W + a_E - S_p$	$-\dfrac{2kA}{\delta x}$	$\dfrac{2kA}{\delta x}T_B$

The discretisation process has yielded one equation for each of the nodal points 1 to 5. Substitution of numerical values gives $kA/\delta x = 100$, and the coefficients of each discretised equation can easily be worked out. Their values are given in Table 4.1.

The resulting set of algebraic equations for this example is

$$\begin{aligned}
300T_1 &= 100T_2 + 200T_A\\
200T_2 &= 100T_1 + 100T_3\\
200T_3 &= 100T_2 + 100T_4\\
200T_4 &= 100T_3 + 100T_5\\
300T_5 &= 100T_4 + 200T_B
\end{aligned} \tag{4.22}$$

Table 4.1

Node	a_W	a_E	S_u	S_P	$a_P = a_W + a_E - S_P$
1	0	100	$200T_A$	−200	300
2	100	100	0	0	200
3	100	100	0	0	200
4	100	100	0	0	200
5	100	0	$200T_B$	−200	300

This set of equations can be rearranged as

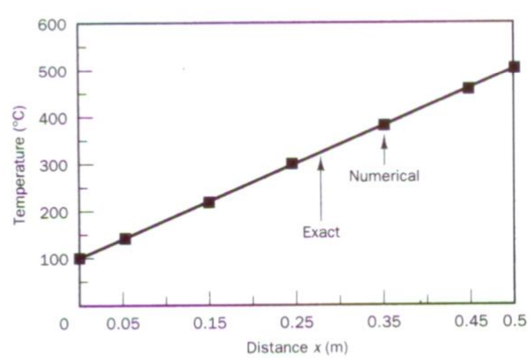
$$\begin{bmatrix} 300 & -100 & 0 & 0 & 0 \\ -100 & 200 & -100 & 0 & 0 \\ 0 & -100 & 200 & -100 & 0 \\ 0 & 0 & -100 & 200 & -100 \\ 0 & 0 & 0 & -100 & 300 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 200T_A \\ 0 \\ 0 \\ 0 \\ 200T_B \end{bmatrix} \quad (4.23)$$

The above set of equations yields the steady state temperature distribution of the given situation. For simple problems involving a small number of nodes the resulting matrix equation can easily be solved with a software package such as MATLAB (1992). For $T_A = 100$ and $T_B = 500$ the solution of (4.23) can obtained by using, for example, Gaussian elimination:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 140 \\ 220 \\ 300 \\ 380 \\ 460 \end{bmatrix} \quad (4.24)$$

The exact solution is a linear distribution between the specified boundary temperatures: $T = 800x + 100$. Figure 4.5 shows that the exact solution and the numerical results coincide.

Figure 4.5 Comparison of the numerical result with the analytical solution



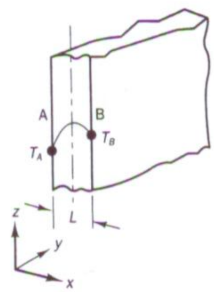
Example 4.2

Now we discuss a problem that includes sources other than those arising from boundary conditions. Figure 4.6 shows a large plate of thickness $L = 2$ cm with constant thermal conductivity $k = 0.5$ W/m.K and uniform heat generation $q = 1000$ kW/m^3. The faces A and B are at temperatures of 100°C and 200°C respectively. Assuming that the dimensions in the y- and

z-directions are so large that temperature gradients are significant in the x-direction only, calculate the steady state temperature distribution. Compare the numerical result with the analytical solution. The governing equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(k\frac{\mathrm{d}T}{\mathrm{d}x}\right) + q = 0 \tag{4.25}$$

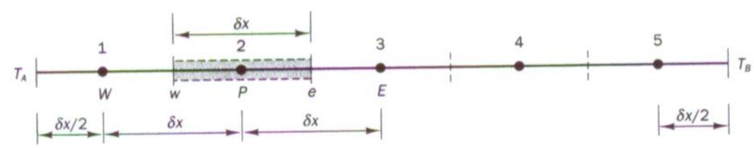
Figure 4.6



Solution

As before, the method of solution is demonstrated using a simple grid. The domain is divided into five control volumes (see Figure 4.7), giving $\delta x = 0.004$ m; a unit area is considered in the y–z plane.

Figure 4.7 The grid used



Formal integration of the governing equation over a control volume gives

$$\int_{\Delta V}\frac{\mathrm{d}}{\mathrm{d}x}\left(k\frac{\mathrm{d}T}{\mathrm{d}x}\right)\mathrm{d}V + \int_{\Delta V} q\,\mathrm{d}V = 0 \tag{4.26}$$

We treat the first term of the above equation as in the previous example. The second integral, the source term of the equation, is evaluated by calculating the average generation (i.e. $\bar{S}\Delta V = q\Delta V$) within each control volume. Equation (4.26) can be written as

$$\left[\left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right)_e - \left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right)_w\right] + q\Delta V = 0 \tag{4.27}$$

$$\left[k_eA\left(\frac{T_E - T_P}{\delta x}\right) - k_wA\left(\frac{T_P - T_W}{\delta x}\right)\right] + qA\delta x = 0 \tag{4.28}$$

The above equation can be rearranged as

$$\left(\frac{k_eA}{\delta x} + \frac{k_wA}{\delta x}\right)T_P = \left(\frac{k_wA}{\delta x}\right)T_W + \left(\frac{k_eA}{\delta x}\right)T_E + qA\delta x \tag{4.29}$$

This equation is written in the general form of (4.11):

$$\boxed{a_P T_P = a_W T_W + a_E T_E + S_u} \tag{4.30}$$

Since $k_e = k_w = k$ we have the following coefficients:

a_W	a_E	a_P	S_P	S_u
$\dfrac{kA}{\delta x}$	$\dfrac{kA}{\delta x}$	$a_W + a_E - S_P$	0	$qA\delta x$

Equation (4.30) is valid for control volumes at **nodal points 2, 3 and 4**.

To incorporate the boundary conditions at nodes 1 and 5 we apply the linear approximation for temperatures between a boundary point and the adjacent nodal point. At node 1 the temperature at the west boundary is known. Integration of equation (4.25) at the control volume surrounding node 1 gives

$$\left[\left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right)_e - \left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right)_w\right] + q\Delta V = 0 \tag{4.31}$$

Introduction of the linear approximation for temperatures between A and P yields

$$\left[k_e A\left(\frac{T_E - T_P}{\delta x}\right) - k_A A\left(\frac{T_P - T_A}{\delta x/2}\right)\right] + qA\delta x = 0 \tag{4.32}$$

The above equation can be rearranged, using $k_e = k_A = k$, to yield the discretised equation for **boundary node 1**:

$$\boxed{a_P T_P = a_W T_W + a_E T_E + S_u} \tag{4.33}$$

where

a_W	a_E	a_P	S_P	S_u
0	$\dfrac{kA}{\delta x}$	$a_W + a_E - S_P$	$-\dfrac{2kA}{\delta x}$	$qA\delta x + \dfrac{2kA}{\delta x}T_A$

At nodal point 5, the temperature on the east face of the control volume is known. The node is treated in a similar way to boundary node 1. At boundary point 5 we have

$$\left[\left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right)_e - \left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right)_w\right] + q\Delta V = 0 \tag{4.34}$$

$$\left[k_B A\left(\frac{T_B - T_P}{\delta x/2}\right) - k_w A\left(\frac{T_P - T_W}{\delta x}\right)\right] + qA\delta x = 0 \tag{4.35}$$

The above equation can be rearranged, noting that $k_B = k_w = k$, to give the discretised equation for **boundary node 5**:

$$\boxed{a_P T_P = a_W T_W + a_E T_E + S_u} \tag{4.36}$$

where

a_W	a_E	a_P	S_P	S_u
$\dfrac{kA}{\delta x}$	0	$a_W + a_E - S_P$	$-\dfrac{2kA}{\delta x}$	$qA\delta x + \dfrac{2kA}{\delta x} T_B$

Substitution of numerical values for $A = 1$, $k = 0.5$ W/m.K, $q = 1000$ kW/m^3 and $\delta x = 0.004$ m everywhere gives the coefficients of the discretised equations summarised in Table 4.2.

Table 4.2

Node	a_W	a_E	S_u	S_P	$a_P = a_W + a_E - S_P$
1	0	125	$4000 + 250T_A$	−250	375
2	125	125	4000	0	250
3	125	125	4000	0	250
4	125	125	4000	0	250
5	125	0	$4000 + 250T_B$	−250	375

Given directly in matrix form the equations are

$$\begin{bmatrix} 375 & -125 & 0 & 0 & 0 \\ -125 & 250 & -125 & 0 & 0 \\ 0 & -125 & 250 & -125 & 0 \\ 0 & 0 & -125 & 250 & -125 \\ 0 & 0 & 0 & -125 & 375 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 29000 \\ 4000 \\ 4000 \\ 4000 \\ 54000 \end{bmatrix} \tag{4.37}$$

The solution to the above set of equations is

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 150 \\ 218 \\ 254 \\ 258 \\ 230 \end{bmatrix} \tag{4.38}$$

Comparison with the analytical solution

The analytical solution to this problem may be obtained by integrating equation (4.25) twice with respect to x and by subsequent application of the boundary conditions. This gives

$$T = \left[\frac{T_B - T_A}{L} + \frac{q}{2k}(L - x) \right] x + T_A \tag{4.39}$$

The comparison between the finite volume solution and the exact solution is shown in Table 4.3 and Figure 4.8 and it can be seen that, even with a coarse grid of five nodes, the agreement is very good.

Table 4.3

Node number	*1*	*2*	*3*	*4*	*5*
x (m)	0.002	0.006	0.01	0.014	0.018
Finite volume solution	150	218	254	258	230
Exact solution	146	214	250	254	226
Percentage error	2.73	1.86	1.60	1.57	1.76

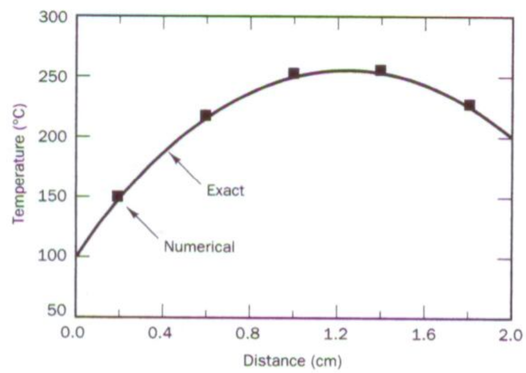


Figure 4.8 Comparison of the numerical result with the analytical solution

Example 4.3

In the final worked example of this chapter we discuss the cooling of a circular fin by means of convective heat transfer along its length. Convection gives rise to a temperature-dependent heat loss or sink term in the governing equation. Shown in Figure 4.9 is a cylindrical fin with uniform cross-sectional area A. The base is at a temperature of 100°C (T_B) and the end is insulated. The fin is exposed to an ambient temperature of 20°C. One-dimensional heat transfer in this situation is governed by

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) - hP(T - T_\infty) = 0 \tag{4.40}$$

where h is the convective heat transfer coefficient, P the perimeter, k the thermal conductivity of the material and T_∞ the ambient temperature. Calculate the temperature distribution along the fin and compare the results with the analytical solution given by

$$\frac{T - T_\infty}{T_B - T_\infty} = \frac{\cosh[n(L - x)]}{\cosh(nL)} \tag{4.41}$$

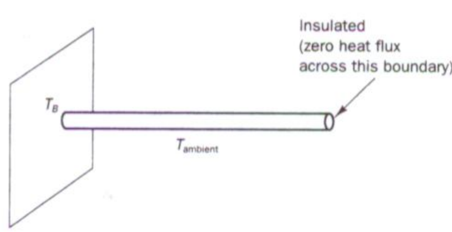


Figure 4.9 The geometry for Example 4.3