Condução Transiente: Método Capacitivo Cap. 5

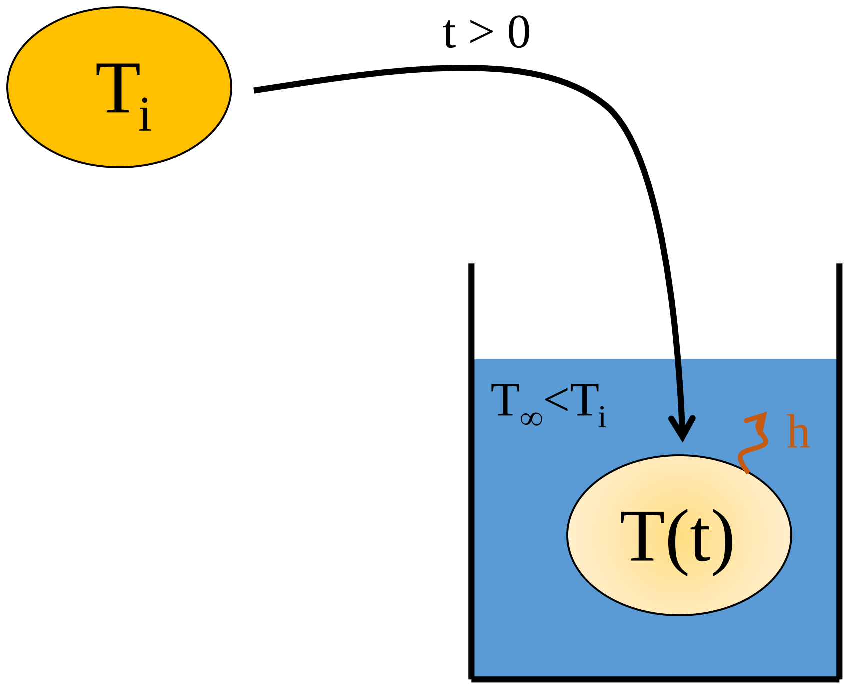
Condução Transiente

- Alterações na condição de equilíbrio térmico:
  - convecção na superfície (h, T_∞),
  - radiação na superfície (h_r, T_{sur}),
  - temperatura ou fluxo de calor imposto na superfície
  - geração interna de calor.

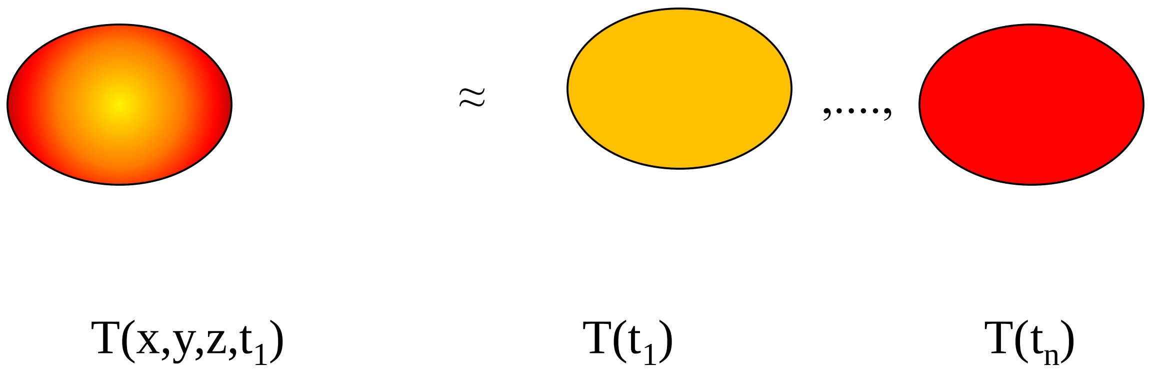
- Técnicas de solução
  - Método Capacitivo, T=T(t)
  - Soluções Exatas, T=T(x,y,z,t)
  - Métodos numéricos.

Método Capacitivo

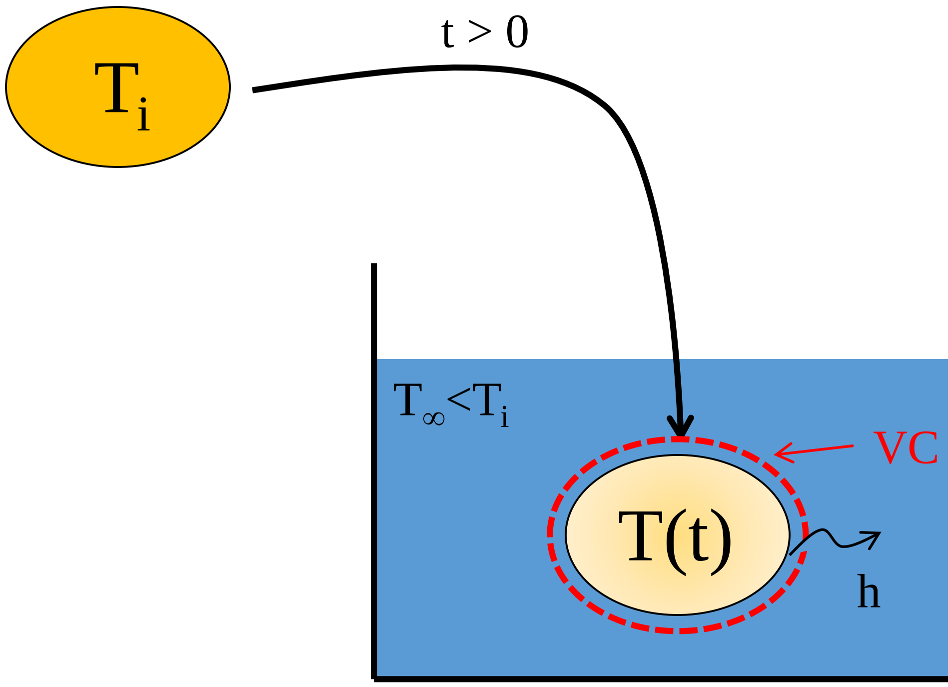
$$T\left(\vec{r},t\right)\approx T\left(t\right)$$



$$T\left(\vec{r},t\right) \approx T\left(t\right)$$



Método Capacitivo



Balanço de Energia:

$$\dot{E}_e + \dot{E}_g - \dot{E}_s = \frac{dE_{ar}}{dt} = \dot{E}_{ac}$$

$$-hA_s\left(T(t) - T_\infty\right) = mc\frac{dT}{dt}$$

$$\dot{E}_e + \dot{E}_g - \dot{E}_s = \frac{dE_{ar}}{dt} = \dot{E}_{ac}$$

$$-hA_s\left(T(t) - T_\infty\right) = mc\frac{dT}{dt}$$

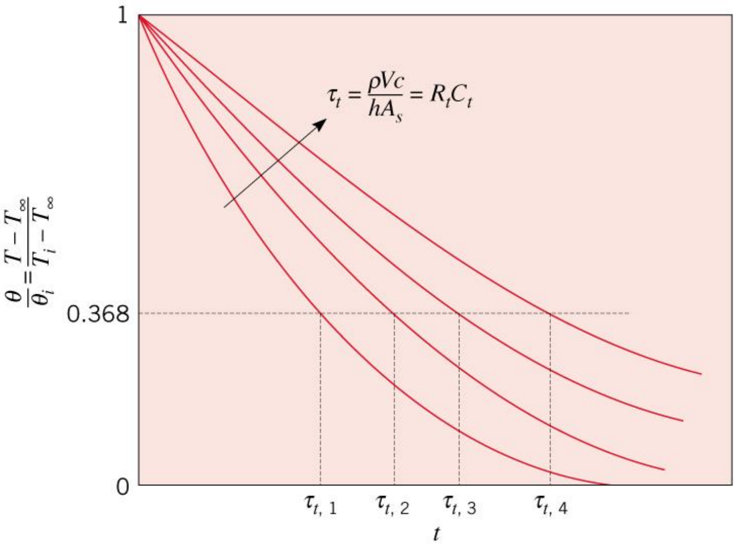
$$\rho \forall c \frac{dT}{dt} = -hA_{s,c}\left(T - T_\infty\right) \tag{5.2}$$

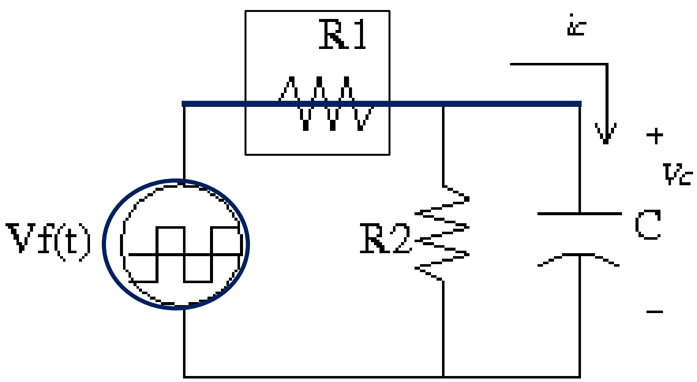
$$\frac{\rho \forall c}{hA_{s,c}} \int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = -\int_0^t dt$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = exp\left[-\left(\frac{hA_{s,c}}{\rho \forall c}\right)t\right] = \exp\left[-\frac{t}{\tau_t}\right]$$

A constante de tempo térmica é definida como

$$\tau_t \equiv \underbrace{\left(\frac{1}{hA_{s,c}}\right)}_{\text{Resistência térmica, } R_t}\underbrace{(\rho\forall c)}_{\text{Capacitância térmica, } C_t} \tag{5.7}$$

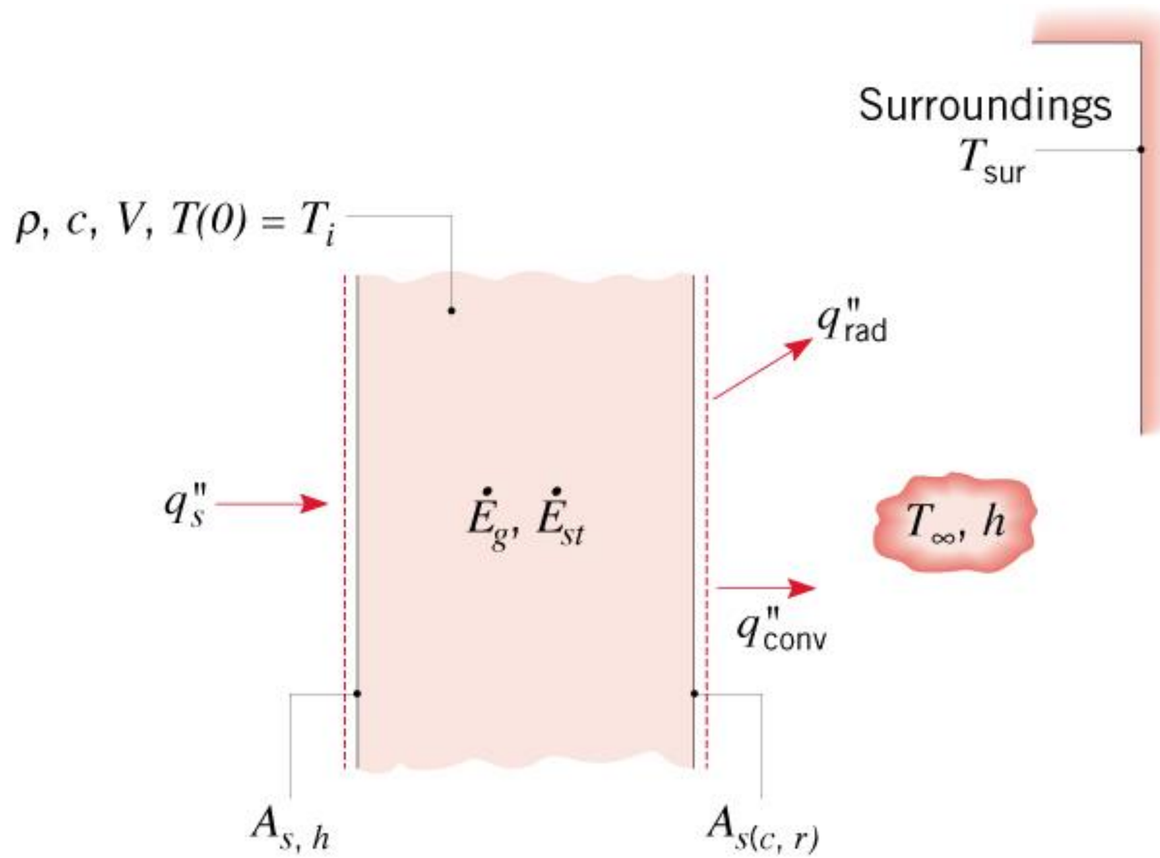


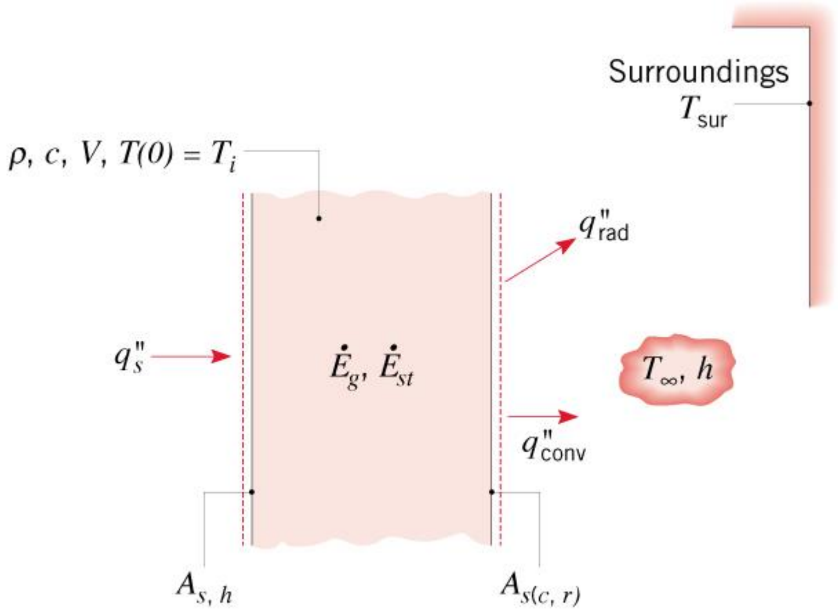


Calor trocado durante o processo

$$\Delta E_{st} \equiv -Q = -\int_0^t \dot{E}_{out}\,dt = -hA_{s,c}\int_0^t \theta\,dt = -(\rho\forall c)\theta_i\left[1-\exp\left(-\frac{t}{\tau_t}\right)\right] \tag{5.8}$$

Análise do método Capacitivo para uma parede plana





- Balanço de Energia:

$$\frac{dE_{st}}{dt} = \rho \forall c \frac{dT}{dt} = \dot{E}_{in} - \dot{E}_{out} + \dot{E}_g$$

$$\rho \forall c \frac{dT}{dt} = q''_s A_{s,h} - h A_{s,c} \left(T - T_\infty\right) - h_r A_{s,r} \left(T - T_{sur}\right) + \dot{E}_g$$

- Casos especiais (Sol. Exata) $T(0) \equiv T_i$

  - Sem radiação $\left(\theta \equiv T - T_\infty,\ \theta' \equiv \theta - b/a\right):$

$$a \equiv hA_{s,c} / \rho \forall c \qquad b \equiv \left(q_s'' A_{s,h} + \dot{E}_g \right) / \rho \forall c$$

$$\frac{d\theta'}{dt} = -a\theta'$$

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp(-at) + \frac{b/a}{T_i - T_\infty}\left[1 - \exp(-at)\right] \qquad (5.25)$$

- Sem radiação e sem geração $\left(h >> h_r,\ \dot{E}_g = 0,\ q''_s = 0 \right)$:

$$\rho \forall c \frac{dT}{dt} = -hA_{s,c}\left(T - T_\infty\right) \tag{5.2}$$

$$\frac{\rho \forall c}{hA_{s,c}} \int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = -\int_0^t dt$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = exp\left[-\left(\frac{hA_{s,c}}{\rho \forall c} \right) t \right] = \exp\left[-\frac{t}{\tau_t} \right]$$

➢Sem convecção e sem geração $\left(h_r >> h,\ \dot{E}_g = 0,\ q''_s = 0 \right)$:

$$\rho \forall c \frac{dT}{dt} = -\varepsilon A_{s,r} \sigma \left(T^4 - T_{sur}^4 \right)$$

$$\frac{\varepsilon A_{s,r} \sigma}{\rho \forall c} \int_0^t dt = \int_{T_i}^{T} \frac{dT}{T_{sur}^4 - T^4}$$

$$t = \frac{\rho \forall c}{4 \varepsilon A_{s,r} \sigma T_{sur}^3} \left\{ \ln \left| \frac{T_{sur} + T}{T_{sur} - T} \right| - \ln \left| \frac{T_{sur} + T_i}{T_{sur} - T_i} \right| \right.$$

$$\left. + 2 \left[\tan^{-1} \left(\frac{T}{T_{sur}} \right) - \tan^{-1} \left(\frac{T_i}{T_{sur}} \right) \right] \right\} \qquad (5.18)$$

Número de Biot e validade do método Capacitivo

- O Número de Biot : análise adimensional.

  - Definição:

$$\boxed{Bi \equiv \frac{hL_c}{k}} \qquad L_C = \frac{V}{A_S}$$

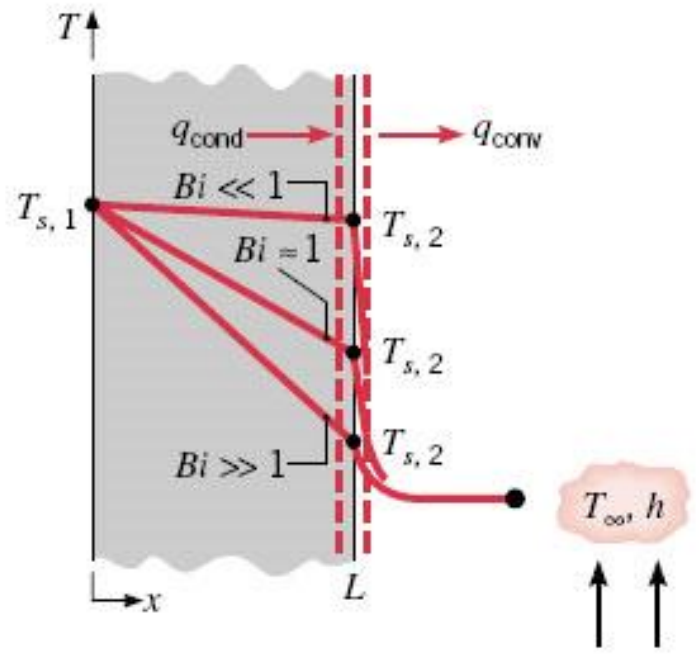
L_c=Comprimento característico

  - Interpretação física:

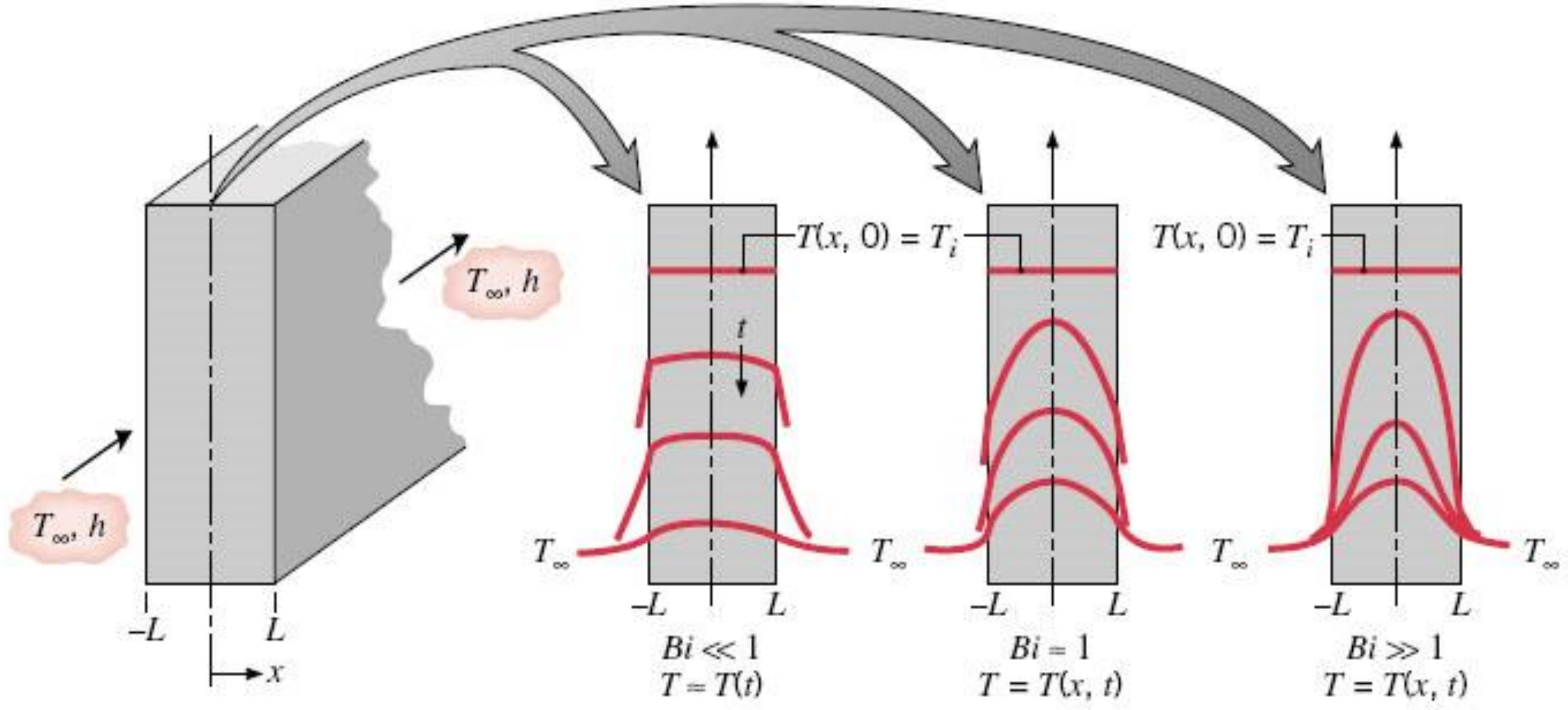
$$q_{cond} = q_{conv}$$

$$\frac{kA}{L}\left(T_{S,1} - T_{S,2}\right) = hA\left(T_{S,2} - T_{\infty}\right)$$

$$\beta i = \frac{T_{S,1} - T_{S,2}}{T_{S,2} - T_{\infty}} = \frac{L_C / kA_S}{1/hA_S} \approx \frac{R_{cond}}{R_{conv}}$$



Critério de aplicação do método capacitivo



- Critério de aplicação do método capacitivo:

Admite-se Bi<0,1

Comprimento Característico

$$L_C = \frac{V}{A_S}$$

-Para uma parede de espessura 2L, L_c=L
-Para um cilindro de raio R, L_c=R/2
-Para uma esfera de raio R, L_c=R/3

Análise Adimensional

$$\frac{\theta}{\theta_i} = e^{-\frac{hA_s}{\rho Vc}t}$$

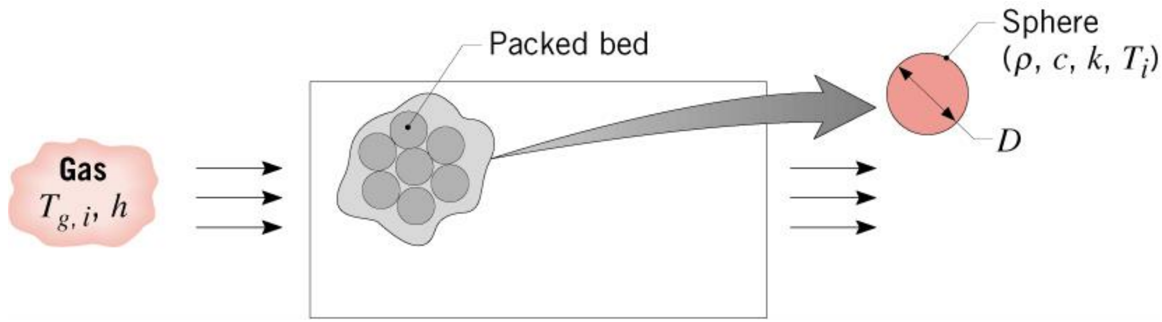
$$\frac{hA_s}{\rho Vc}t = \frac{ht}{\rho c L_C} = \frac{hL_C}{k}\frac{k}{\rho c}\frac{t}{L_C^2}$$

$$= \frac{hL_C}{k}\frac{\alpha t}{L_C^2}$$

$$= \beta\mathrm{i}.Fo$$

$$\frac{\theta}{\theta_i} = e^{-\beta\mathrm{i}.Fo}$$

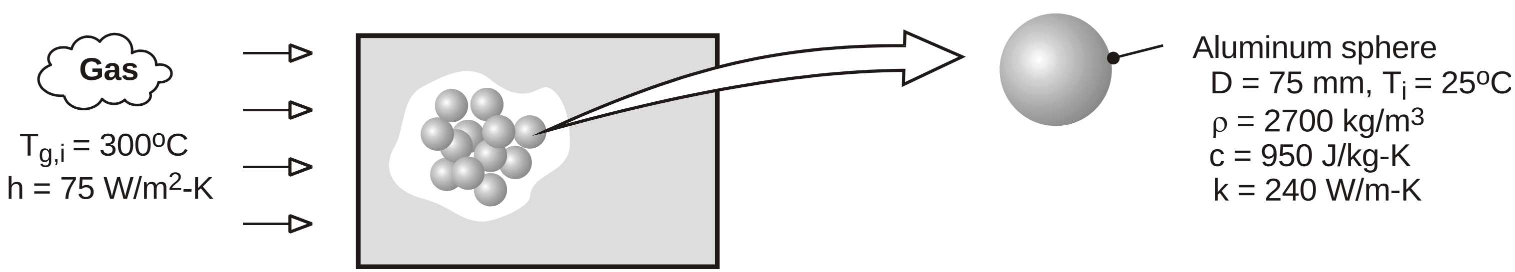
Problem 5.12: Charging a thermal energy storage system consisting of a packed bed of aluminum spheres.



KNOWN: Diameter, density, specific heat and thermal conductivity of aluminum spheres used in packed bed thermal energy storage system. Convection coefficient and inlet gas temperature.

FIND: Time required for sphere at inlet to acquire 90% of maximum possible thermal energy and the corresponding center temperature.

Schematic:



ASSUMPTIONS: (1) Negligible heat transfer to or from a sphere by radiation or conduction due to contact with other spheres, (2) Constant properties.

ANALYSIS: To determine whether a lumped capacitance analysis can be used, first compute Bi = $h(r_o/3)/k$ = 75 W/m^2·K (0.025m)/150 W/m·K = 0.013 <<1.

Hence, the lumped capacitance approximation may be made, and a uniform temperature may be assumed to exist in the sphere at any time.

From Eq. 5.8a, achievement of 90% of the maximum possible thermal energy storage corresponds to

$$-\frac{\Delta E_{st}}{\rho c V \theta} = 0.90 = 1 - \exp\left(-t/\tau_t\right)$$

$$\tau_t = \rho V c / hA_s = \rho D c / 6h = \frac{2700\,\text{kg/m}^3 \times 0.075\text{m} \times 950\,\text{J/kg}\cdot\text{K}}{6 \times 75\,\text{W/m}^2\cdot\text{K}} = 427\text{s}.$$

$$t = -\tau_t \ln(0.1) = 427\text{s} \times 2.30 = 984\text{s}$$

From Eq. (5.6), the corresponding temperature at any location in the sphere is

$$T(984\text{s}) = T_{g,i} + \left(T_i - T_{g,i}\right)\exp\left(-6ht/\rho Dc\right)$$

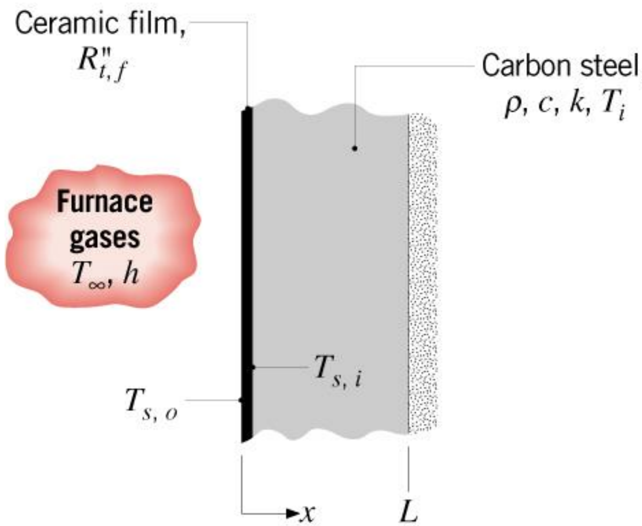
$$T(984\text{s}) = 300°\text{C} - 275°\text{C}\ \exp\left(-6 \times 75\,\text{W/m}^2\cdot\text{K} \times 984\text{s} / 2700\,\text{kg/m}^3 \times 0.075\text{m} \times 950\,\text{J/kg}\cdot\text{K}\right)$$

$$T(984\text{s}) = 272.5°\text{C}$$

If the product of the density and specific heat of copper is $(\rho c)_{Cu} \approx$ 8900 kg/m^3 × 400 J/kg·K = 3.56 × 10^6 J/m^3·K, is there any advantage to using copper spheres of equivalent diameter in lieu of aluminum spheres?

Does the time required for a sphere to reach a prescribed state of thermal energy storage change with increasing distance from the bed inlet? If so, how and why?

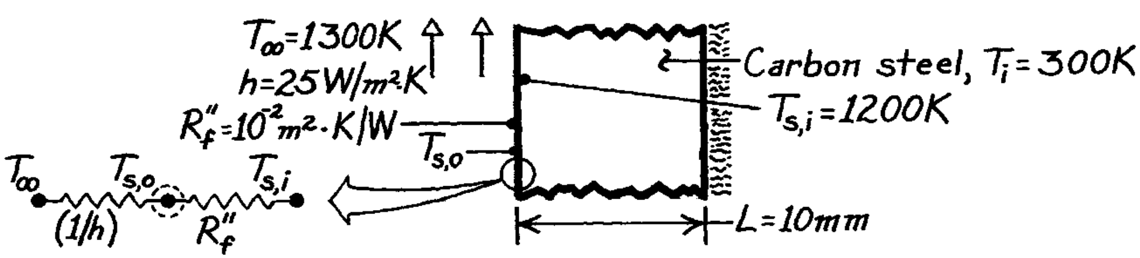
Problem 5.16: Heating of coated furnace wall during start-up.



KNOWN: Thickness and properties of furnace wall. Thermal resistance of ceramic coating on surface of wall exposed to furnace gases. Initial wall temperature.

FIND: (a) Time required for surface of wall to reach a prescribed temperature, (b) Corresponding value of coating surface temperature.

Schematic:



Problem: Furnace Start-up

ASSUMPTIONS: (1) Constant properties, (2) Negligible coating thermal capacitance, (3) Negligible radiation.

PROPERTIES: Carbon steel: ρ = 7850 kg/m^3, c = 430 J/kg·K, k = 60 W/m·K.

ANALYSIS: Heat transfer to the wall is determined by the total resistance to heat transfer from the gas to the surface of the steel, and not simply by the convection resistance.

Hence, with

$$U = \left(R''_{tot}\right)^{-1} = \left(\frac{1}{h} + R''_f\right)^{-1} = \left(\frac{1}{25\ \mathrm{W/m^2 \cdot K}} + 10^{-2}\ \mathrm{m^2 \cdot K/W}\right)^{-1} = 20\ \mathrm{W/m^2 \cdot K}.$$

$$Bi = \frac{UL}{k} = \frac{20\ \mathrm{W/m^2 \cdot K} \times 0.01\ \mathrm{m}}{60\ \mathrm{W/m \cdot K}} = 0.0033 << 1$$

and the lumped capacitance method can be used.

(a) From Eqs. (5.6) and (5.7),

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp\left(-t/\tau_t\right) = \exp\left(-t/R_t C_t\right) = \exp\left(-Ut/\rho Lc\right)$$

$$t = -\frac{\rho Lc}{U} \ln \frac{T - T_\infty}{T_i - T_\infty} = -\frac{7850\ \mathrm{kg/m^3}\left(0.01\ \mathrm{m}\right) 430\ \mathrm{J/kg \cdot K}}{20\ \mathrm{W/m^2 \cdot K}} \ln \frac{1200 - 1300}{300 - 1300}$$

$$t = 3886\mathrm{s} = 1.08\mathrm{h}.$$

(b) Performing an energy balance at the outer surface (s,o),

$$h\left(T_\infty - T_{s,o}\right) = \left(T_{s,o} - T_{s,i}\right) / R''_f$$

$$T_{s,o} = \frac{hT_\infty + T_{s,i} / R''_f}{h + \left(1/R''_f\right)} = \frac{25\ \mathrm{W/m^2 \cdot K} \times 1300\ \mathrm{K} + 1200\ \mathrm{K}/10^{-2}\mathrm{m^2 \cdot K/W}}{\left(25 + 100\right)\mathrm{W/m^2 \cdot K}}$$

$$T_{s,o} = 1220\ \mathrm{K}.$$

How does the coating affect the thermal time constant?

Condução transiente: consideração dos efeitos espaciais e as soluções analíticas

Soluções analíticas para problemas transientes quando o método da capacitância global não é válido (*Bi* não é <<1)

$$T = T(x,y,z,t)$$

- Para uma placa placa, com propriedades constantes e para as seguintes condições de contorno:

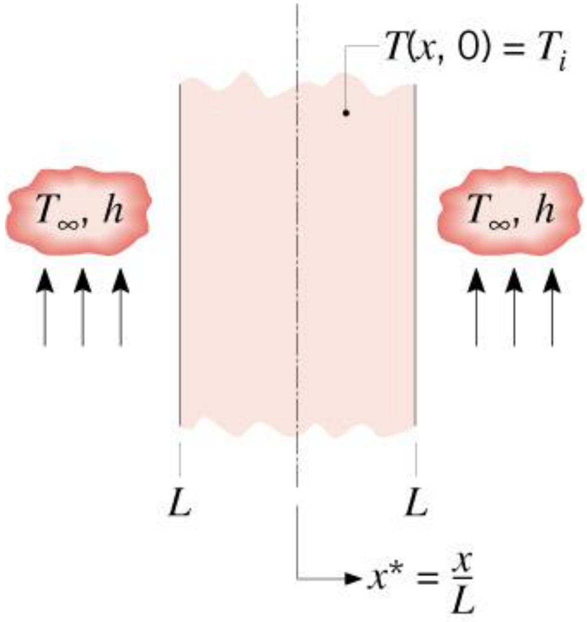
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{5.26}$$

$$T(x,0) = T_i \tag{5.27}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \tag{5.28}$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T(L,t) - T_\infty\right] \tag{5.29}$$

$$T = T(x,t,T_i,T_\infty,k,\alpha,h) \tag{5.30}$$



- Solução adimensional:

Diferença de Temp adimensional: $$\theta^* \equiv \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty}$$

Coordenadas adim.: $$x^* \equiv \frac{x}{L}$$

Tempo adim.: $$t^* \equiv \frac{\alpha t}{L^2} \equiv Fo$$

Número de Biot: $$Bi \equiv \frac{hL}{k_{solid}}$$

$$\theta^* = f\left(x^*, Fo, Bi\right)$$

- Solução Exata:

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right)\cos\left(\zeta_n x^*\right) \quad \text{(5.39a)}$$

$$C_n = \frac{4\sin\zeta_n}{2\zeta_n + \sin\left(2\zeta_n\right)} \qquad \zeta_n \tan\zeta_n = Bi \quad \text{(5.39b,c)}$$

Apendice B.3 para as raízes (eigenvalues $\zeta_1, \ldots, \zeta_4$)

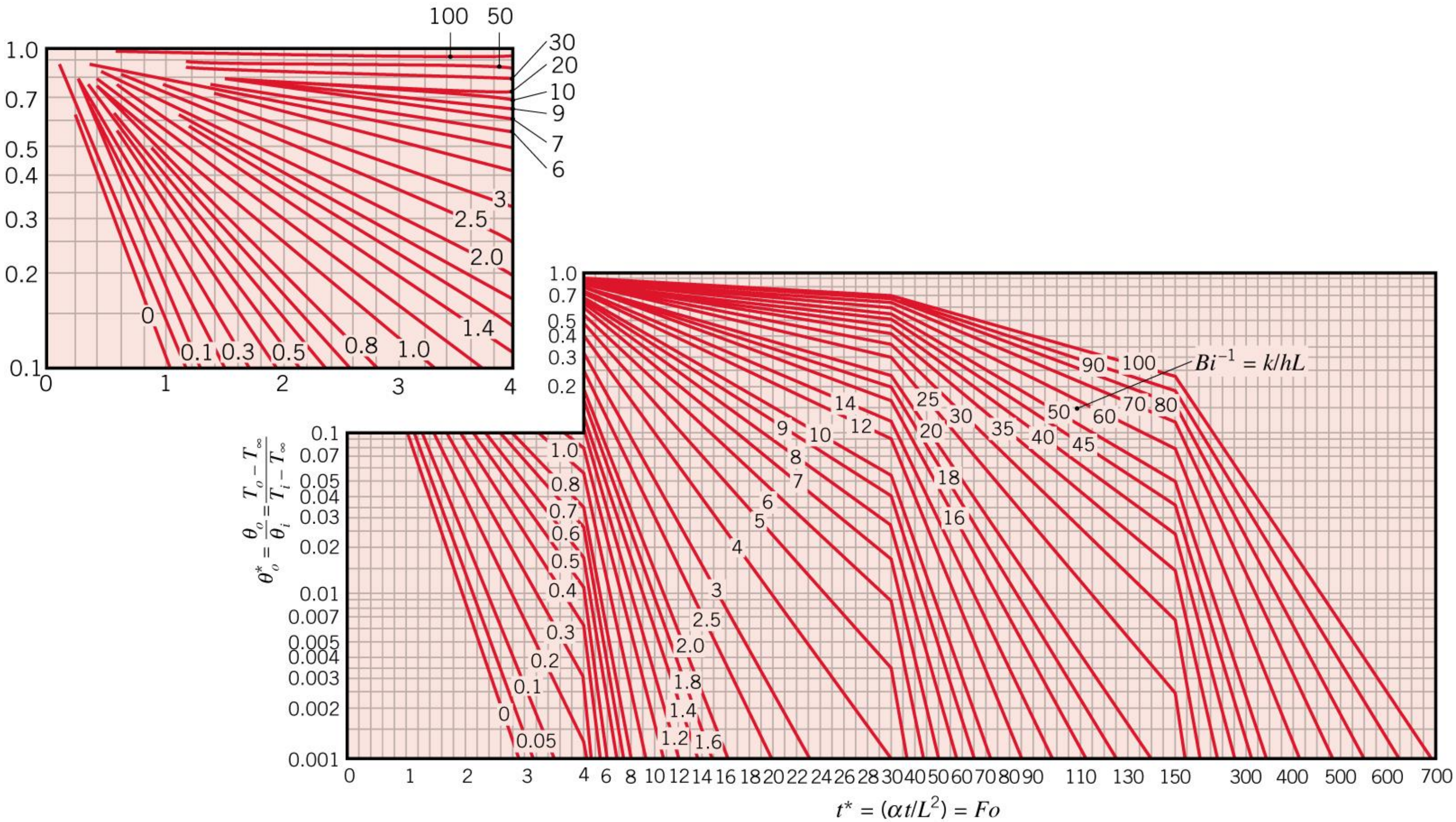
TABLE 5.1 Coefficients used in the one-term approximation to the series solutions for transient one-dimensional conduction

	Plane Wall		Infinite Cylinder		Sphere	
Bi[a]	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.03	0.1723	1.0049	0.2440	1.0075	0.2991	1.0090
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.05	0.2218	1.0082	0.3143	1.0124	0.3854	1.0149
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.07	0.2615	1.0114	0.3709	1.0173	0.4551	1.0209
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.09	0.2956	1.0145	0.4195	1.0222	0.5150	1.0268
0.10	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.15	0.3779	1.0237	0.5376	1.0365	0.6609	1.0445
0.20	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.25	0.4801	1.0382	0.6856	1.0598	0.8447	1.0737
0.30	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0932	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0919	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5994	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2402	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	1.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2881	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2733	2.4050	1.6018	3.1415	2.0000

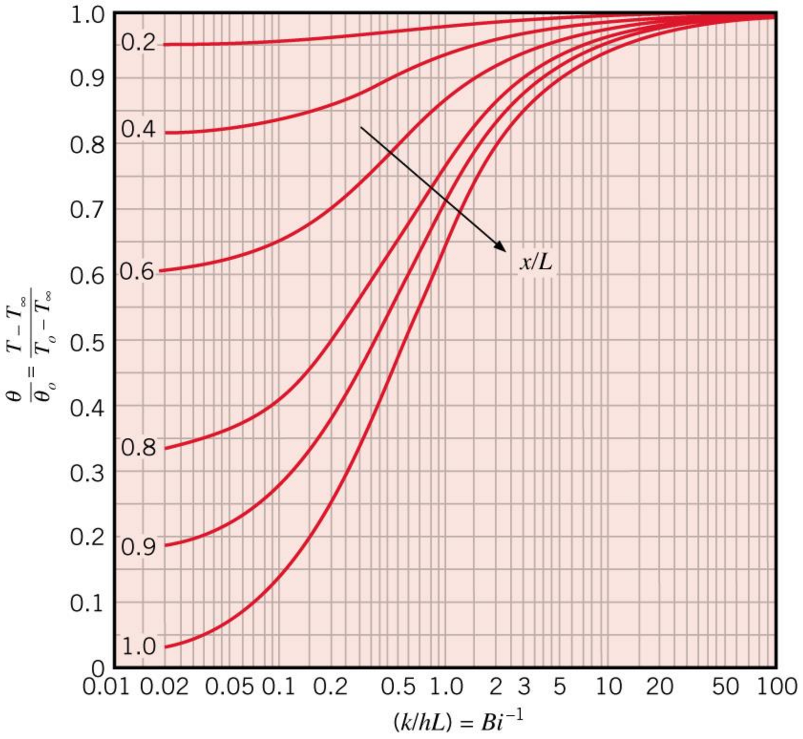
[a] $Bi = hL/k$ for the plane wall and hr_o/k for the infinite cylinder and sphere. See Figure 5.6.

Representação Gráfica para parede plana
Heisler Cartas

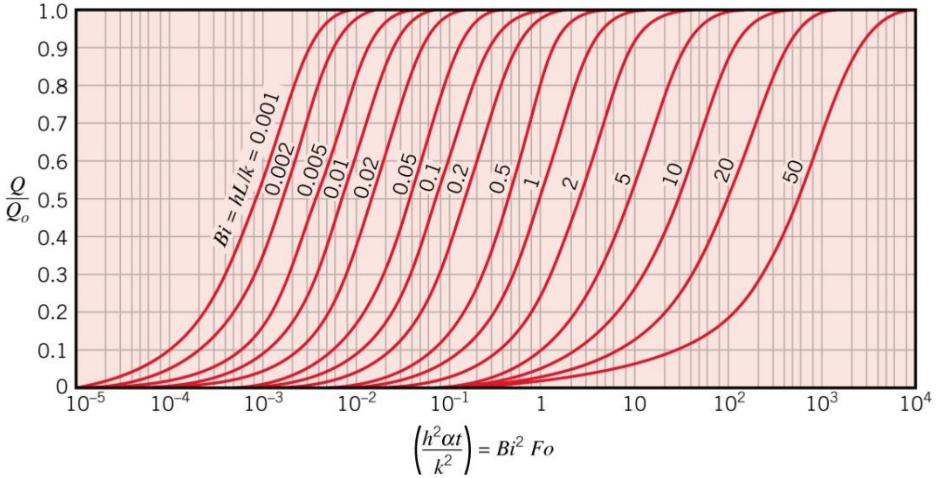
•Temperatura média do meio função do tempo (método capacitivo):



- Distribuição de Temp.:



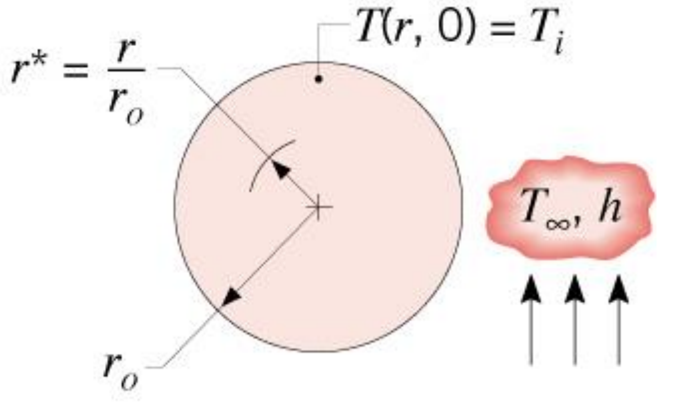
- Variações no calor transferido:



Sistemas Radiais

•

$$Bi = hr_o / k$$
$$Fo = \alpha t / r_o^2$$



- One-Term Approximations:
  Long Rod: Eqs. (5.49) and (5.51)
  Sphere: Eqs. (5.50) and (5.52)

$C_1, \zeta_1 \rightarrow$ Table 5.1

- Graphical Representations:
  Long Rod: Figs. 5 S.4 – 5 S.6
  Sphere: Figs. 5 S.7 – 5 S.9

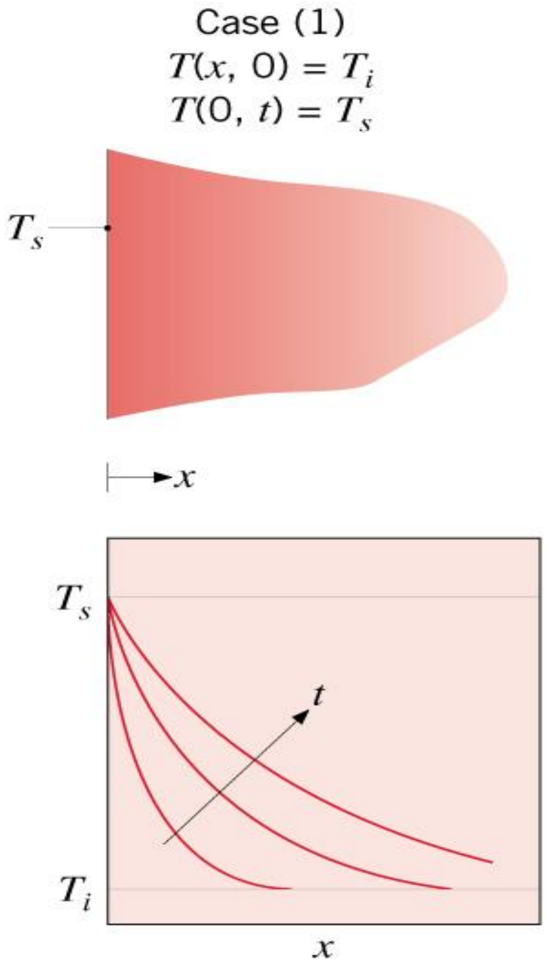
Sólido Semi-Infinito

•Caso 1: mudança na Temp. de superfície (T_s)

$$T(0,t) = T_s \neq T(x,0) = T_i$$

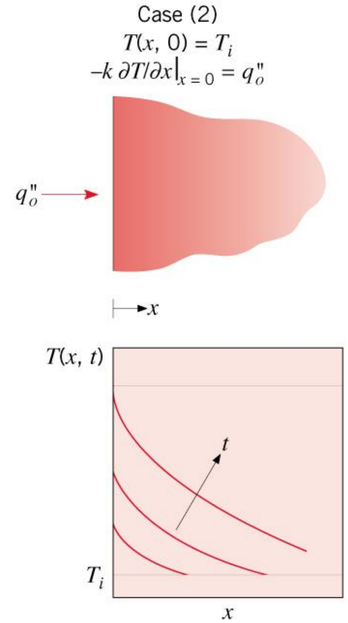
$$\frac{T(x,t) - T_s}{T_i - T_s} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) \quad (5.57)$$

$$q''_s = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}} \quad (5.58)$$



Caso 2: Fluxo de calor cte $\left(q''_s = q''_o\right)$

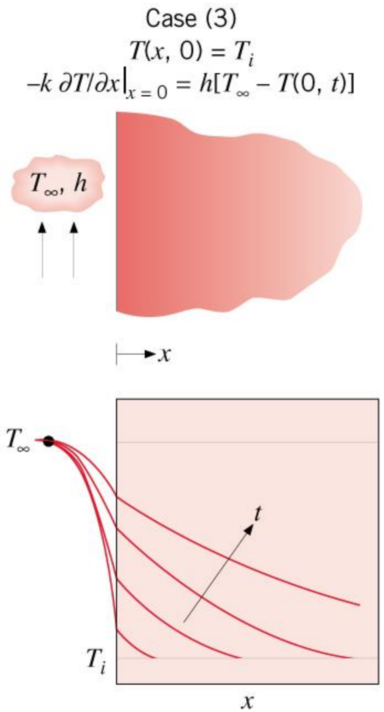
$$T\left(x,t\right) - T_i = \frac{2q''_o\left(\alpha t/\pi\right)^{1/2}}{k}\exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{q''_o x}{k}\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \qquad (5.59)$$



Caso 3: Convecção na sup. $\left(h, T_\infty\right)$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h\left[T_\infty - T\left(0,t\right)\right]$$

$$\frac{T\left(x,t\right) - T_i}{T_\infty - T_i} = erfc\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\right]\left[erfc\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right] \qquad (5.60)$$



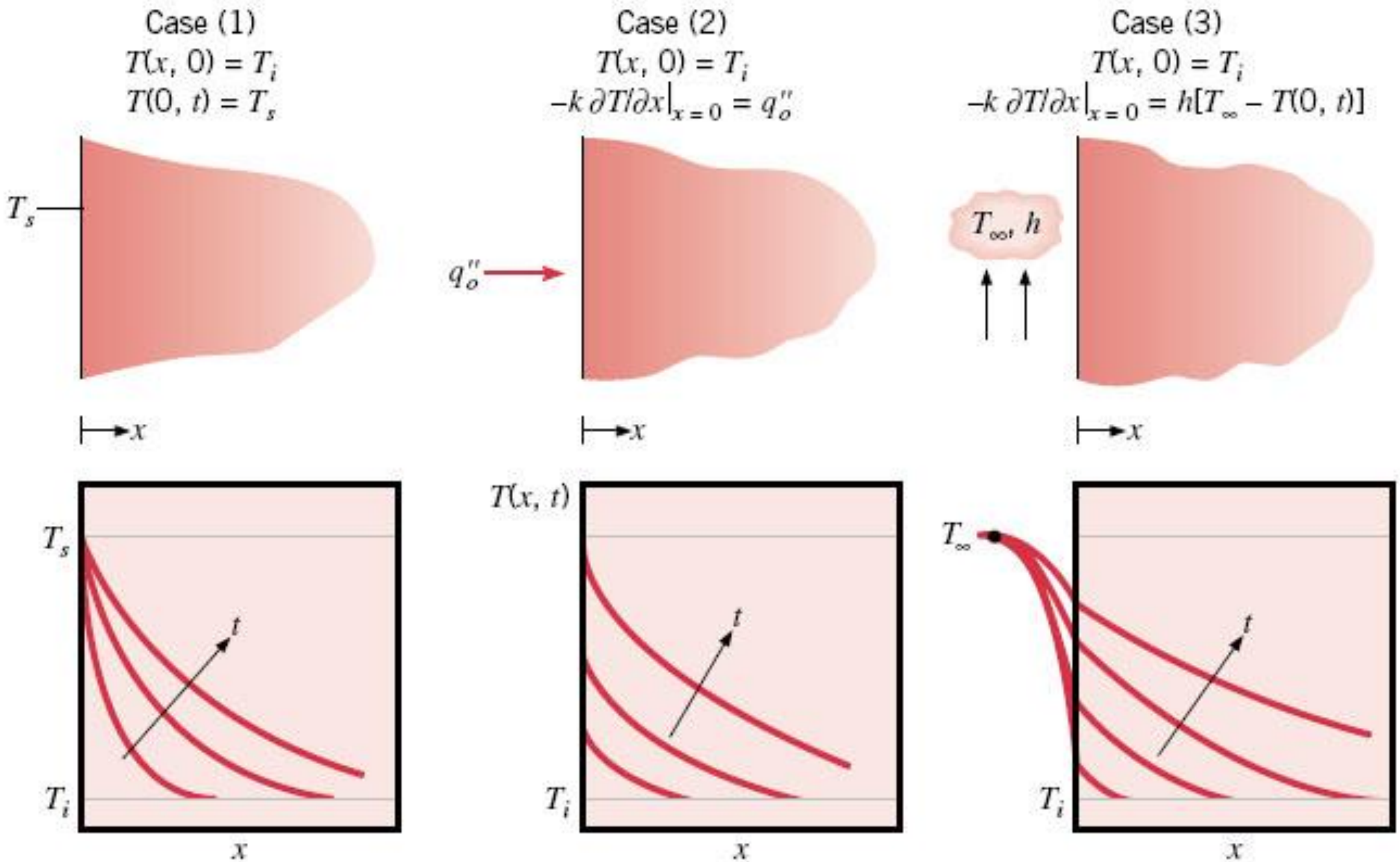


FIGURE 5.7 Transient temperature distributions in a semi-infinite solid for three surface conditions: constant surface temperature, constant surface heat flux, and surface convection.

Problemas com interface de contato

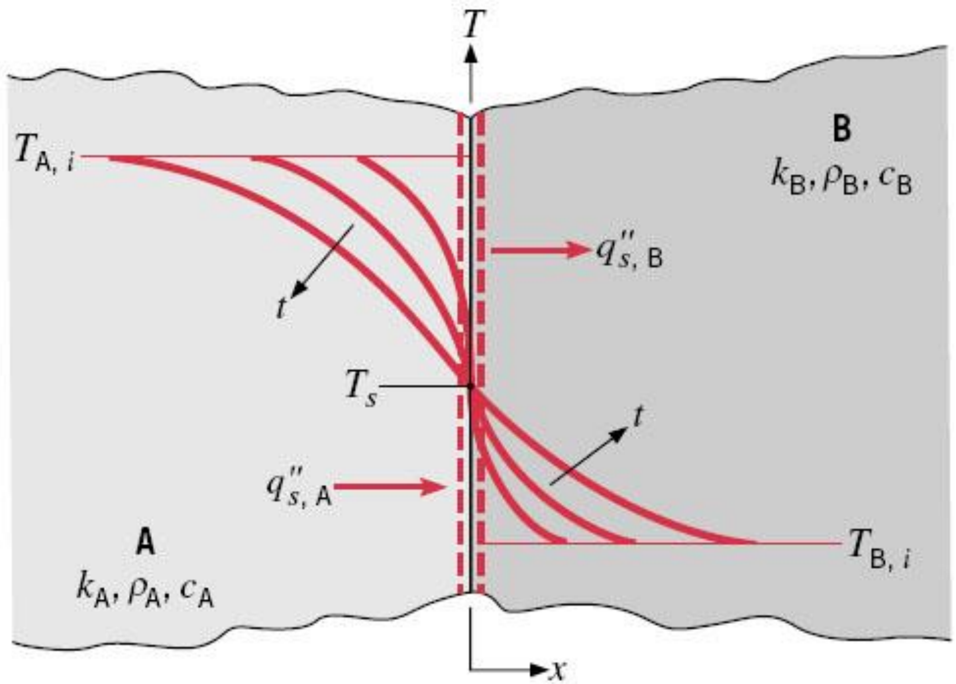


FIGURE 5.9
Interfacial contact between two semi-infinite solids at different initial temperatures.

Aquecimento periódico na superfície

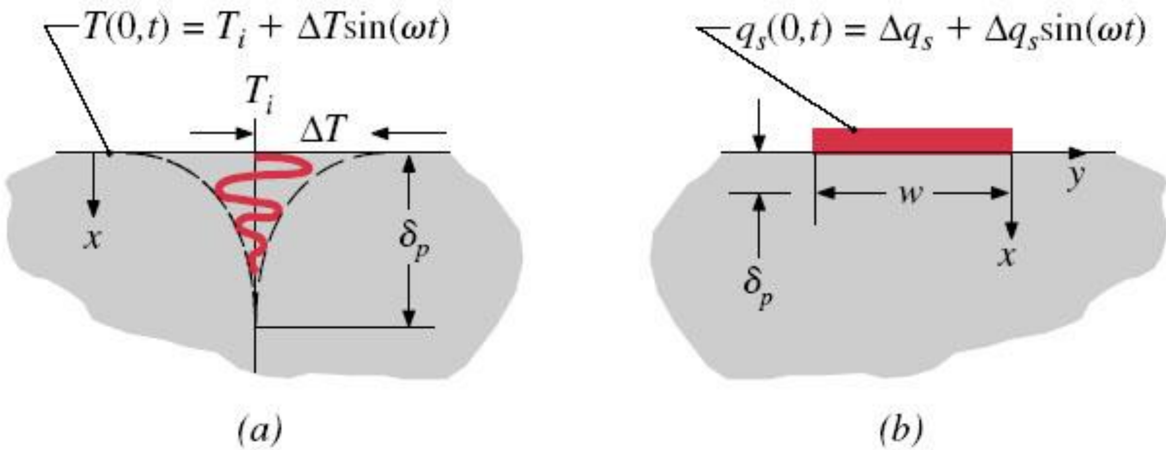


FIGURE 5.11 Schematic of (*a*) a periodically heated, one-dimensional semi-infinite solid and (*b*) a periodically heated strip attached to a semi-infinite solid.

TABLE 5.2a Summary of transient heat transfer results for constant surface temperature cases[a]

Geometry	Length Scale, L_c	$q^*(Fo)$ Exact Solutions	Approximate Solutions $Fo < 0.2$	Approximate Solutions $Fo \geq 0.2$	Maximum Error (%)
Semi-infinite	L (arbitrary)	$\dfrac{1}{\sqrt{\pi Fo}}$	Use exact solution.		none
Interior Cases					
Plane wall of thickness $2L$	L	$2\sum_{n=1}^{\infty} \exp(-\zeta_n^2 Fo) \quad \zeta_n = (n - \frac{1}{2})\pi$	$\dfrac{1}{\sqrt{\pi Fo}}$	$2 \exp(-\zeta_1^2 Fo) \quad \zeta_1 = \pi/2$	1.7
Infinite cylinder	r_o	$2\sum_{n=1}^{\infty} \exp(-\zeta_n^2 Fo) \quad J_0(\zeta_n) = 0$	$\dfrac{1}{\sqrt{\pi Fo}} - 0.50 - 0.65\,Fo$	$2 \exp(-\zeta_1^2 Fo) \quad \zeta_1 = 2.4050$	0.8
Sphere	r_o	$2\sum_{n=1}^{\infty} \exp(-\zeta_n^2 Fo) \quad \zeta_n = n\pi$	$\dfrac{1}{\sqrt{\pi Fo}} - 1$	$2 \exp(-\zeta_1^2 Fo) \quad \zeta_1 = \pi$	6.3
Exterior Cases					
Sphere	r_o	$\dfrac{1}{\sqrt{\pi Fo}} + 1$	Use exact solution.		none
Various shapes (Table 4.1, cases 12–15)	$(A_s/4\pi)^{1/2}$	none	$\dfrac{1}{\sqrt{\pi Fo}} + q_s^*, \quad q_s^*$ from Table 4.1		7.1

[a] $q^* \equiv q_s'' L_c / k(T_s - T_i)$ and $Fo \equiv \alpha t / L_c^2$ where L_c is the length scale given in the table, T_s is the object surface temperature, and T_i is (a) the initial object temperature for the interior cases and (b) the temperature of the infinite medium for the exterior cases.

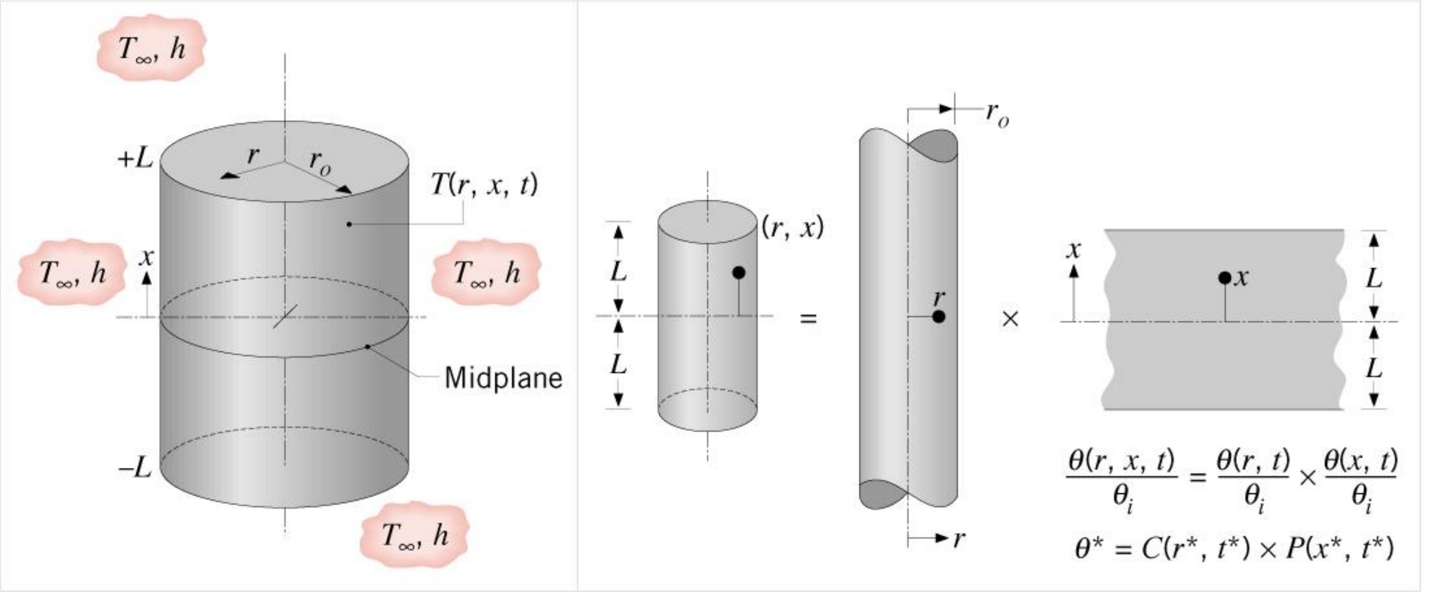
TABLE 5.2b Summary of transient heat transfer results for constant surface heat flux cases[a]

Geometry	Length Scale, L_c	$q^*(Fo)$ Exact Solutions		Approximate Solutions $Fo < 0.2$	Approximate Solutions $Fo \geq 0.2$	Maximum Error (%)
Semi-infinite	L (arbitrary)	$\frac{1}{2}\sqrt{\frac{\pi}{Fo}}$		Use exact solution.		none
Interior Cases						
Plane wall of thickness $2L$	L	$\left[Fo + \frac{1}{3} - 2\sum_{n=1}^{\infty}\frac{\exp(-\zeta_n^2 Fo)}{\zeta_n^2}\right]^{-1}$	$\zeta_n = n\pi$	$\frac{1}{2}\sqrt{\frac{\pi}{Fo}}$	$\left[Fo + \frac{1}{3}\right]^{-1}$	5.3
Infinite cylinder	r_o	$\left[2Fo + \frac{1}{4} - 2\sum_{n=1}^{\infty}\frac{\exp(-\zeta_n^2 Fo)}{\zeta_n^2}\right]^{-1}$	$J_1(\zeta_n) = 0$	$\frac{1}{2}\sqrt{\frac{\pi}{Fo}} - \frac{\pi}{8}$	$\left[2Fo + \frac{1}{4}\right]^{-1}$	2.1
Sphere	r_o	$\left[3Fo + \frac{1}{5} - 2\sum_{n=1}^{\infty}\frac{\exp(-\zeta_n^2 Fo)}{\zeta_n^2}\right]^{-1}$	$\tan(\zeta_n) = \zeta_n$	$\frac{1}{2}\sqrt{\frac{\pi}{Fo}} - \frac{\pi}{4}$	$\left[3Fo + \frac{1}{5}\right]^{-1}$	4.5
Exterior Cases						
Sphere	r_o	$[1 - \exp(Fo)\mathrm{erfc}(Fo^{1/2})]^{-1}$		$\frac{1}{2}\sqrt{\frac{\pi}{Fo}} + \frac{\pi}{4}$	$\frac{0.77}{\sqrt{Fo}} + 1$	3.2
Various shapes (Table 4.1, cases 12–15)	$(A_s/4\pi)^{1/2}$	none		$\frac{1}{2}\sqrt{\frac{\pi}{Fo}} + \frac{\pi}{4}$	$\frac{0.77}{\sqrt{Fo}} + q^*_{ss}$	unknown

[a] $q^* \equiv q_s'' L_c/k(T_s - T_i)$ and $Fo \equiv \alpha t/L_c^2$, where L_c is the length scale given in the table, T_s is the object surface temperature, and T_i is (a) the initial object temperature for the interior cases and (b) the temperature of the infinite medium for the exterior cases.

Efeitos Multidimensionais

- Superposição de soluções : método de separação de vairiáveis:



Transient Conduction: Finite-Difference Equations and Solutions

Não será cobrado

The Finite-Difference Method

- An approximate method for determining temperatures at discrete (nodal) points of the physical system and at discrete times during the transient process.

- Procedure:
  - Represent the physical system by a nodal network, with an m, n notation used to designate the location of discrete points in the network, and discretize the problem in time by designating a time increment Δt and expressing the time as $t = p\Delta t$, where p assumes integer values, ($p = 0, 1, 2, \ldots$).

  - Use the energy balance method to obtain a finite-difference equation for each node of unknown temperature.

  - Solve the resulting set of equations for the nodal temperatures at $t = \Delta t, 2\Delta t, 3\Delta t, \ldots$, until steady-state is reached.

What is represented by the temperature, $T_{m,n}^{p}$?

Energy Balance and Finite-Difference Approximation for the Storage Term

- For any nodal region, the energy balance is

$$\dot{E}_{in} + \dot{E}_g = \dot{E}_{st} \qquad (5.81)$$

where, according to convention, all heat flow is assumed to be into the region.

- Discretization of temperature variation with time:

$$\left.\frac{\partial T}{\partial t}\right|_{m,n} \approx \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} \qquad (5.74)$$

- Finite-difference form of the storage term:

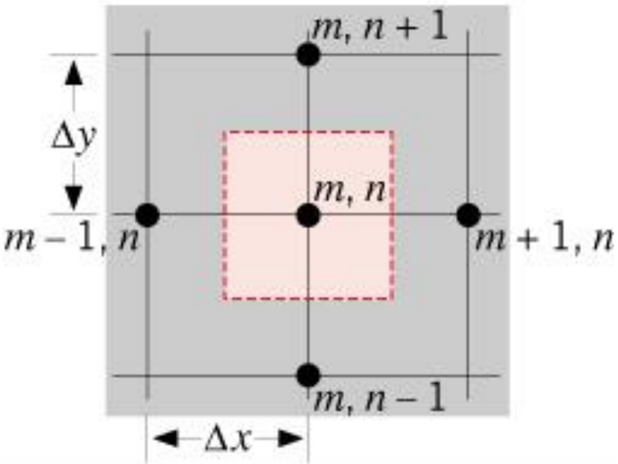
$$\dot{E}_{st(m,n)} = \rho \forall c \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t}$$

- Existence of two options for the time at which all other terms in the energy balance are evaluated: p or $p+1$.

The Explicit Method of Solution

- All other terms in the energy balance are evaluated at the preceding time corresponding to p. Equation (5.74) is then termed a *forward-difference approximation*.

- Example: Two-dimensional conduction for an interior node with $\Delta x = \Delta y$.



$$T_{m,n}^{p+1} = Fo\left(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}\right) + (1-4Fo)T_{m,n}^{p} \qquad (5.76)$$

$$Fo = \frac{\alpha \Delta t}{(\Delta x)^2} \rightarrow \text{finite-difference form of Fourier number}$$

- Unknown nodal temperatures at the new time, $t = (p+1)\Delta t$, are determined exclusively by known nodal temperatures at the preceding time, $t = p\Delta t$, hence the term explicit solution.

- How is solution accuracy affected by the choice of Δx and Δt?
- Do other factors influence the choice of Δt?
- What is the nature of an unstable solution?
- *Stability criterion*: *Determined by requiring the coefficient for the node of interest at the previous time to be greater than or equal to zero.*

For a finite-difference equation of the form,

$$T_{m,n}^{p+1} = \dots\dots\dots\dots\dots\dots + AT_{m,n}^{p}$$

$$A \geq 0$$

Hence, for the two-dimensional interior node:

$$(1-4Fo) \geq 0$$

$$Fo \leq 1/4$$

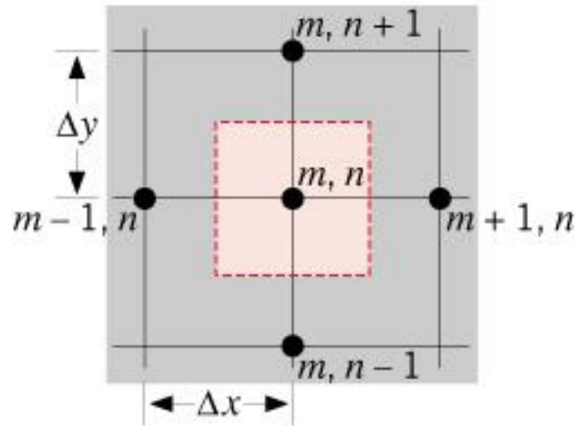
$$\Delta t \leq \frac{(\Delta x)^2}{4\alpha}$$

- Table 5.3 → finite-difference equations for other common nodal regions.

The Implicit Method of Solution

- All other terms in the energy balance are evaluated at the new time corresponding to $p+1$. Equation (5.74) is then termed a *backward-difference approximation.*

- Example: Two-dimensional conduction for an interior node with $\Delta x = \Delta y$.



$$\left(1+4Fo\right)T_{m,n}^{p+1} - Fo\left(T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}\right) = T_{m,n}^{p} \qquad (5.92)$$

- System of N finite-difference equations for N unknown nodal temperatures may be solved by matrix inversion or Gauss-Seidel iteration.

- Solution is unconditionally stable.

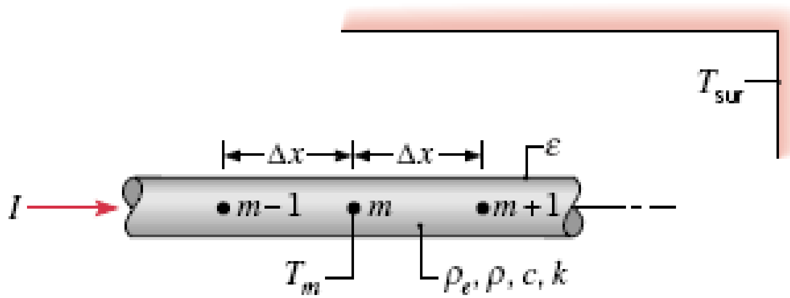
Table 5.3 $\rightarrow$ finite-difference equations for other common nodal regions.

Marching Solution

- Transient temperature distribution is determined by a marching solution. beginning with known initial conditions.

	p	t	T_1	T_2	T_3	………………	T_N
Known →	0	0	$T_{1,i}$	$T_{2,i}$	$T_{3,i}$	………………	$T_{N,i}$
	1	Δt	--	--	--	……………	--
	2	$2\Delta t$	--	--	--	……………	--
	3	$3\Delta t$	--	--	--	……………	--
	.		.				
	.		.				
	.		.				
	.		.				
	.		.				
	.		.				
	Steady-state	--	--	--	--	……………	--

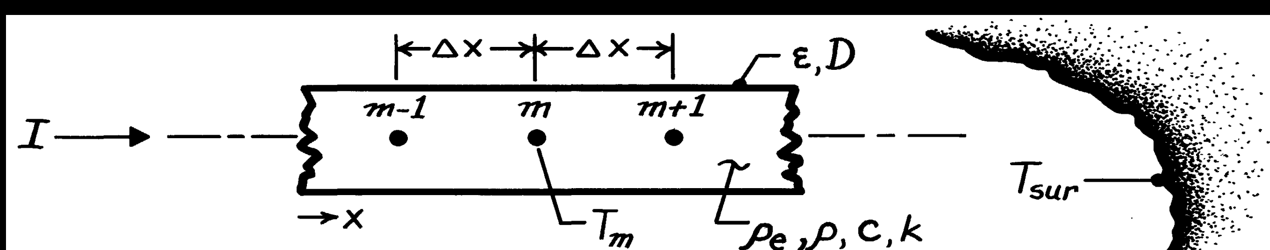
Problem 5.93: Derivation of explicit form of finite-difference equation for a nodal point in a thin, electrically conducting rod confined by a vacuum enclosure.



KNOWN: Thin rod of diameter D, initially in equilibrium with its surroundings, T_{sur}, suddenly passes a current I; rod is in vacuum enclosure and has prescribed electrical resistivity, ρ_e, and other thermophysical properties.

FIND: Transient, finite-difference equation for node m.

SCHEMATIC:



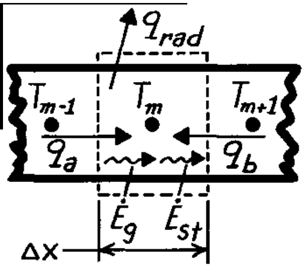
Problem: Finite-Difference Equation

ASSUMPTIONS: (1) One-dimensional, transient conduction in rod, (2) Surroundings are much larger than rod, (3) Constant properties.

ANALYSIS: Applying conservation of energy to a nodal region of volume $\forall = A_c \Delta x$, where $A_c = \pi D^2/4$,

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

Hence, with $\dot{E}_g = I^2 R_e$, where $R_e = \rho_e \Delta x / A_c$, and use of the forward-difference representation for the time derivative,



$$q_a + q_b - q_{rad} + I^2 R_e = \rho\, cV \frac{T_m^{p+1} - T_m^p}{\Delta t}$$

$$kA_c \frac{T_{m-1}^p - T_m^p}{\Delta x} + kA_c \frac{T_{m+1}^p - T_m^p}{\Delta x} - \varepsilon \pi D \Delta x \sigma \left[\left(T_m^p \right)^4 - T_{sur}^4 \right]^4 + I^2 \frac{\rho_e \Delta x}{A_c} = \rho\, c A_c \Delta x \frac{T_m^{p+1} - T_m^p}{\Delta t}.$$

Dividing each term by $\rho c A_c \Delta x / \Delta t$ and solving for T_m^{p+1},

$$T_m^{p+1} = \frac{k}{\rho\, c} \cdot \frac{\Delta t}{\Delta x^2} \left(T_{m-1}^p + T_{m+1}^p \right) - \left[2 \cdot \frac{k}{\rho\, c} \cdot \frac{\Delta t}{\Delta x^2} - 1 \right] T_m^p$$

$$- \frac{\varepsilon P \sigma}{A_c} \cdot \frac{\Delta t}{\rho\, c} \left[\left(T_m^p \right)^4 - T_{sur}^4 \right] + \frac{I^2 \rho_e}{A_c^2} \cdot \frac{\Delta t}{\rho\, c}.$$

Problem: Finite-Difference Equation

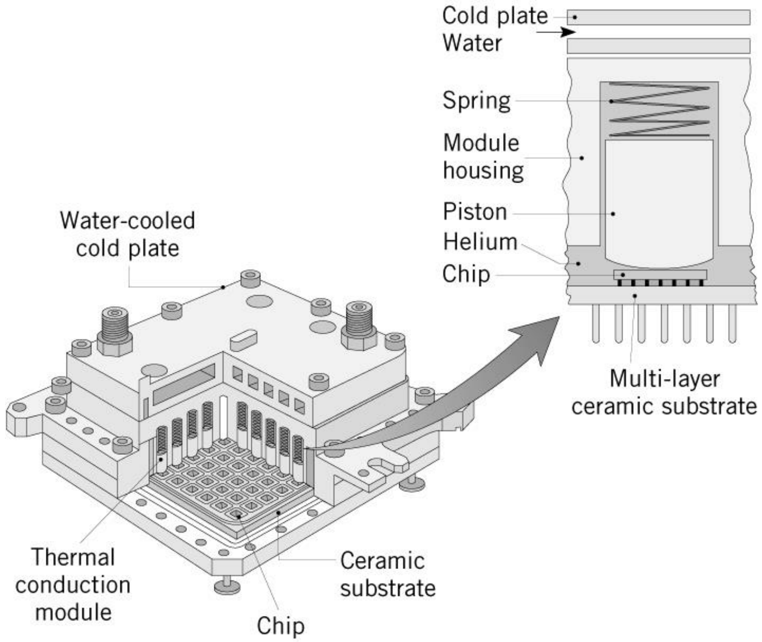
or, with Fo = $\alpha\, \Delta t/\Delta x^2$,

$$T_m^{p+1} = Fo\left(T_{m-1}^p + T_{m+1}^p\right) + \left(1 - 2\ Fo\right)T_m^p - \frac{\varepsilon P \sigma \Delta x^2}{kA_c} \cdot Fo\left[\left(T_m^p\right)^4 - T_{sur}^4\right] + \frac{I^2 \rho_e \Delta x^2}{kA_c^2} \cdot Fo.$$

Basing the stability criterion on the coefficient of the T_m^p term, it would follow that Fo ≤ ½.

However, stability is also affected by the nonlinear term, $\left(T_m^p\right)^4$, and smaller values of Fo may be needed to insure its existence.

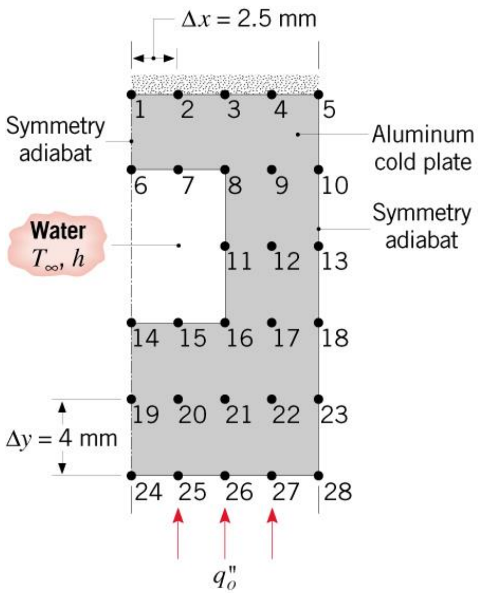
Problem 5.127: Use of implicit finite-difference method with a time interval of $\Delta t = 0.1s$ to determine transient response of a water-cooled cold plate attached to IBM multi-chip thermal conduction module.



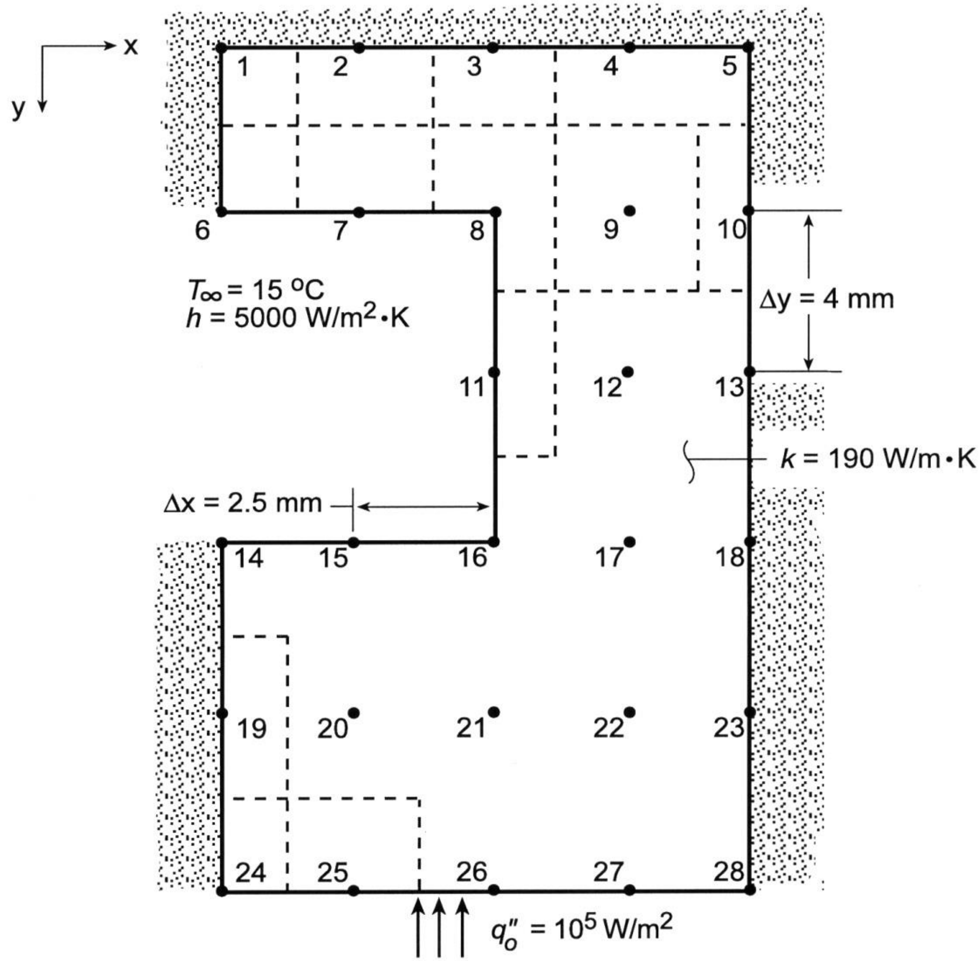
Features:

- Cold plate is at a uniform temperature, T_i=15°C, when a uniform heat flux of is applied to its base due to activation of chips.

- During the transient process, heat transfer into the cold plate increases its thermal energy while providing for heat transfer by convection to the water . Steady state is reached when .



Problem: Cold Plate (cont.)



Problem: Cold Plate (cont.)

ANALYSIS:

Nodes 1 and 5:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_1^{p+1}-\frac{2\alpha\Delta t}{\Delta x^2}T_2^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_6^{p+1}=T_1^p$$

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_5^{p+1}-\frac{2\alpha\Delta t}{\Delta x^2}T_4^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{10}^{p+1}=T_5^p$$

Nodes 2, 3, 4:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{m,n}^{p+1}-\frac{\alpha\Delta t}{\Delta x^2}T_{m-1,n}^{p+1}-\frac{\alpha\Delta t}{\Delta x^2}T_{m+1,n}^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{m,n-1}^{p+1}=T_{m,n}^p$$

Nodes 6 and 14:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2h\alpha\Delta t}{k\Delta y}\right)T_6^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_1^{p+1}-\frac{2\alpha\Delta t}{\Delta x^2}T_7^{p+1}=\frac{2h\alpha\Delta t}{k\Delta y}T_\infty+T_6^p$$

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2h\alpha\Delta t}{k\Delta y}\right)T_{14}^{p+1}-\frac{2\alpha\Delta t}{\Delta x^2}T_{15}^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{19}^{p+1}=\frac{2h\alpha\Delta t}{k\Delta y}T_\infty+T_{14}^p$$

Nodes 7 and 15:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2h\alpha\Delta t}{k\Delta y}\right)T_7^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_2^{p+1}-\frac{\alpha\Delta t}{\Delta x^2}T_6^{p+1}-\frac{\alpha\Delta t}{k\Delta x^2}T_8^{p+1}=\frac{2h\alpha\Delta t}{k\Delta y}T_\infty+T_7^p$$

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2h\alpha\Delta t}{k\Delta y}\right)T_{15}^{p+1}-\frac{\alpha\Delta t}{\Delta x^2}T_{14}^{p+1}-\frac{\alpha\Delta t}{\Delta x^2}T_{16}^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{20}^{p+1}=\frac{2h\alpha\Delta t}{k\Delta y}T_\infty+T_{15}^p$$

Nodes 8 and 16:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2}{3}\ \frac{h\alpha\Delta t}{k\Delta x}+\frac{2}{3}\frac{h\alpha\Delta t}{k\Delta y}\right)T_8^{p+1}-\frac{4}{3}\frac{\alpha\Delta t}{\Delta y^2}T_3^{p+1}-\frac{2}{3}\frac{\alpha\Delta t}{\Delta x^2}T_7^{p+1}$$
$$-\frac{4}{3}\frac{\alpha\Delta t}{\Delta x^2}T_9^{p+1}-\frac{2}{3}\frac{\alpha\Delta t}{\Delta y^2}T_{11}^{p+1}=\frac{2}{3}\frac{h\alpha\Delta t}{k}\left(\frac{1}{\Delta x}+\frac{1}{\Delta y}\right)T_\infty+T_8^p$$

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2}{3}+\frac{h\alpha\Delta t}{k\Delta x}+\frac{2}{3}\frac{h\alpha\Delta t}{k\Delta y}\right)T_{16}^{p+1}-\frac{2}{3}\frac{\alpha\Delta t}{\Delta y^2}T_{11}^{p+1}-\frac{2}{3}\frac{\alpha\Delta t}{\Delta x^2}T_{15}^{p+1}$$
$$-\frac{4}{3}\frac{\alpha\Delta t}{\Delta x^2}T_{17}^{p+1}-\frac{4}{3}\frac{\alpha\Delta t}{\Delta y^2}T_{21}^{p+1}=\frac{2}{3}\frac{h\alpha\Delta t}{k}\left(\frac{1}{\Delta x}+\frac{1}{\Delta y}\right)T_\infty+T_{16}^p$$

Problem: Cold Plate (cont.)

Node 11:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}+\frac{2h\alpha\Delta t}{k\Delta x}\right)T_{11}^{p+1}-\frac{\alpha\Delta t}{\Delta y^2}T_{8}^{p+1}-2\alpha\frac{\Delta t}{\Delta x^2}T_{12}^{p+1}-\frac{\alpha\Delta t}{\Delta y^2}T_{16}^{p+1}=\frac{2h\alpha\Delta t}{k\Delta x}T_{\infty}+T_{11}^{p}$$

Nodes 9, 12, 17, 20, 21, 22:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{m,n}^{p+1}-\frac{\alpha\Delta t}{\Delta y^2}\left(T_{m,n+1}^{p+1}+T_{m,n-1}^{p+1}\right)-\frac{\alpha\Delta t}{\Delta x^2}\left(T_{m-1,n}^{p+1}+T_{m+1,n}^{p+1}\right)=T_{m,n}^{p}$$

Nodes 10, 13, 18, 23:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{m,n}^{p+1}-\frac{\alpha\Delta t}{\Delta y^2}\left(T_{m,n+1}^{p+1}+T_{m,n-1}^{p+1}\right)-\frac{2\alpha\Delta t}{\Delta x^2}T_{m-1,n}^{p+1}=T_{m,n}^{p}$$

Node 19:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{19}^{p+1}-\frac{\alpha\Delta t}{\Delta y^2}\left(T_{14}^{p+1}+T_{24}^{p+1}\right)-\frac{2\alpha\Delta t}{\Delta x^2}T_{20}^{p+1}=T_{19}^{p}$$

Nodes 24, 28:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{24}^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{19}^{p+1}-\frac{2\alpha\Delta t}{\Delta x^2}T_{25}^{p+1}=\frac{2q_o''\alpha\Delta t}{k\Delta y}+T_{24}^{p}$$

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{28}^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{23}^{p+1}-\frac{2\alpha\Delta t}{\Delta x^2}T_{27}^{p+1}=\frac{2q_o''\alpha\Delta t}{k\Delta y}+T_{28}^{p}$$

Problem: Cold Plate (cont.)

Nodes 25, 26, 27:

$$\left(1+\frac{2\alpha\Delta t}{\Delta x^2}+\frac{2\alpha\Delta t}{\Delta y^2}\right)T_{m,n}^{p+1}-\frac{2\alpha\Delta t}{\Delta y^2}T_{m,n+1}^{p+1}-\frac{\alpha\Delta t}{\Delta x^2}\left(T_{m-1,n}^{p+1}+T_{m+1,n}^{p+1}\right)=\frac{2q_o''\alpha\Delta t}{k\Delta y}+T_{m,n}^{p+1}$$

The convection heat rate per unit length is

$$q'_{conv}=h\left[(\Delta x/2)(T_6-T_\infty)+\Delta x(T_7-T_\infty)+(\Delta x+\Delta y)(T_8-T_\infty)/2+\Delta y(T_{11}-T_\infty)+(\Delta x+\Delta y)(T_{16}-T_\infty)/2+\Delta x(T_{15}-T_\infty)+(\Delta x/2)(T_{14}-T_\infty)\right]=q'_{out}.$$

The heat input per unit length is

$$q'_{in}=q_o''(4\Delta x)$$

On a percentage basis, the ratio of convection to heat in is

$$n\equiv\left(q'_{conv}/q'_{in}\right)\times 100.$$

Results of the calculations (in °C) are as follows:

Time: 5.00 s; n = 60.57%

19.612	19.712	19.974	20.206	20.292
19.446	19.597	20.105	20.490	20.609
		21.370	21.647	21.730
24.217	24.074	23.558	23.494	23.483
25.658	25.608	25.485	25.417	25.396
27.581	27.554	27.493	27.446	27.429

Time: 10.00 s; n = 85.80%

22.269	22.394	22.723	23.025	23.137
21.981	22.167	22.791	23.302	23.461
		24.143	24.548	24.673
27.216	27.075	26.569	26.583	26.598
28.898	28.851	28.738	28.690	28.677
30.901	30.877	30.823	30.786	30.773

Time: 15.00 s; n = 94.89%

23.228	23.363	23.716	24.042	24.165
22.896	23.096	23.761	24.317	24.491
		25.142	25.594	25.733
28.294	28.155	27.652	27.694	27.719
30.063	30.018	29.908	29.867	29.857
32.095	32.072	32.021	31.987	31.976

Time: 20.00 s; n = 98.16%

23.574	23.712	24.073	24.409	24.53
23.226	23.430	24.110	24.682	24.86
		25.502	25.970	26.11
28.682	28.543	28.042	28.094	28.12
30.483	30.438	30.330	30.291	30.28
32.525	32.502	32.452	32.419	32.40

Time: 23.00 s; n = 99.00%

23.663	23.802	24.165	24.503	24.63
23.311	23.516	24.200	24.776	24.95
		25.595	26.067	26.21
28.782	28.644	28.143	28.198	28.22
30.591	30.546	30.438	30.400	30.39
32.636	32.613	32.563	32.531	32.52

Temperatures at t = 23 s are everywhere within 0.13°C of the final steady-state values