Section 7.11 Coordinates with Respect to a Basis

NONRECTANGULAR COORDINATE SYSTEMS IN R^2 AND R^3

Our first goal in this section is to extend the notion of a coordinate system from R^2 and R^3 to R^n by adopting a vector point of view.

For this purpose, recall that in a rectangular xy-coordinate system in R^2 we associate an ordered pair of coordinates (a, b) with a point P by projecting the point onto the coordinate axes and finding the signed distances of the projections from the origin.

This establishes a ***one-to-one correspondence*** between points in the plane and ordered pairs of real numbers. The same one-to-one correspondence can be obtained by considering the unit vectors $\mathbf{i}$ and $\mathbf{j}$ in the positive x- and y-directions, and expressing the vector $\overrightarrow{OP}$ as the linear combination

$\overrightarrow{OP} = a\mathbf{i} + b\mathbf{j}$ (Figure 7.11.1a).

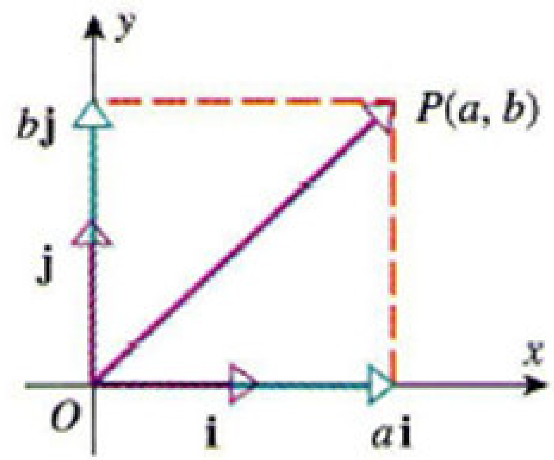


Figure 7.11.1 (a)

Since the coefficients in this linear combination are the same as the coordinates obtained using the coordinate axes, we can view the coordinates of a point P in a rectangular xy-coordinate system as the ordered pair of coefficients that result when $\overrightarrow{OP}$ is expressed in terms of the ordered basis* $B = \{\mathbf{i}, \mathbf{j}\}$.

Similarly, the coordinates (a, b, c) of a point P in a rectangular xyz-coordinate system can be viewed as the coefficients in the linear combination

$$\overrightarrow{OP} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

of the ordered basis $B = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ (Figure 7.11.1b).

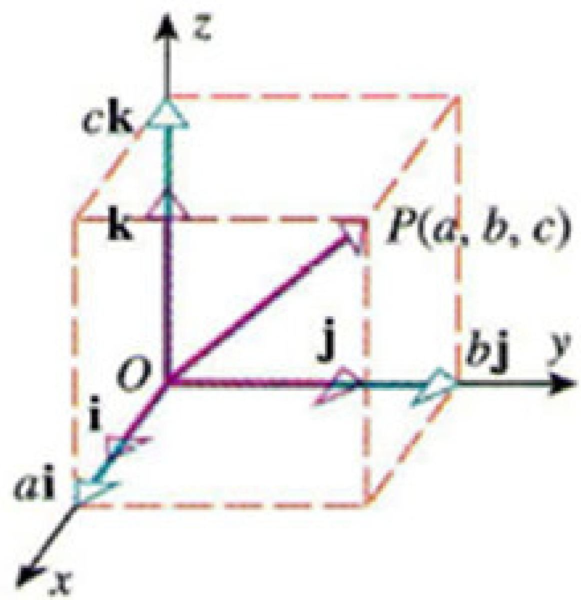


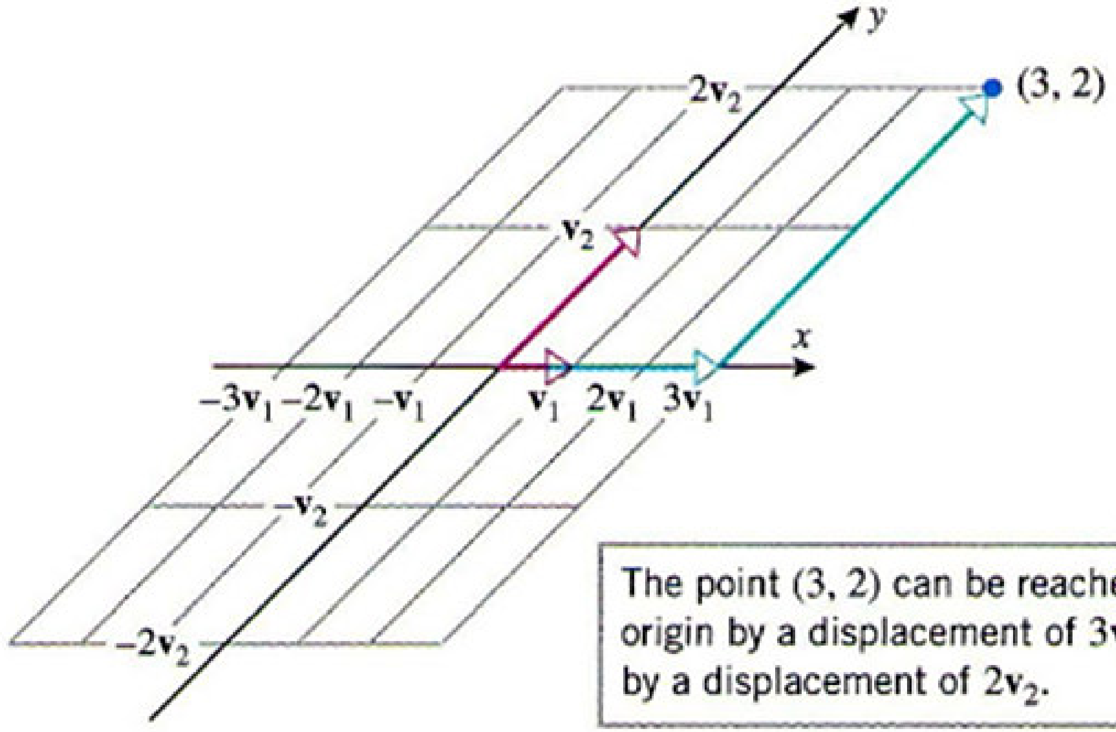
Figure 7.11.1 (b)

*The order in which the basis vectors are listed here is important, since changing the order of the vectors would change the order of the coordinates. In general, when the order in which the basis vectors are listed must be adhered to, the basis is called an ***ordered basis.*** In this section we will consider all bases to be ordered.

For the purpose of establishing a one-to-one correspondence between points in the plane and ordered pairs of real numbers, it is not essential that the basis vectors be orthogonal or have length 1. For example, if $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ is *any* ordered basis for R^2, then for each point P in the plane, there is exactly one way to express the vector $\overrightarrow{OP}$ as a linear combination

$$\overrightarrow{OP} = a\mathbf{v}_1 + b\mathbf{v}_2$$

Thus, the ordered basis B associates a unique ordered pair of numbers (a, b) with the point P, and conversely. Accordingly, we can think of the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ as defining a "generalized coordinate system" in which the coordinates (a, b) of a point P tell us how to reach the point P from the origin by a displacement $a\mathbf{v}_1$ followed by a displacement $b\mathbf{v}_2$ (Figure 7.11.2).



The point (3, 2) can be reached from the origin by a displacement of $3\mathbf{v}_1$ followed by a displacement of $2\mathbf{v}_2$.

Figure 7.11.2

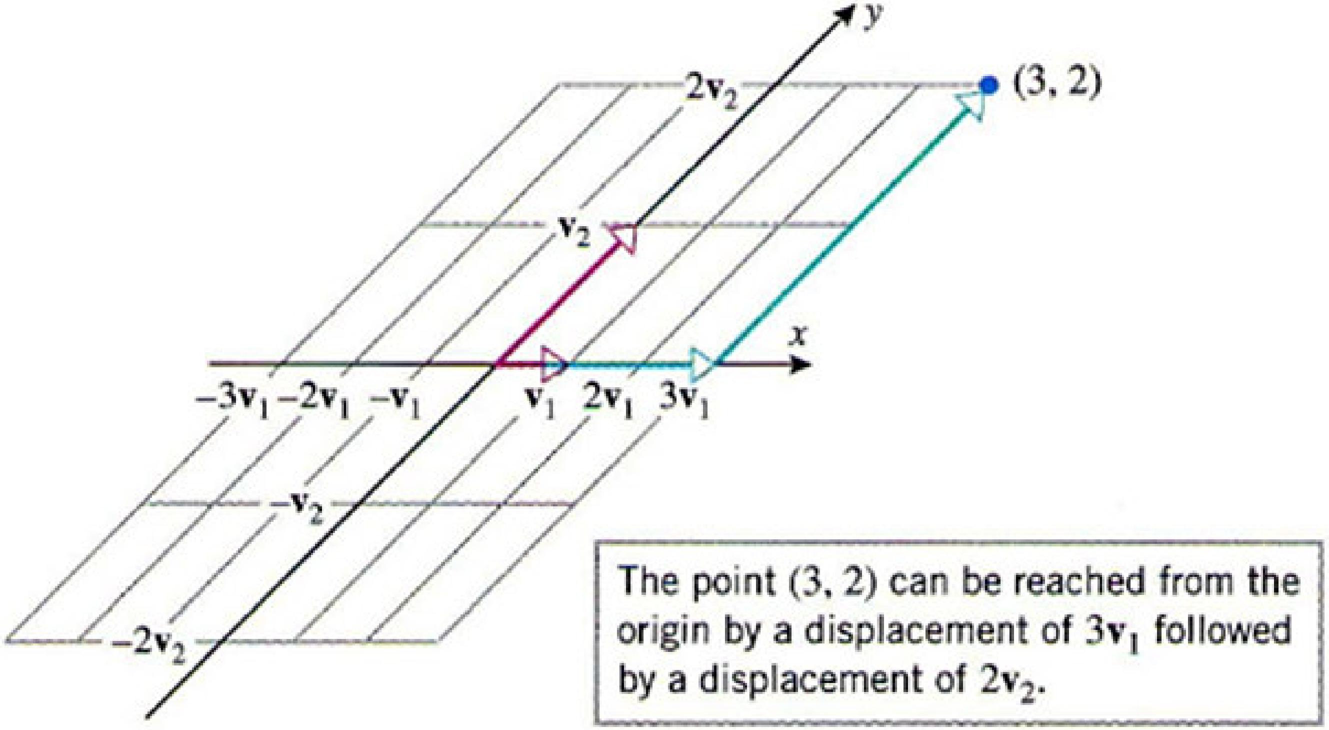


Figure 7.11.2

More generally, we can think of an ordered basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for R^n as defining a generalized coordinate system in which each point

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

in R^n is represented by the n-tuple of "coordinates" $(a_1, a_2, \ldots, a_n)$.

Definition 7.11.1 If $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an ordered basis for a subspace W of R^n, and if

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$$

is the expression for a vector $\mathbf{w}$ in W as a linear combination of the vectors in B, then we call

$$a_1, a_2, \ldots, a_n$$

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the ***coordinates of* w *with respect to* B**; and more specifically, we call a_j the **$\mathbf{v}_j$*-coordinate of* w**. We denote the ordered k-tuple of coordinates by

$$(\mathbf{w})_B = (a_1, a_2, \ldots, a_k)$$

and call it the ***coordinate vector*** for $\mathbf{w}$ with respect to B; and we denote the column vector of coordinates by

$$[\mathbf{w}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

and call it the ***coordinate matrix*** for $\mathbf{w}$ with respect to B.

EXAMPLE 1 Finding Coordinates

In Example 2(a) of Section 7.2 we showed that the vectors

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (1, -1, 3), \quad \mathbf{v}_3 = (1, 1, 4)$$

form a basis for R^3.

(a) Find the coordinate vector and coordinate matrix for the vector $\mathbf{w} = (4, 9, 8)$ with respect to the ordered basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) Find the vector $\mathbf{w}$ in R^3 whose coordinate vector relative to B is $(\mathbf{w})_B = (1, 2, -3)$.

Solution (a) In Example 2(b) of Section 7.2 we showed that $\mathbf{w}$ can be expressed as

$$\mathbf{w} = 3\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3$$

(by solving a linear system Ax = b;
v's as columns of A and **w** in b)

Thus,

$$(\mathbf{w})_B = (3, -1, 2) \quad \text{and} \quad [\mathbf{w}]_B = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

Solution (b) The entries in the coordinate vector tell us how to express $\mathbf{w}$ as a linear combination of the basis vectors:

$$\mathbf{w} = \mathbf{v}_1 + 2\mathbf{v}_2 - 3\mathbf{v}_3 = (1, 2, 1) + 2(1, -1, 3) - 3(1, 1, 4) = (0, -3, -5)$$ ■

EXAMPLE 2 Coordinates with Respect to the Standard Basis

If $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard basis for R^n, and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, then $\mathbf{w}$ can be expressed as a linear combination of the standard basis vectors as

$$\mathbf{w} = w_1(1, 0, \ldots, 0) + w_2(0, 1, \ldots, 0) + \cdots + w_n(0, 0, \ldots, 1) = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + \cdots + w_n\mathbf{e}_n$$

Thus,

$$(\mathbf{w})_S = (w_1, w_2, \ldots, w_n) = \mathbf{w} \tag{1}$$

That is, the components of $\mathbf{w}$ are the same as its coordinates with respect to the standard basis. If $\mathbf{w}$ is written in column form, then

$$[\mathbf{w}]_S = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \mathbf{w} \tag{2}$$

When we want to think of the components of a vector in R^n as coordinates with respect to the standard basis, we will call them the ***standard coordinates*** of the vector. ■

COORDINATES WITH RESPECT TO AN ORTHONORMAL BASIS

Recall from Theorem 7.9.4 that if $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthonormal basis for a subspace W of R^n, and if $\mathbf{w}$ is a vector in W, then the expression for $\mathbf{w}$ as a linear combination of the vectors in B is

$$\mathbf{w} = (\mathbf{w} \cdot \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{w} \cdot \mathbf{v}_2)\mathbf{v}_2 + \cdots + (\mathbf{w} \cdot \mathbf{v}_k)\mathbf{v}_k$$

Thus, the coordinate vector for $\mathbf{w}$ with respect to B is

$$(\mathbf{w})_B = \big((\mathbf{w} \cdot \mathbf{v}_1), (\mathbf{w} \cdot \mathbf{v}_2), \ldots, (\mathbf{w} \cdot \mathbf{v}_k)\big) \tag{3}$$

This result is noteworthy because it tells us that the components of a vector with respect to an orthonormal basis can be obtained by computing appropriate inner products, whereas for a general basis it is usually necessary to solve a linear system (as in Example 1). This is yet another computational advantage of orthonormal bases.

EXAMPLE 3 Finding Coordinates with Respect to an Orthonormal Basis

We showed in Example 3 of Section 7.9 that the vectors

$$\mathbf{v}_1 = \left(\tfrac{3}{7}, -\tfrac{6}{7}, \tfrac{2}{7}\right), \quad \mathbf{v}_2 = \left(\tfrac{2}{7}, \tfrac{3}{7}, \tfrac{6}{7}\right), \quad \mathbf{v}_3 = \left(\tfrac{6}{7}, \tfrac{2}{7}, -\tfrac{3}{7}\right)$$

form an orthonormal basis for R^3. Find the coordinate vector for $\mathbf{w} = (1, -1, 1)$ with respect to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution We leave it for you to show that

$$\mathbf{w} \cdot \mathbf{v}_1 = \tfrac{11}{7}, \quad \mathbf{w} \cdot \mathbf{v}_2 = \tfrac{5}{7}, \quad \mathbf{w} \cdot \mathbf{v}_3 = \tfrac{1}{7}$$

Thus, $(\mathbf{w})_B = \left(\tfrac{11}{7}, \tfrac{5}{7}, \tfrac{1}{7}\right)$, or in column form,

$$[\mathbf{w}]_B = \begin{bmatrix} \frac{11}{7} \\ \frac{5}{7} \\ \frac{1}{7} \end{bmatrix}$$

■

Theorem 7.11.2 *If B is an orthonormal basis for a k-dimensional subspace W of R^n, and if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in W with coordinate vectors*

$$(\mathbf{u})_B = (u_1, u_2, \ldots, u_k), \quad (\mathbf{v})_B = (v_1, v_2, \ldots, v_k), \quad (\mathbf{w})_B = (w_1, w_2, \ldots, w_k)$$

then:

norm of a vector = norm of its coordinate vector

(a) $\|\mathbf{w}\| = \sqrt{w_1^2 + w_2^2 + \cdots + w_k^2} = \|(\mathbf{w})_B\|$

(b) $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_k v_k = (\mathbf{u})_B \cdot (\mathbf{v})_B$

inner product of two vectors =
inner product of their coordinate vectors

EXAMPLE 4 Computing with Coordinates

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the orthonormal basis for R^3 that was given in Example 3, and let $\mathbf{w} = (1, -1, 1)$. We showed in Example 3 that $(\mathbf{w})_B = \left(\frac{11}{7}, \frac{5}{7}, \frac{1}{7}\right)$. Thus,

$$\|(\mathbf{w})_B\| = \sqrt{\left(\tfrac{11}{7}\right)^2 + \left(\tfrac{5}{7}\right)^2 + \left(\tfrac{1}{7}\right)^2} = \sqrt{3} = \|\mathbf{w}\|$$

as guaranteed by part (a) of Theorem 7.11.2. ■

CHANGE OF BASIS FOR R^n

The Change of Basis Problem If $\mathbf{w}$ is a vector in R^n, and if we change the basis for R^n from a basis B to a basis B', how are the coordinate matrices $[\mathbf{w}]_B$ and $[\mathbf{w}]_{B'}$ related?

To solve this problem, it will be convenient to refer to B as the "old basis" and B' as the "new basis." Thus, our objective is to find a relationship between the old and new coordinates of a fixed vector $\mathbf{w}$. For notational simplicity, we will solve the problem in R^2. For this purpose, let

$$B = [\mathbf{v}_1, \mathbf{v}_2] \quad \text{and} \quad B' = [\mathbf{v}'_1, \mathbf{v}'_2]$$

be the old and new bases, respectively, and suppose that the coordinate matrices for the old basis vectors with respect to the new basis are

$$[\mathbf{v}_1]_{B'} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad [\mathbf{v}_2]_{B'} = \begin{bmatrix} c \\ d \end{bmatrix} \tag{4}$$

That is,

$$\begin{aligned} \mathbf{v}_1 &= a\mathbf{v}'_1 + b\mathbf{v}'_2 \\ \mathbf{v}_2 &= c\mathbf{v}'_1 + d\mathbf{v}'_2 \end{aligned} \tag{5}$$

Now let $\mathbf{w}$ be any vector in W, and let

$$[\mathbf{w}]_B = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \tag{6}$$

be the old coordinate matrix; that is,

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 \tag{7}$$

To find the new coordinate matrix for $\mathbf{w}$ we must express $\mathbf{w}$ in terms of the new basis B'. For this purpose we substitute (5) into (7) to obtain

$$\mathbf{w} = k_1(a\mathbf{v}'_1 + b\mathbf{v}'_2) + k_2(c\mathbf{v}'_1 + d\mathbf{v}'_2)$$

which we can rewrite as

$$\mathbf{w} = (k_1a + k_2c)\mathbf{v}'_1 + (k_1b + k_2d)\mathbf{v}'_2$$

Thus,

$$[\mathbf{w}]_{B'} = \begin{bmatrix} k_1a + k_2c \\ k_1b + k_2d \end{bmatrix}$$

Now using (6), we can express this as

$$[\mathbf{w}]_{B'} = \begin{bmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} [\mathbf{w}]_B \tag{8}$$

which is the relationship we were looking for, since it tells us that the new coordinate matrix can be obtained by multiplying the old coordinate matrix by

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = [[\mathbf{v}_1]_{B'} \mid [\mathbf{v}_2]_{B'}] \tag{9}$$

We will denote this matrix by $P_{B \to B'}$ to suggest that it transforms B-coordinates to B'-coordinates. Using this notation, we can express (8) as

$$[\mathbf{w}]_{B'} = P_{B \to B'} [\mathbf{w}]_B$$

Although this relationship was derived for R^2 for notational simplicity, the same relationship holds for R^n. Here is the general result.

Theorem 7.11.3 ***(Solution of the Change of Basis Problem)*** *If* $\mathbf{w}$ *is a vector in* R^n, *and if* $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *and* $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$ *are bases for* R^n, *then the coordinate matrices of* $\mathbf{w}$ *with respect to the two bases are related by the equation*

$$[\mathbf{w}]_{B'} = P_{B\to B'}[\mathbf{w}]_B \tag{10}$$

where

$$P_{B\to B'} = \big[[\mathbf{v}_1]_{B'} \mid [\mathbf{v}_2]_{B'} \mid \cdots \mid [\mathbf{v}_n]_{B'}\big] \tag{11}$$

This matrix is called the ***transition matrix*** *(or the* ***change of coordinates matrix****) from* B *to* B'.

REMARK Formula (10) can be confusing when different letters are used for the bases or when the roles of B and B' are reversed, but you won't go wrong if you keep in mind that the columns in the transition matrix are coordinate matrices of the basis you are *transforming from* with respect to the basis you are *transforming to*.

$$P_{B\to B'} = \big[[\mathbf{v}_1]_{B'} \mid [\mathbf{v}_2]_{B'} \mid \cdots \mid [\mathbf{v}_n]_{B'}\big] \qquad (11)$$

EXAMPLE 5 Transition Matrices

Consider the bases $B_1 = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $B_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where

$$\mathbf{e}_1 = (1, 0), \quad \mathbf{e}_2 = (0, 1), \quad \mathbf{v}_1 = (1, 1), \quad \mathbf{v}_2 = (2, 1)$$

(a) Find the transition matrix from B_1 to B_2.

(b) Use the transition matrix from B_1 to B_2 to find $[\mathbf{w}]_{B_2}$ given that

$$[\mathbf{w}]_{B_1} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \tag{12}$$

(c) Find the transition matrix from B_2 to B_1.

(d) Use the transition matrix from B_2 to B_1 to recover the vector $[\mathbf{w}]_{B_1}$ from the vector $[\mathbf{w}]_{B_2}$.

Solution **(*a*)** Since we are transforming *to* B_2-coordinates, the form of the required transition matrix is

$$P_{B_1 \to B_2} = \left[[\mathbf{e}_1]_{B_2} \mid [\mathbf{e}_2]_{B_2} \right] \tag{13}$$

We leave it for you to show that

$$\begin{aligned} \mathbf{e}_1 &= -\mathbf{v}_1 + \mathbf{v}_2 \\ \mathbf{e}_2 &= 2\mathbf{v}_1 - \mathbf{v}_2 \end{aligned}$$

From v_1 and v_2 as column vectors of a coefficient matrix 'A', and e_1 and e_2 as 'b' vectors.

from which we obtain

$$[\mathbf{e}_1]_{B_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad [\mathbf{e}_2]_{B_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Thus, the transition matrix from B_1 to B_2 is

$$P_{B_1 \to B_2} = \left[[\mathbf{e}_1]_{B_2} \mid [\mathbf{e}_2]_{B_2} \right] = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \tag{14}$$

Solution (b) Using (12) and (14) we obtain

$$[\mathbf{w}]_{B_2} = P_{B_1 \to B_2}[\mathbf{w}]_{B_1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix} \tag{15}$$

As a check, (12) and (15) should correspond to the same vector $\mathbf{w}$. This is in fact the case, since (12) yields

That should be clear!

$$\mathbf{w} = 7\mathbf{e}_1 + 2\mathbf{e}_2 = 7\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

and (15) yields

$$\mathbf{w} = -3\mathbf{v}_1 + 5\mathbf{v}_2 = -3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

As it should be!

Solution (c) Since we are transforming *to* B_1-coordinates, the form of the required transition matrix is

$$P_{B_2 \to B_1} = \left[[\mathbf{v}_1]_{B_1} \mid [\mathbf{v}_2]_{B_1} \right]$$

But B_1 is the standard basis, so if $\mathbf{v}_1$ and $\mathbf{v}_2$ are written in column form, then $[\mathbf{v}_1]_{B_1} = \mathbf{v}_1$ and $[\mathbf{v}_2]_{B_1} = \mathbf{v}_2$. Thus,

$$P_{B_2 \to B_1} = \left[[\mathbf{v}_1]_{B_1} \mid [\mathbf{v}_2]_{B_1} \right] = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad (16)$$

No linear systems are required.

Solution (d) Using (15) and (16) we obtain

$$[\mathbf{w}]_{B_1} = P_{B_2 \to B_1} [\mathbf{w}]_{B_2} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

which is consistent with (12). ■

INVERTIBILITY OF TRANSITION MATRICES

If B_1, B_2, and B_3 are bases for R^n, then it is reasonable, though we will not formally prove it, that

$$P_{B_2 \to B_3} P_{B_1 \to B_2} = P_{B_1 \to B_3} \tag{17}$$

This is because multiplication by $P_{B_1 \to B_2}$ maps B_1-coordinates into B_2-coordinates and multiplication by $P_{B_2 \to B_3}$ maps B_2-coordinates into B_3-coordinates, so the effect of first multiplying by $P_{B_1 \to B_2}$ and then by $P_{B_2 \to B_3}$ is to map B_1-coordinates into B_3-coordinates.

In particular, if B and B' are two bases for W, then

$$P_{B' \to B} P_{B \to B'} = P_{B \to B} \tag{18}$$

But $P_{B \to B} = I$ (why?), so (18) implies that $P_{B' \to B}$ and $P_{B \to B'}$ are invertible and are inverses of one another.

Theorem 7.11.4 *If B and B' are bases for R^n, then the transition matrices $P_{B' \to B}$ and $P_{B \to B'}$ are invertible and are inverses of one another; that is,*

$$(P_{B' \to B})^{-1} = P_{B \to B'} \quad \textit{and} \quad (P_{B \to B'})^{-1} = P_{B' \to B}$$

EXAMPLE 6

Inverse of a Transition Matrix

For the bases in Example 5 we found that

$$P_{B_2 \to B_1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad P_{B_1 \to B_2} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

A simple multiplication will show that these matrices are inverses, as guaranteed by Theorem 7.11.4. ■

A GOOD TECHNIQUE FOR FINDING TRANSITION MATRICES

Our next goal is to develop an efficient technique for finding transition matrices. For this purpose, let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$ be bases for R^n, and consider how the columns of the transition matrix

$$P_{B \to B'} = \left[[\mathbf{v}_1]_{B'} \mid [\mathbf{v}_2]_{B'} \mid \cdots \mid [\mathbf{v}_n]_{B'} \right] \tag{19}$$

are computed. The entries of $[\mathbf{v}_j]_{B'}$ are the coefficients that are required to express $\mathbf{v}_j$ as a linear combination of $\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n$, and hence can be obtained by solving the linear system

$$\begin{bmatrix} \mathbf{v}'_1 & \mathbf{v}'_2 & \cdots & \mathbf{v}'_n \end{bmatrix} \mathbf{x} = \mathbf{v}_j \tag{20}$$

whose augmented matrix is

$$\left[\mathbf{v}'_1 \quad \mathbf{v}'_2 \quad \cdots \quad \mathbf{v}'_n \mid \mathbf{v}_j \right] \tag{21}$$

Since $\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n$ are linearly independent, the coefficient matrix in (20) has rank n, so its reduced row echelon form is the identity matrix; hence, the reduced row echelon form of (21) is

$$\left[I \mid [\mathbf{v}_j]_{B'} \right]$$

Elementary row operations do not change
the solution of the associated linear system.

Thus, we have shown that $[\mathbf{v}_j]_{B'}$ is the matrix that results on the right side when row operations are applied to (21) to reduce the left side to the identity matrix.

However, rather than compute one column at a time, we can obtain all of the columns of (19) at once by applying row operations to the matrix

$$\left[\mathbf{v}'_1 \quad \mathbf{v}'_2 \quad \cdots \quad \mathbf{v}'_n \mid \mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n\right] \tag{22}$$

to reduce it to

$$\left[I \mid [\mathbf{v}_1]_{B'} \quad [\mathbf{v}_2]_{B'} \quad \cdots \quad [\mathbf{v}_n]_{B'}\right] = \left[I \mid P_{B\to B'}\right] \tag{23}$$

In summary, if we call $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ the old basis and $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$ the new basis, then the process of obtaining (23) from (22) by row operations is captured in the diagram

$$[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row operations}} [I \mid \text{transition from old to new}] \tag{24}$$

In summary, we have the following procedure for finding transition matrices by row reduction.

A Procedure for Computing $P_{B \to B'}$

Step 1. Form the matrix $[B' \mid B]$.

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.

Step 3. The resulting matrix will be $[I \mid P_{B \to B'}]$.

Step 4. Extract the matrix $P_{B \to B'}$ from the right side of the matrix in Step 3.

EXAMPLE 7 Transition Matrices by Row Reduction

In Example 5 we found the transition matrices between the bases $B_1 = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $B_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2, where

$$\mathbf{e}_1 = (1, 0), \quad \mathbf{e}_2 = (0, 1), \quad \mathbf{v}_1 = (1, 1), \quad \mathbf{v}_2 = (2, 1)$$

Find these transition matrices using (24).

$$[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row operations}} [I \mid \text{transition from old to new}] \tag{24}$$

Solution To find the transition matrix $P_{B_1 \to B_2}$ we must reduce the matrix

$$[B_2 \mid B_1] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

to make the left side the identity matrix. This yields (verify)

$$[I \mid P_{B_1 \to B_2}] = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

which agrees with (14). To find the transition matrix $P_{B_2 \to B_1}$ we must reduce the matrix

$$[B_1 \mid B_2] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

to make the left side the identity matrix.

However, it is already in that form, so there is nothing to do; we see immediately that

$$P_{B_2 \to B_1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

which agrees with (16). ■

REMARK The second part of this example illustrates the general fact that if $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for R^n, then the transition matrix from B to the standard basis S for R^n is

$$P_{B \to S} = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n] \tag{25}$$

COORDINATE MAPS

If B is a basis for R^n, then the transformation

$$\mathbf{x} \to (\mathbf{x})_B, \quad \text{or in column notation,} \quad \mathbf{x} \to [\mathbf{x}]_B$$

is called the ***coordinate map*** for B.

The coordinate map ("aplicação de coordenadas") is the operation of assigning coordinates to a vector.

In the exercises we will ask you to show that the following relationships hold for any scalar c and for any vectors $\mathbf{v}$ and $\mathbf{w}$ in R^n:

$$(c\mathbf{v})_B = c(\mathbf{v})_B \quad \text{and} \quad [c\mathbf{v}]_B = c[\mathbf{v}]_B \tag{26}$$

$$(\mathbf{v} + \mathbf{w})_B = (\mathbf{v})_B + (\mathbf{w})_B \quad \text{and} \quad [\mathbf{v} + \mathbf{w}]_B = [\mathbf{v}]_B + [\mathbf{w}]_B \tag{27}$$

Theorem 7.11.5 *If B is a basis for R^n, then the coordinate map $\mathbf{x} \to (\mathbf{x})_B$ (or $\mathbf{x} \to [\mathbf{x}]_B$) is a one-to-one linear operator on R^n. Moreover, if B is an orthonormal basis for R^n, then it is an orthogonal operator.*

see theorem 7.11.2

Theorem 7.11.6 *If A and C are $m \times n$ matrices, and if B is any basis for R^n, then $A = C$ if and only if $A[\mathbf{x}]_B = C[\mathbf{x}]_B$ for every $\mathbf{x}$ in R^n.*

It is a way of using coordinate matrices to determine whether two matrices are equal in cases their entries are not known explicitly.

Definition 6.3.9 A transformation $T: R^n \to R^m$ is said to be ***onto*** if its range is the entire codomain R^m; that is, every vector in R^m is the image of at least one vector in R^n.

Definition 6.3.10 A transformation $T: R^n \to R^m$ is said to be ***one-to-one*** (sometimes written 1–1) if T maps distinct vectors in R^n into distinct vectors in R^m.

TRANSITION BETWEEN ORTHONORMAL BASES

Theorem 7.11.7 *If B and B' are orthonormal bases for R^n, then the transition matrices $P_{B \to B'}$ and $P_{B' \to B}$ are orthogonal.*

Proof Since $P_{B \to B'}$ and $P_{B' \to B}$ are inverses, we need only prove that $P_{B \to B'}$ is orthogonal, since the orthogonality of $P_{B' \to B}$ will then follow from Theorem 6.2.3.

Accordingly, suppose that B and B' are orthonormal bases for W and that $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. To prove that

$$P_{B \to B'} = \left[[\mathbf{v}_1]_{B'} \mid [\mathbf{v}_2]_{B'} \mid \cdots \mid [\mathbf{v}_n]_{B'} \right] \tag{28}$$

is an orthogonal matrix, we will show that the column vectors are orthonormal (see Theorem 6.2.5). But this follows from Theorem 7.11.2, since

$$\|[\mathbf{v}_j]_{B'}\| = \|\mathbf{v}_j\| = 1 \quad \text{and} \quad [\mathbf{v}_i]_{B'} \cdot [\mathbf{v}_j]_{B'} = \mathbf{v}_i \cdot \mathbf{v}_j = 0 \qquad (i \neq j)$$ ■

Theorem 6.2.3

(*b*) *The inverse of an orthogonal matrix is orthogonal.*

Theorem 6.2.5 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(*a*) *A is orthogonal.*

(*d*) *The column vectors of A are orthonormal.*

EXAMPLE 8 A Rotation of the Standard Basis for R^2

Let $S = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for R^2, and let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis that results when the vectors in S are rotated about the origin through the angle θ. From (25) and Figure 7.11.3,

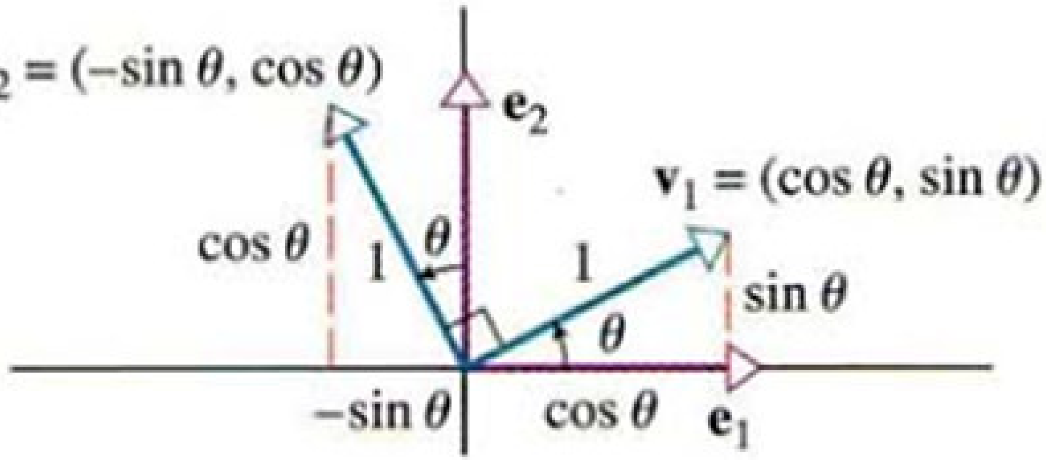


Figure 7.11.3

we see that the transition matrix from B to S is

$$P = P_{B\to S} = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This matrix is orthogonal, as guaranteed by Theorem 7.11.7, and hence the transition matrix from S to B is

$$P^T = P_{S\to B} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$ ■

EXAMPLE 9 Rotation of Coordinate Axes in R^2

Suppose that a rectangular xy-coordinate system is given and an $x'y'$-coordinate system is obtained by rotating the xy-coordinate system about the origin through an angle θ. Since there are now two coordinate systems, each point Q in the plane has two pairs of coordinates, a pair of coordinates (x, y) with respect to the xy-system and a pair of coordinates (x', y') with respect to the $x'y'$-system (Figure 7.11.4a).

To find a relationship between the two pairs of coordinates, we will treat the axis rotation as a change of basis from the standard basis $S = \{\mathbf{e}_1, \mathbf{e}_2\}$ to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ run along the positive x- and y-axes, respectively, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are the unit vectors along the positive x'- and y'-axes, respectively (Figure 7.11.4b).

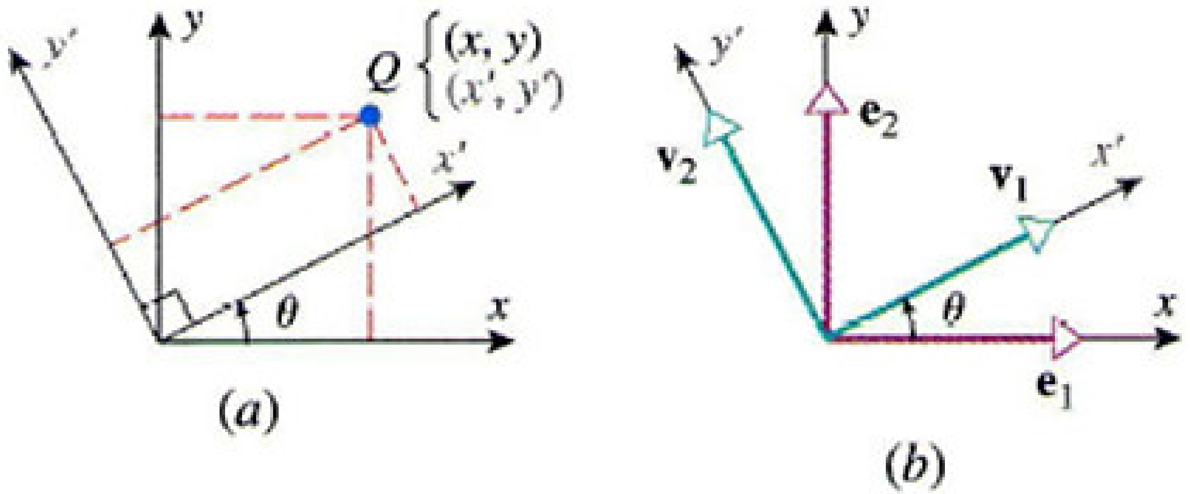


Figure 7.11.4

Since the vectors in B result from rotating the vectors in S about the origin through the angle θ, it follows from Example 8 that the transition matrices between the two bases are

$$P_{B\to S} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad P_{S\to B} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{29–30}$$

Thus, the relationship between two pairs of coordinates can be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{31–32}$$

If preferred, these matrix relationships can be expressed in equation form as

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned} \quad \text{or} \quad \begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned} \tag{33–34}$$

These are sometimes called the ***rotation equations*** for the plane. ■

Theorem 7.11.8 *If P is an invertible $n \times n$ matrix with column vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, then P is the transition matrix from the basis $B = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ for R^n to the standard basis $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for R^n.*

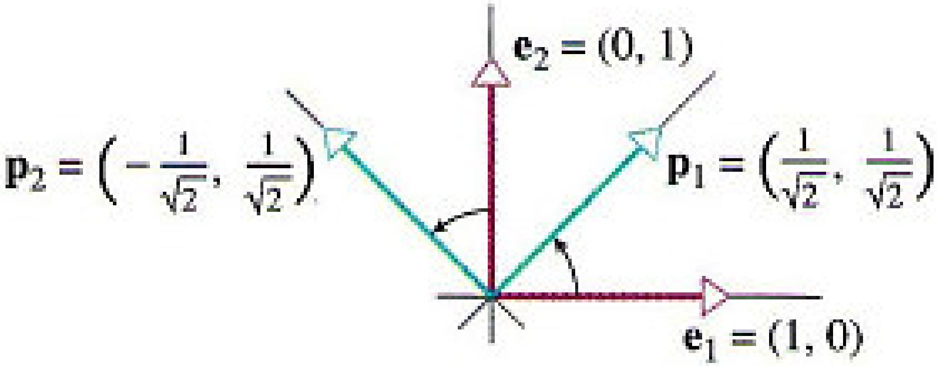
In the special case where P is a 2×2 or 3×3 orthogonal matrix and $\det(P) = 1$, the matrix P represents a rotation, which we can view either as a rotation of vectors or as a change in coordinates resulting from a rotation of coordinate axes. For example, if $P = [\mathbf{p}_1 \ \mathbf{p}_2]$ is 2×2, and if we view P as the standard matrix for a linear operator, then multiplication by P represents the rotation of R^2 that rotates $\mathbf{e}_1$ and $\mathbf{e}_2$ into $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively.

Alternatively, if we view the same matrix P as a transition matrix, then it follows from Theorem 7.11.3 that multiplication by P changes coordinates relative to the rectangular coordinate system whose positive axes are in the directions of $\mathbf{p}_1$ and $\mathbf{p}_2$ into those relative to the rotated coordinate system whose positive axes are in the directions of $\mathbf{e}_1$ and $\mathbf{e}_2$; hence, multiplication by $P^{-1} = P^T$ changes coordinates relative to the system whose positive axes are in the directions of $\mathbf{e}_1$ and $\mathbf{e}_2$ into those relative to the system whose positive axes are in the directions of $\mathbf{p}_1$ and $\mathbf{p}_2$.

These two interpretations of

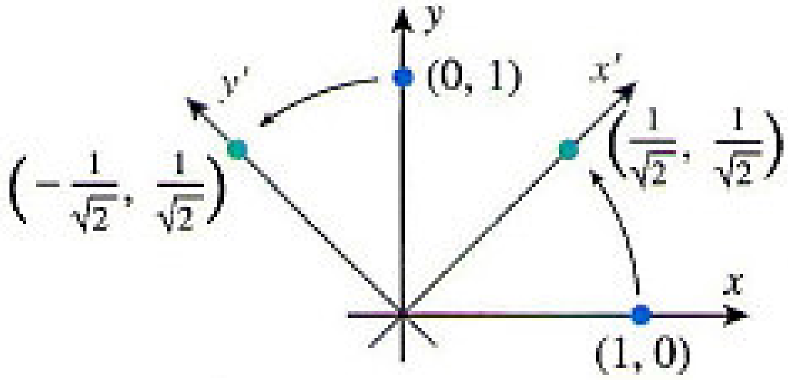
$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad P^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

are illustrated in Figure 7.11.5.



Multiplication by P as a rotation of vectors.

(a)



Multiplication by $P^{-1} = P^T$ as a change in coordinates resulting from a rotation of coordinate axes.

(b)

Figure 7.11.5