Section 8.3 Orthogonal Diagonalizability; Functions of a Matrix

ORTHOGONAL SIMILARITY

Recall from the last section that two $n \times n$ matrices A and C are said to be *similar* if there exists an invertible matrix P such that $C = P^{-1}AP$. The special case in which there is an orthogonal matrix P such that $C = P^{-1}AP = P^TAP$ is of special importance and has some terminology associated with it.

Definition 8.3.1 If A and C are square matrices with the same size, then we say that ***C is orthogonally similar to A*** if there exists an orthogonal matrix P such that $C = P^TAP$.

Theorem 8.3.2 *Two matrices are orthogonally similar if and only if there exist orthonormal bases with respect to which the matrices represent the same linear operator.*

This is the analog of Theorem 8.2.2.

The Orthogonal Diagonalization Problem Given a square matrix A, does there exist an orthogonal matrix P for which $P^T A P$ is a diagonal matrix, and if so, how does one find such a P? If such a matrix P exists, then A is said to be ***orthogonally diagonalizable***, and P is said to ***orthogonally diagonalize*** A.

REMARK If you think of A as the standard matrix for a linear operator, then the orthogonal diagonalization problem is equivalent to asking whether this operator can be represented by a diagonal matrix with respect to some orthonormal basis.

The first observation we should make about orthogonal diagonalization is that there is no hope of orthogonally diagonalizing a *nonsymmetric* matrix. To see why this is so, suppose that

$$D = P^T A P \tag{1}$$

where P is orthogonal and D is diagonal. Since $P^T P = PP^T = I$, we can rewrite (1) as

$$A = PDP^T$$

Transposing both sides of this equation and using the fact that $D^T = D$ yields

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$$

which shows that an orthogonally diagonalizable matrix must be symmetric.

Of course, this still leaves open the question of which symmetric matrices, if any, are orthogonally diagonalizable. The following analog of Theorem 8.2.6 will help us to answer this question.

Theorem 8.3.3 *An $n \times n$ matrix A is orthogonally diagonalizable if and only if there exists an orthonormal set of n eigenvectors of A.*

Proof We will show first that if A is orthogonally diagonalizable, then there exists an orthonormal set of n eigenvectors of A. The orthogonal diagonalizability of A implies that there exists an orthogonal matrix P and a diagonal matrix D such that $P^TAP = D$.

However, since the column vectors of an orthogonal matrix are orthonormal, and since the column vectors of P are eigenvectors of A (see the proof of Theorem 8.2.6), we have established that the column vectors of P form an orthonormal set of n eigenvectors of A.

Conversely, assume that there exists an orthonormal set $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ of n eigenvectors of A. We showed in the proof of Theorem 8.2.6 that the matrix

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$$

diagonalizes A. However, in this case P is an orthogonal matrix, since its column vectors are orthonormal. Thus, P orthogonally diagonalizes A. ■

REMARK Recalling that an orthonormal set of n vectors in R^n is an orthonormal basis for R^n, Theorem 8.3.3 is equivalent to saying that *an* $n \times n$ *matrix* A *is orthogonally diagonalizable if and only if there is an orthonormal basis for* R^n *consisting of eigenvectors of* A.

Theorem 8.3.4

(a) *A matrix is orthogonally diagonalizable if and only if it is symmetric.*

(b) *If* A *is a symmetric matrix, then eigenvectors from different eigenspaces are orthogonal.*

We will prove part (b); the proof of part (a) is outlined in the exercises.

Proof (b) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2, respectively. The proof that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ will be facilitated by using Formula (26) of Section 3.1 to write $\lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2) = (\lambda_1\mathbf{v}_1) \cdot \mathbf{v}_2$ as the matrix product $(\lambda_1\mathbf{v}_1)^T\mathbf{v}_2$. The rest of the proof now consists of manipulating this expression in the right way:

$$
\begin{aligned}
\lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2) = (\lambda_1\mathbf{v}_1)^T\mathbf{v}_2 = (A\mathbf{v}_1)^T\mathbf{v}_2 &= (\mathbf{v}_1^T A^T)\mathbf{v}_2 && [\mathbf{v}_1 \text{ is an eigenvector corresponding to } \lambda_1.] \\
&= (\mathbf{v}_1^T A)\mathbf{v}_2 && [\text{Symmetry of } A] \\
&= \mathbf{v}_1^T(A\mathbf{v}_2) \\
&= \mathbf{v}_1^T(\lambda_2\mathbf{v}_2) && [\mathbf{v}_2 \text{ is an eigenvector corresponding to } \lambda_2.] \\
&= \lambda_2\mathbf{v}_1^T\mathbf{v}_2 \\
&= \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2) && [\text{Formula (26) of Section 3.1}]
\end{aligned}
$$

This implies that $(\lambda_1 - \lambda_2)(\mathbf{v}_1 \cdot \mathbf{v}_2) = 0$, so $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ as a result of the fact that $\lambda_1 \neq \lambda_2$. ■

$$\boxed{\mathbf{u}^T\mathbf{v} = \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = \mathbf{v}^T\mathbf{u} \qquad (26)}$$

A METHOD FOR ORTHOGONALLY DIAGONALIZING A SYMMETRIC MATRIX

Orthogonally Diagonalizing an $n \times n$ Symmetric Matrix

Step 1. Find a basis for each eigenspace of A.

Step 2. Apply the Gram–Schmidt process to each of these bases to produce orthonormal bases for the eigenspaces.

Step 3. Form the matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$ whose columns are the vectors constructed in Step 2. The matrix P will orthogonally diagonalize A, and the eigenvalues on the diagonal of $D = P^T AP$ will be in the same order as their corresponding eigenvectors in P.

EXAMPLE 1

Orthogonally Diagonalizing a Symmetric Matrix

Find a matrix P that orthogonally diagonalizes the symmetric matrix

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Solution The characteristic equation of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 4 & -2 & -2 \\ -2 & \lambda - 4 & -2 \\ -2 & -2 & \lambda - 4 \end{bmatrix} = (\lambda - 2)^2(\lambda - 8) = 0 \qquad (2)$$

Thus, the eigenvalues of A are $\lambda = 2$ and $\lambda = 8$.

Using the method given in Example 3 of Section 8.2, it can be shown that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

form a basis for the eigenspace corresponding to $\lambda = 2$ and that

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{4}$$

is a basis for the eigenspace corresponding to $\lambda = 8$.

It is applied to each basis!

Applying the Gram–Schmidt process to the bases $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_3\}$ yields the orthonormal bases $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Thus, A is orthogonally diagonalized by the matrix

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

As a check, we leave it for you to confirm that

$$P^TAP = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \quad \blacksquare$$

SPECTRAL DECOMPOSITION

If A is a symmetric matrix that is orthogonally diagonalized by

$$P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]$$

and if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A corresponding to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$, then we know that $D = P^TAP$, where D is a diagonal matrix with the eigenvalues in the diagonal positions. It follows from this that the matrix A can be expressed as

$$A = PDP^T = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} = [\lambda_1\mathbf{u}_1 \quad \lambda_2\mathbf{u}_2 \quad \cdots \quad \lambda_n\mathbf{u}_n] \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

$$A = PDP^T = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} = [\lambda_1\mathbf{u}_1 \quad \lambda_2\mathbf{u}_2 \quad \cdots \quad \lambda_n\mathbf{u}_n] \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

Multiplying out using the column-row rule (Theorem 3.8.1), we obtain the formula

$$A = \lambda_1\mathbf{u}_1\mathbf{u}_1^T + \lambda_2\mathbf{u}_2\mathbf{u}_2^T + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^T \tag{5}$$

which is called a ***spectral decomposition of A*** or an ***eigenvalue decomposition of A*** (sometimes abbreviated as the EVD of A).*

spectrum: set of all eigenvalues

REMARK The spectral decomposition (5) expresses a symmetric matrix A as a linear combination of rank 1 matrices in which the coefficients of the matrices are the eigenvalues of A.

EXAMPLE 2 A Geometric Interpretation of the Spectral Decomposition

The matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$ with corresponding eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(verify). Normalizing these basis vectors yields

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

so a spectral decomposition of A is

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T = (-3) \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} + (2) \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$= (-3) \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix} + (2) \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} \tag{6}$$

where the 2×2 matrices on the right are the standard matrices for the orthogonal projections onto the eigenspaces.

See Theorem 7.7.3.

symmetric, rank 1.

Now let us see what this decomposition tells us about the image of the vector $\mathbf{x} = (1, 1)$ under multiplication by A. Writing $\mathbf{x}$ in column form, it follows that

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \tag{7}$$

and from (6) that

$$\begin{aligned} A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= (-3) \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (2) \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= (-3) \begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{bmatrix} + (2) \begin{bmatrix} \frac{6}{5} \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ -\frac{6}{5} \end{bmatrix} + \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix} \end{aligned} \tag{8}$$

$$Ax = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad (7)$$

$$Ax = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-3) \begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{bmatrix} + (2) \begin{bmatrix} \frac{6}{5} \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ -\frac{6}{5} \end{bmatrix} + \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix} \quad (8)$$

It follows from (7) that the image of (1, 1) under multiplication by A is (3, 0), and it follows from (8) that this image can also be obtained by projecting (1, 1) onto the eigenspaces corresponding to $\lambda_1 = -3$ and $\lambda_2 = 2$ to obtain the vectors $\left(-\frac{1}{5}, \frac{2}{5}\right)$ and $\left(\frac{6}{5}, \frac{3}{5}\right)$, then scaling by the eigenvalues to obtain $\left(\frac{3}{5}, -\frac{6}{5}\right)$ and $\left(\frac{12}{5}, \frac{6}{5}\right)$, and then adding these vectors (see Figure 8.3.1). ■

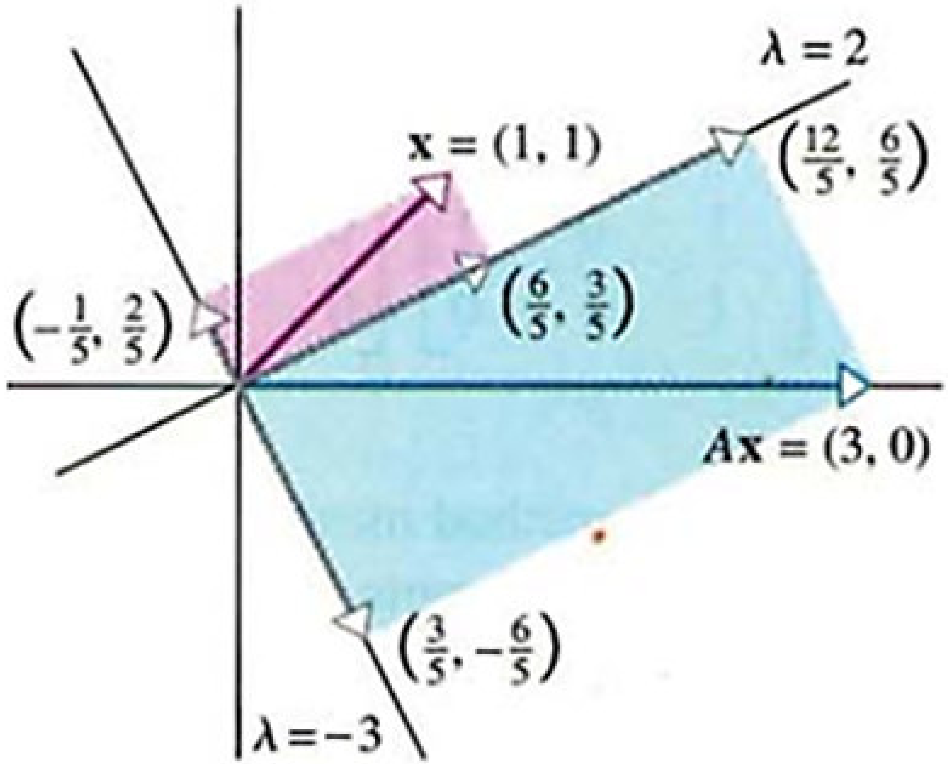

Figure 8.3.1

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T \tag{5}$$

REMARK The spectral decomposition (5) expresses a symmetric matrix A as a linear combination of rank 1 matrices in which the coefficients of the matrices are the eigenvalues of A.

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis for R^n, and if A can be expressed as

$$A = c_1 \mathbf{u}_1 \mathbf{u}_1^T + c_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + c_n \mathbf{u}_n \mathbf{u}_n^T$$

then A is symmetric and has eigenvalues $c_1, c_2, \ldots, c_n$.

POWERS OF A DIAGONALIZABLE MATRIX

There are many applications that require the computation of high powers of square matrices. Since such computations can be time consuming and subject to roundoff error, there is considerable interest in techniques that can reduce the amount of computation involved.

We will now consider an important method for computing high powers of diagonalizable matrices (symmetric matrices, for example). To explain the idea, suppose that A is an $n \times n$ matrix and P is an invertible $n \times n$ matrix.

A is an $n \times n$ matrix and P is an invertible $n \times n$ matrix. Then

$$(P^{-1}AP)^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}APP^{-1}AP = P^{-1}AIAP = P^{-1}A^2P$$

and more generally, if k is any positive integer, then

$$(P^{-1}AP)^k = P^{-1}A^kP \tag{9}$$

In particular, if A is diagonalizable and $P^{-1}AP = D$ is a diagonal matrix, then it follows from (9) that

$$P^{-1}A^kP = D^k \tag{10}$$

which we can rewrite as

$$A^k = PD^kP^{-1} \tag{11}$$

EXAMPLE 3

Powers of a Diagonalizable Matrix

Use Formula (11) to find A^{13} for the diagonalizable matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution We showed in Example 4 of Section 8.2 that

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Thus,

$$A^{13} = \underbrace{\begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 2^{13} \end{bmatrix}}_{D^{13}} \underbrace{\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}}_{P^{-1}} = \begin{bmatrix} -8190 & 0 & -16{,}382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16{,}383 \end{bmatrix} \quad (12)$$

With this method most of the work is diagonalizing A. Once that work is done, it need not be repeated to compute other powers of A. For example, to compute A^{1000} we need only change the exponents from 13 to 1000 in (12). ■

In the special case where A is a symmetric matrix with a spectral decomposition

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$$

the matrix

$$P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]$$

orthogonally diagonalizes A, so (11) can be expressed as

$$A^k = PD^k P^T$$

We leave it for you to show that this equation can be written as

$$A^k = \lambda_1^k \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2^k \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n^k \mathbf{u}_n \mathbf{u}_n^T \qquad \longrightarrow \quad \text{see Eq. (5).} \tag{13}$$

from which it follows that A^k is a symmetric matrix whose eigenvalues are the kth powers of the eigenvalues of A.

Theorem 8.3.5 (*Cayley–Hamilton Theorem*) *Every square matrix satisfies its characteristic equation; that is, if A is an $n \times n$ matrix whose characteristic equation is*

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

then

$$A^n + c_1 A^{n-1} + \cdots + c_n I = 0 \tag{14}$$

The Cayley–Hamilton theorem makes it possible to express all positive integer powers of an $n \times n$ matrix A in terms of $I, A, \ldots, A^{n-1}$ by solving (14) for A^n.

In the case where A is invertible, it also makes it possible to express A^{-1} (and hence all negative powers of A) in terms of $I, A, \ldots, A^{n-1}$ by rewriting (14) as

$$A\left(-\frac{1}{c_n}A^{n-1} - \frac{c_1}{c_n}A^{n-2} - \cdots - \frac{c_{n-1}}{c_n}I\right) = I \tag{15}$$

(verify), from which it follows that A^{-1} is the parenthetical expression on the left.

REMARK We are guaranteed that $c_n \neq 0$ in (15), for otherwise $\lambda = 0$ would be a root of the characteristic equation, contradicting the invertibility of A [see parts (c) and (h) of Theorem 7.4.4].

EXAMPLE 4

We showed in Example 2 of Section 8.2 that the characteristic polynomial of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & 5 & 3 \end{bmatrix}$$

is $p(\lambda) = (\lambda - 2)(\lambda - 3)^2 = \lambda^3 - 8\lambda^2 + 21\lambda - 18$

so the Cayley–Hamilton theorem implies that

$$A^3 - 8A^2 + 21A - 18I = 0 \tag{16}$$

This equation can be used to express A^3 and all higher powers of A in terms of I, A, and A^2.

$$A^3 - 8A^2 + 21A - 18I = 0 \tag{16}$$

For example,

$$A^3 = 8A^2 - 21A + 18I$$

and using this equation we can write

$$\begin{aligned} A^4 = AA^3 = 8A^3 - 21A^2 + 18A &= 8(8A^2 - 21A + 18I) - 21A^2 + 18A \\ &= 43A^2 - 150A + 144I \end{aligned}$$

Equation (16) can also be used to express A^{-1} as a polynomial in A by rewriting it as

$$A(A^2 - 8A + 21I) = 18I$$

from which it follows that (verify) $A^{-1} = \frac{1}{18}(A^2 - 8A + 21I) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 \\ \frac{7}{9} & -\frac{5}{9} & \frac{1}{3} \end{bmatrix}$ ■

EXPONENTIAL OF A MATRIX

(Calculus Required)

$$\boxed{f(A) = e^{tA}}$$

In Section 3.2 we defined polynomial functions of square matrices. Recall from that discussion that if A is an $n \times n$ matrix and

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$

then the matrix $p(A)$ is defined as

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m$$

Other functions of square matrices can be defined using power series. For example, if the function f is represented by its Maclaurin series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{m}(0)}{m!}x^m + \cdots \tag{17}$$

on some interval, then we define $f(A)$ to be

$$f(A) = f(0)I + f'(0)A + \frac{f''(0)}{2!}A^2 + \cdots + \frac{f^{m}(0)}{m!}A^m + \cdots \tag{18}$$

where we interpret this to mean that the ijth entry of $f(A)$ is the sum of the series of the ijth entries of the terms on the right.*

$$f(A) = f(0)I + f'(0)A + \frac{f''(0)}{2!}A^2 + \cdots + \frac{f^m(0)}{m!}A^m + \cdots \tag{18}$$

In the special case where A is a diagonal matrix, say

$$A = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

and f is defined at the points $d_1, d_2, \ldots, d_k$, each matrix on the right side of (18) is diagonal, and hence so is $f(A)$. In this case, equating corresponding diagonal entries on the two sides of (18) yields

$$(f(A))_{kk} = f(0) + f'(0)d_k + \frac{f''(0)}{2!}d_k^2 + \cdots + \frac{f^m(0)}{m!}d_k^m + \cdots = f(d_k)$$

Thus, we can avoid the series altogether in the diagonal case and compute $f(A)$ directly as

$$f(A) = \begin{bmatrix} f(d_1) & 0 & \cdots & 0 \\ 0 & f(d_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(d_n) \end{bmatrix} \tag{19}$$

For example, if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \text{then} \quad e^A = \begin{bmatrix} e & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^{-2} \end{bmatrix}$$

Now let us consider how we might use these ideas to find functions of diagonalizable matrices without summing infinite series. If A is an $n \times n$ diagonalizable matrix and $P^{-1}AP = D$, where

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

then (10) and (18) suggest that

$$\begin{aligned} P^{-1}f(A)P &= f(0)I + f'(0)(P^{-1}AP) + \frac{f''(0)}{2!}(P^{-1}A^2P) + \cdots + \frac{f^m(0)}{m!}(P^{-1}A^mP) + \cdots \\ &= f(0)I + f'(0)D + \frac{f''(0)}{2!}D^2 + \cdots + \frac{f^m(0)}{m!}D^m + \cdots \\ &= f(D) \end{aligned}$$

This tells us that $f(A)$ can be expressed as

$$f(A) = Pf(D)P^{-1} \tag{20}$$

which suggests the following theorem.

Theorem 8.3.6 *Suppose that A is an $n \times n$ diagonalizable matrix that is diagonalized by P and that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A corresponding to the successive column vectors of P. If f is a real-valued function whose Maclaurin series converges on some interval containing the eigenvalues of A, then*

$$f(A) = P \begin{bmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\lambda_n) \end{bmatrix} P^{-1} \tag{21}$$

EXAMPLE 5 Exponentials of Diagonalizable Matrices

Find e^{tA} for the diagonalizable matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution We showed in Example 8.3 of Section 8.2 that

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

so applying Formula (21) with $f(A) = e^{tA}$ implies that

$$e^{tA} = P \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} P^{-1} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{t} - e^{2t} & 0 & 2e^{t} - 2e^{2t} \\ e^{2t} - e^{t} & e^{2t} & e^{2t} - e^{t} \\ e^{2t} - e^{t} & 0 & 2e^{2t} - e^{t} \end{bmatrix}$$

■

In the special case where A is a symmetric matrix with a spectral decomposition

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$$

the matrix

$$P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]$$

orthogonally diagonalizes A, so (20) can be expressed as

$$f(A) = Pf(D)P^T$$

We will ask you to show in the exercises that this equation can be written as

$$f(A) = f(\lambda_1)\mathbf{u}_1 \mathbf{u}_1^T + f(\lambda_2)\mathbf{u}_2 \mathbf{u}_2^T + \cdots + f(\lambda_n)\mathbf{u}_n \mathbf{u}_n^T \tag{22}$$

(Exercise P3), which tells us that $f(A)$ is a symmetric matrix whose eigenvalues can be obtained by evaluating f at the eigenvalues of A.

DIAGONALIZATION AND LINEAR SYSTEMS

The problem of diagonalizing a square matrix A is closely related to the problem of solving the linear system $A\mathbf{x} = \mathbf{b}$. For example, suppose that A is diagonalizable and $P^{-1}AP = D$. If we define a new vector $\mathbf{y} = P^{-1}\mathbf{x}$, and if we substitute

$$\mathbf{x} = P\mathbf{y} \tag{23}$$

in $A\mathbf{x} = \mathbf{b}$, then we obtain a new linear system $AP\mathbf{y} = \mathbf{b}$ in the unknown $\mathbf{y}$.

Multiplying both sides of this equation by P^{-1} and using the fact that $P^{-1}AP = D$ yields

$$D\mathbf{y} = P^{-1}\mathbf{b}$$

Since this system has a diagonal coefficient matrix, the solution for $\mathbf{y}$ can be read off immediately, and the vector $\mathbf{x}$ can then be computed using (23).

Many algorithms for solving large-scale linear systems are based on this idea. Such algorithms are particularly effective in cases in which the coefficient matrix can be orthogonally diagonalized since multiplication by orthogonal matrices does not magnify roundoff error.

THE NONDIAGONALIZABLE CASE

In cases where A is not diagonalizable it is still possible to achieve considerable simplification in the form of $P^{-1}AP$ by choosing the matrix P appropriately. We will consider two such theorems for matrices with real entries that involve orthogonal similarity. The proofs will be omitted.

The first theorem, due to the German mathematician Issai Schur (1875–1941), states that every square matrix A with real eigenvalues is orthogonally similar to an *upper triangular matrix* that has the eigenvalues of A on the main diagonal.

Theorem 8.3.7 (***Schur's Theorem***) *If A is an $n \times n$ matrix with real entries and real eigenvalues, then there is an orthogonal matrix P such that P^TAP is an upper triangular matrix of the form*

$$P^TAP = \begin{bmatrix} \lambda_1 & \times & \times & \cdots & \times \\ 0 & \lambda_2 & \times & \cdots & \times \\ 0 & 0 & \lambda_3 & \cdots & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{24}$$

in which $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the matrix A repeated according to multiplicity.

It is common to denote the upper triangular matrix in (24) by S (for Schur), in which case that equation can be rewritten as

$$A = PSP^T \tag{25}$$

which is called a ***Schur decomposition*** of A.

The next theorem, due to the German mathematician Gerhard Hessenberg (1894–1925), states that every square matrix with real entries is orthogonally similar to a matrix in which each entry below the first ***subdiagonal*** is zero (Figure 8.3.2). Such a matrix is said to be in ***upper Hessenberg form***.

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}$$

First subdiagonal

Figure 8.3.2

Theorem 8.3.8 **(*Hessenberg's Theorem*)** *Every square matrix with real entries is orthogonally similar to a matrix in upper Hessenberg form; that is, if A is an $n \times n$ matrix, then there is an orthogonal matrix P such that P^TAP is a matrix of the form*

$$P^TAP = \begin{bmatrix} \times & \times & \cdots & \times & \times & \times \\ \times & \times & \cdots & \times & \times & \times \\ 0 & \times & \ddots & \times & \times & \times \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \times & \times & \times \\ 0 & 0 & \cdots & 0 & \times & \times \end{bmatrix} \tag{26}$$

REMARK The diagonal entries in (26) will usually not be the eigenvalues of A.

$$P^TAP = \begin{bmatrix} \times & \times & \cdots & \times & \times & \times \\ \times & \times & \cdots & \times & \times & \times \\ 0 & \times & \ddots & \times & \times & \times \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \times & \times & \times \\ 0 & 0 & \cdots & 0 & \times & \times \end{bmatrix} \tag{26}$$

It is common to denote the upper Hessenberg matrix in (26) by H (for Hessenberg), in which case that equation can be rewritten as

$$A = PHP^T \tag{27}$$

which is called an ***upper Hessenberg decomposition*** of A.

In many numerical LU- and QR-algorithms the initial matrix is first converted to upper Hessenberg form, thereby reducing the amount of computation in the algorithm itself. Some computer programs have built-in commands for finding Schur or Hessenberg decompositions.