APPENDIX A HOW TO READ THEOREMS

Since many of the most important concepts in linear algebra occur as theorem statements, it is important to be familiar with the various ways in which theorems can be structured. This appendix will help you to do that.

CONTRAPOSITIVE FORM OF A THEOREM

The simplest theorems are of the form

If H is true, then C is true. (1)

where H is a statement, called the ***hypothesis***, and C is a statement, called the ***conclusion***. *The theorem is true if the conclusion is true whenever the hypothesis is true, and the theorem is false if there is some case where the hypothesis is true but the conclusion is false.* It is common to denote a theorem of form (1) as

$$H \Rightarrow C \tag{2}$$

(read, "H implies C"). As an example, the theorem

If a and b are both positive numbers, then ab is a positive number. (3)

is of form (2), where

$H = a$ and b are both positive numbers (4)

$C = ab$ is a positive number (5)

Sometimes it is desirable to phrase theorems in a *negative* way. For example, the theorem in (3) can be rephrased equivalently as

If ab is not a positive number, then a and b are not both positive numbers. (6)

If we write $\sim H$ to mean that (4) is false and $\sim C$ to mean that (5) is false, then the structure of the theorem in (6) is

$$\sim C \Rightarrow \sim H \tag{7}$$

In general, any theorem of form (2) can be rephrased in form (7), which is called the ***contrapositive*** of (2). If a theorem is true, then so is its contrapositive, and vice versa.

CONVERSE OF A THEOREM

The ***converse*** of a theorem is the statement that results when the hypothesis and conclusion are interchanged. Thus, the converse of the theorem $H \Rightarrow C$ is the statement $C \Rightarrow H$. Whereas the contrapositive of a true theorem must itself be a true theorem, the converse of a true theorem may or may not be true. For example, the converse of (3) is the *false* statement

If ab is a positive number, then a and b are both positive numbers.

but the converse of the true theorem

If $a > b$, then $2a > 2b$. (8)

is the *true* theorem

If $2a > 2b$, then $a > b$. (9)

EQUIVALENT STATEMENTS

If a theorem $H \Rightarrow C$ and its converse $C \Rightarrow H$ are both true, then we say that H and C are ***equivalent*** statements, which we denote by writing

$$H \Leftrightarrow C$$

(read, "H and C are equivalent"). There are various ways of phrasing equivalent statements as a single theorem. Here are three ways in which (8) and (9) can be combined into a single theorem.

Form 1 If $a > b$, then $2a > 2b$, and conversely, if $2a > 2b$, then $a > b$.

Form 2 $a > b$ if and only if $2a > 2b$.

Form 3 The following statements are equivalent.

(i) $a > b$
(ii) $2a > 2b$

THEOREMS INVOLVING THREE OR MORE STATEMENTS

Sometimes two true theorems will give you a third true theorem for free. Specifically, if $H \Rightarrow C$ is a true theorem, and $C \Rightarrow D$ is a true theorem, then $H \Rightarrow D$ must also be a true theorem. For example, the theorems

If opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.

and

Opposite sides of a parallelogram have equal lengths.

imply the third theorem

If opposite sides of a quadrilateral are parallel, then they have equal lengths.

Sometimes three theorems yield equivalent statements for free. For example, if

$$H \Rightarrow C, \quad C \Rightarrow D, \quad D \Rightarrow H \tag{10}$$

then we have the *implication loop* in Figure A.1 from which we can conclude that

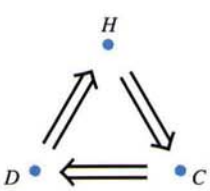

Figure A.1

$$C \Rightarrow H, \quad D \Rightarrow C, \quad H \Rightarrow D \tag{11}$$

Combining this with (10) we obtain

$$H \Leftrightarrow C, \quad C \Leftrightarrow D, \quad D \Leftrightarrow H \tag{12}$$

In summary, if you want to prove the three equivalences in (12), you need only prove the three implications in (10).