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A MODIFIED STRONGLY IMPLICIT PROCEDURE FOR THE NUMERICAL SOLUTION OF FIELD PROBLEMS

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A modified strongly implicit procedure for solving the system of algebraic equations that arise in the finite-difference or finite-analytic description of field problems is presented. The method is derived for a nine-point difference scheme and can readily be applied to the more conventional five-point scheme simply through the use of the five-point scheme coefficients. The method is demonstrated by application to several examples and a comparison is made between the performance of the modified procedure and that of the strongly implicit procedure, the alternating direction implicit method, and successive over-relaxation. In all cases examined the modified strongly implicit procedure offers superior results when the number of iterations required for convergence or the computational cost required for convergence is used as the measure of performance. The method is also less sensitive to control volume aspect ratio, relaxation parameters, and mesh subdivision than other available procedures. Savings in computational cost for a converged solution range from a factor of two to a factor of four over the strongly implicit procedure. It is felt that the development of this procedure offers a significant advance in the state of the art of solving the finite-difference equations that are used to describe field problems.

INTRODUCTION

In continuum problems, application of conventional finite-difference techniques to the analysis of magnetic, electric, temperature, velocity, and pressure fields within the domain of interest frequently leads to a system of algebraic equations having a well-defined structure. When using a five-point scheme a pentadiagonal coefficient matrix results, and when using a nine-point scheme a nine-diagonal coefficient matrix results. In general, the system of equations can be represented by the matrix equation.

$$[A]\{T\} = \{q\} \tag{1}$$

where in the analysis of thermal fields, the solution vector $\{T\}$ represents the temperature field as determined at discrete locations within the domain.

While the simplest methods of solution of the above matrix equation are direct methods, these methods fail to take advantage of the well-defined structure of the coefficient matrix. While direct methods are suitable for small systems, the cost of obtaining a solution to large sets of equations rapidly becomes prohibitive as the number of equations

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NOMENCLATURE

A, A', A''	coefficient matrices	$\bar{T}_c$	average contact area temperature
B	additional matrix		
ERRAV	average error in domain	T_f	fluid temperature
h	heat transfer coefficient	U	upper triangular matrix
i, j	location indices for control volume	x, y	Cartesian coordinates
		α	iterative parameter
I, J	number of control volumes in x and y directions, respectively	δ	change in temperature for one iteration
k	thermal conductivity	Δ	difference in accompanying variable
k_x, k_y	thermal conductivity in x and y directions, respectively	$\phi^1, \phi^2, \phi^3, \phi^4$	coefficients
L	lower triangular matrix or characteristic length	ω	relaxation parameter
Nc.V.	number of control volumes	**Subscripts**	
NITER	number of iterations		
q	right-hand vector of finite-difference equations	i	x location in grid
		j	y location in grid
$Q, \bar{Q}$	strength of heat source, total heat flow rate	**Superscripts**	
R	residual vector	e	east
RESAV	average residual in domain	n	north, iteration level
R^*	nondimensional thermal resistance $[\equiv kL(\bar{T}_c - T_f)/\bar{Q}]$	p	pole
		s	south
T	temperature	w	west

increases. This realization has provided the major motivation for the development of iterative procedures that recognize the well-defined structure of the coefficient matrix.

One of the more common iterative procedures is successive over-relaxation (SOR), in which an initial guess field is successively improved through application of the equation for each discrete location. Through the application of under- or over-relaxation, the convergence of the guess field toward its final solution can be enhanced above that available through a simple application of the equation for each discrete location in the field. As the problem size increases, however, through finer mesh subdivisions, the convergence of such iterative procedures decreases and a large number of iterations are required. As the number of iterations increases, the cost of solution also increases, and large systems of equations thereby become excessively expensive to solve.

In attempting to alleviate the slow convergence characteristics of SOR procedures, the alternating direction implicit (ADI) procedure was proposed by Peaceman and Rachford [1]. In this procedure the system of equations is successively rearranged so that at each stage of the process a tridiagonal system can be solved efficiently by direct means, using a tridiagonal matrix algorithm for the modified set of equations. Through appropriate selection of the rearrangement procedure, "line-by-line" solutions are obtained in each of the coordinate directions. This procedure has the advantage that the influence of boundary conditions is propagated throughout the entire domain when a complete iteration has been completed, where a complete iteration requires obtaining a line-by-line solution for each of the coordinate directions involved in the problem. The ADI method, being more implicit than the SOR method, generally offers a higher convergence rate than does the SOR method. However, both methods lose their effectiveness when complex problems are encountered and when the equation set becomes large.

More recently, an iterative, strongly implicit procedure (SIP) was proposed by Stone [2]. In this method the iterative procedure involves the direct, simultaneous solution of a set of equations formed by modification of the original matrix equation. The modified matrix is constructed according to two criteria: (1) the equation set must remain more strongly implicit than in the ADI case, and (2) the elimination procedure for the modified equation set must be economically efficient. The modified matrix equation of the SIP procedure has the form

$$[\mathbf{A} + \mathbf{B}]\{\mathbf{T}\} = \{\mathbf{b}\} \tag{2}$$

where

$$\{\mathbf{b}\} \equiv \{\mathbf{q}\} + [\mathbf{B}]\{\mathbf{T}\} \tag{3}$$

The form of [B] is such that $\|[\mathbf{B}]\| \ll \|[\mathbf{A}]\|$ and that the decomposition of [A + B] into a lower and an upper triangular matrix product involves much less computation than the direct decomposition of [A].

Since the right-hand side of Eq. (2) involves the unknown solution vector {T}, iteration is still required. The procedure given by Stone [2] is

$$[\mathbf{A} + \mathbf{B}]\{\mathbf{T}\}^{n+1} = [\mathbf{A} + \mathbf{B}]\{\mathbf{T}\}^{n} - \omega([\mathbf{A}]\{\mathbf{T}\}^{n} - \{\mathbf{q}\}) \tag{4}$$

where a value of unity for ω was used. Although the SIP method can lead to a reduction in computational cost for certain problems, there remain several disadvantages to this procedure. These are

1. The method is restricted to five-point, pentadiagonal systems.
2. Reordering of the equations is required at each step of the iteration.
3. The rate of convergence is sensitive to the control volume or grid aspect ratio.
4. Convergence is highly problem-dependent and in certain cases convergence is slower than that provided by ADI methods.
5. The [L] [U] product matrix is strongly asymmetric.

Dupont et al. [3], and Bracha-Barak and Saylor [4] adopted an approach similar to that proposed by Stone [2] and succeeded in obtaining a symmetric [L] [U] decomposed matrix product. They also provided a discussion of the influence of the parameter ω in Eq. (4). Although they do obtain a symmetric [L] [U] decomposed matrix, equation reordering during the iteration process is still required and they do not discuss the sensitivity of their procedure to control volume aspect ratio or to the complexity of the problem being examined. In addition, their procedure is restricted to the five-point pentadiagonal system.

Saylor [5] proposed a second-order symmetric factorization in which matrices [A] and [B] are both symmetric. His results indicate, however, that this method is unstable and therefore is not of practical utility.

The present work provides a procedure that is a significant departure from the SIP method. The motivation for the development of this modified strongly implicit (MSI) procedure is fourfold:

1. To remove the asymmetry of the [L] [U] decomposed matrix or to weaken its influence.

2. To reduce the computational effort required to obtain a converged solution.
3. To extend the method's capability to include nine-point difference formulations.
4. To reduce the sensitivity of the procedure to grid aspect ratio.

In addition to the above, extension of the method's capability to include the nine-point difference formulations will also extend its utility to include the more recent finite-differential method [6].

In the sections of the paper that follow, the MSI procedure will be developed in detail. Then the sensitivity of the method to grid aspect ratio and iteration parameter will be investigated. Finally, the method will be compared with the SIP, ADI, and SOR methods for several different problems. This comparison, although performed for steady heat conduction with no sources or sinks, demonstrates the advantages of the MSI procedure over those methods previously available.

PROBLEM FORMULATION

The governing differential equation for two-dimensional, steady-state heat conduction is

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right)=-Q \tag{5}$$

By performing an energy balance on the finite control volume of Fig. 1*a* or by simply adopting a Taylor series approximation of the governing differential equation, an algebraic equation is formed for each control volume that takes the form, in the case of a five-point scheme, of

$$A^s_{i,j}T_{i,j-1}+A^w_{i,j}T_{i-1,j}+A^p_{i,j}T_{i,j}+A^e_{i,j}T_{i+1,j}+A^n_{i,j}T_{i,j+1}=q_{i,j} \tag{6}$$

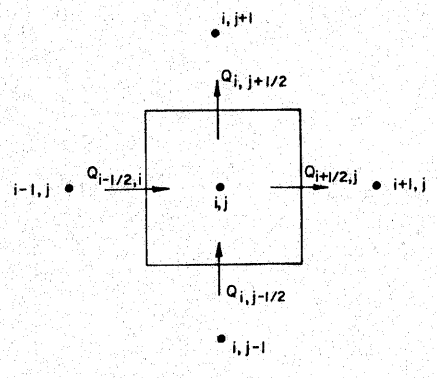
The collection of equations in the form of Eq. (6) written for each location (i, j) in the domain yields the matrix equation given as Eq. (1). For nodes lying outside the boundary as shown in the grid network of Fig. 1*b*, boundary-condition equations replace the conservation balance and relate the temperature outside the boundary to that immediately inside the domain. The equation form, however, can still be viewed as that given by Eq. (6). In matrix form, the system of equations takes a form that clearly indicates a pentadiagonal structure of the matrix [2]. This pentadiagonal structure results directly from the five-point scheme given as Eq. (6) when an appropriate, ordered numbering scheme is adopted for the nodal points in the discretized mesh [2].

SOLUTION PROCEDURE

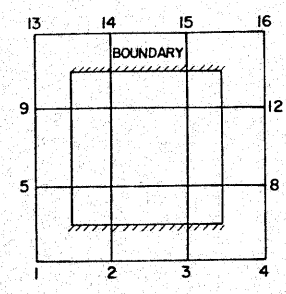
In the SIP method, an **LU** decomposition was proposed by Stone [2] such that

$$[\mathbf{L}]\,[\mathbf{U}] = [\mathbf{A}'] \tag{7}$$

where the **L** and **U** matrices are lower and upper triangular, respectively, with the principal diagonal of **U** being the unity diagonal. Upon forming the **LU** product $\mathbf{A}'$ there are



(a)



(b)

Fig. 1 Control volume energy balance and grid network.

two additional nonzero entries denoted by ϕ^1 and ϕ^2. Denoting the matrix consisting solely of the ϕ^1 and ϕ^2 diagonals by **B**, the matrix **A**′ can be written as

$$[\mathbf{A}'] = [\mathbf{A}] + [\mathbf{B}] \tag{8}$$

Directly from the defining equations determined by forming the **LU** product, the coefficients of the **L** and **U** matrices can be determined as well as the additional nonzero coefficients that appear in the modified matrix **A**′. The details of this determination are given by Stone [2]. In this determination, however, the coefficients of **L** and **U** are determined such that the coefficients in the **A**′ matrix in the locations of the nonzero entries of the original matrix **A** are identical with those of the original **A** matrix. The two additional coefficients obtained from the **LU** product are then simply accepted as their nonzero values. A corrective and iterative procedure is therefore required to nullify the influence of these latter two terms in the modified equation system. Details of one method for accomplishing this goal are given in Stone's paper [2]. However, the asymmetric influence of $\phi^1_{i,j}$ and $\phi^2_{i,j}$ is strong, and requires reversing the numbering scheme for the grid system after every iteration by interchanging the i and j indices.

The solution to the above dilemma is to extend the consideration to a nine-point formulation. This has the additional advantage that the nine-point scheme formulations can be solved by using this technique and that the five-point scheme formulations become simply a special case of the nine-point formulation. In the nine-point formulation, the

coefficient matrix **A** has the form shown in Fig. 2. The **L** and **U** matrices for use in the nine-point scheme are illustrated in Fig. 3, *a* and *b*. The modified coefficient matrix **A**′ has the form shown in Fig. 3*c*, where there are four additional entries, ϕ^1, ϕ^2, ϕ^3, and ϕ^4, in excess of those present in the original matrix **A**. The equations to be used to determine the coefficients of **L** and **U** such that the original nine coefficients in **A** remain unchanged in **A**′ are given by the following equations.

$$a_{i,j} = A^{sw}_{i,j} \tag{9a}$$

$$a_{i,j} f_{i-1,j-1} + b_{i,j} = A^{s}_{i,j} \tag{9b}$$

$$b_{i,j} f_{i,j-1} + c_{i,j} = A^{se}_{i,j} \tag{9c}$$

$$a_{i,j} h_{i-1,j-1} + b_{i,j} g_{i,j-1} + d_{i,j} = A^{w}_{i,j} \tag{9d}$$

$$a_{i,j} u_{i-1,j-1} + b_{i,j} h_{i,j-1} + c_{i,j} g_{i+1,j-1} + d_{i,j} f_{i-1,j} + e_{i,j} = A^{p}_{i,j} \tag{9e}$$

$$b_{i,j} u_{i,j-1} + c_{i,j} h_{i+1,j-1} + e_{i,j} f_{i,j} = A^{e}_{i,j} \tag{9f}$$

$$d_{i,j} h_{i-1,j} + e_{i,j} g_{i,j} = A^{nw}_{i,j} \tag{9g}$$

$$d_{i,j} u_{i-1,j} + e_{i,j} h_{i,j} = A^{n}_{i,j} \tag{9h}$$

$$e_{i,j} u_{i,j} = A^{ne}_{i,j} \tag{9i}$$

while the additional nonzero entries of A' are determined from

$$\phi^1_{i,j} = c_{i,j} f_{i+1,j-1} \tag{10a}$$

$$\phi^2_{i,j} = a_{i,j} g_{i-1,j-1} \tag{10b}$$

$$\phi^3_{i,j} = c_{i,j} u_{i+1,j-1} \tag{10c}$$

$$\phi^4_{i,j} = d_{i,j} g_{i-1,j} \tag{10d}$$

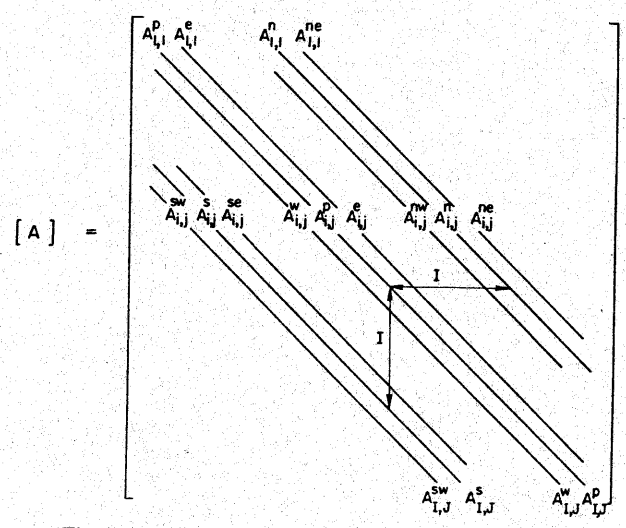
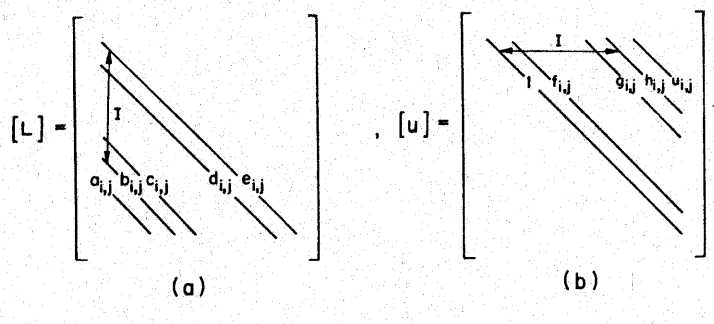


Fig. 2 Coefficient matrix for nine-point formulation.



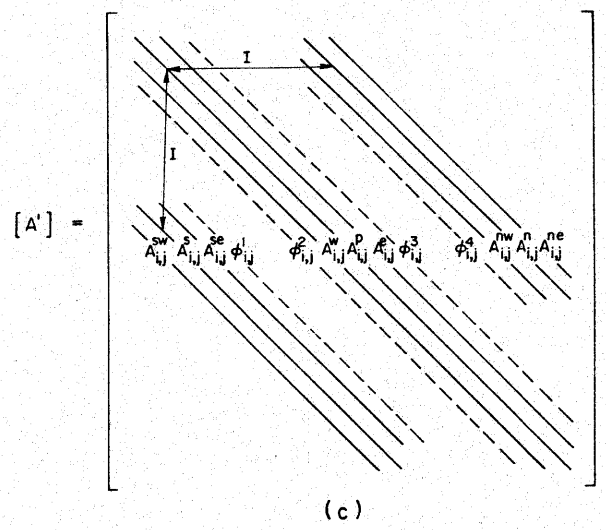


Fig. 3 L, U, and LU matrices for MSI procedure.

It is noted that the subscript i, j refers to location within the grid network rather than the matrix row-column designation and that, as in the original proposal of Stone [2], the additional coefficients ϕ^1, ϕ^2, ϕ^3, and ϕ^4 are simply accepted as their calculated values.

The influence of the additional entries of $\mathbf{A}'$ is again asymmetric, but their location of influence is further removed from the central location and hence their influence is expected to be weaker. Our experience indicates that this is indeed the case. The numerical molecule associated with this modified matrix $\mathbf{A}'$ is shown schematically in Fig. 4*a*, wherein the nine central coefficients (solid circles) are the correct ones and may correspond to a five-point scheme, in which case $A^{nw}_{i,j} = A^{sw}_{i,j} = A^{se}_{i,j} = A^{ne}_{i,j} = 0$.

For boundary-condition application, the same approach is used as previously described. However, since boundary relations are not available for the corner nodes, a five-point formulation is used at the four interior corner control volumes. The influence of the additional coefficients is partially canceled in the same manner as employed in Stone's method and an iterative approach to the true solution of the original matrix

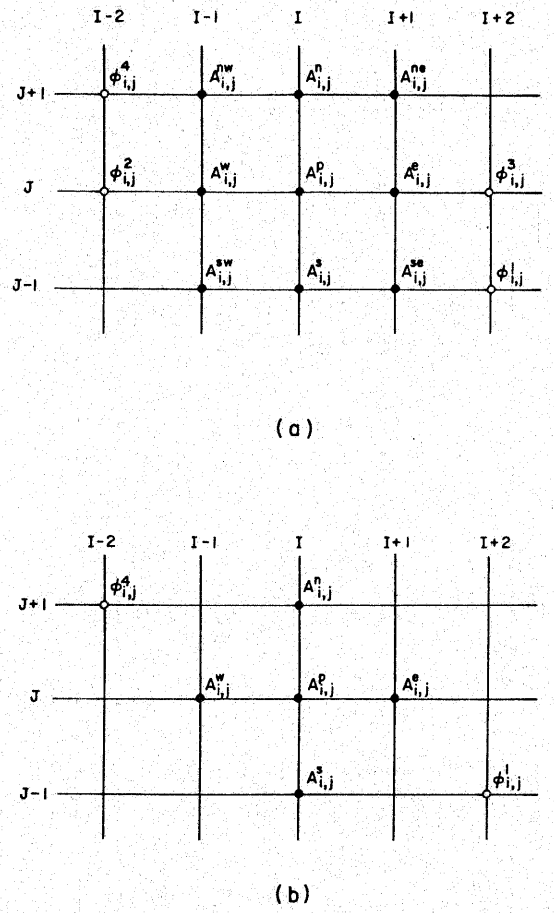


Fig. 4 Numerical molecule representation of MSI procedure: (*a*) nine-point and (*b*) five-point formulations.

equation is performed. The above procedure constitutes the MSI procedure presented in this work.

To partially cancel the influence of the additional terms in the $\mathbf{A}'$ matrix, for the complete nine-point scheme, a Taylor series expansion is used to express the temperature at locations $(i-2,j)$, $(i+2,j)$, $(i-2,j+1)$, and $(i+2,j-1)$ in terms of those temperatures involved in the original matrix equation. Thus the approximations are employed that

$$T_{i-2,j} = -T_{i,j} + 2T_{i-1,j} \tag{11a}$$

$$T_{i+2,j} = -T_{i,j} + 2T_{i+1,j} \tag{11b}$$

$$T_{i+2,j-1} = -2T_{i,j} + 2T_{i+1,j} + T_{i,j-1} \tag{11c}$$

$$T_{i-2,j+1} = -2T_{i,j} + 2T_{i-1,j} + T_{i,j+1} \tag{11d}$$

Higher-order Taylor series expansions have also been employed, but with a negligible improvement in the results. It is noted that the use of the above Taylor series representa-

$$A_p T_p + \Sigma(A_{nb} T_{NB}) - \alpha(\phi^1 T_{SEE} + \phi^2 T_{NW} + \phi^3 T_{EE} + \phi^4 T_{NWW}) = ST$$

tions affects only the approach to convergence of the iteration algorithm and that the influence of these representations vanishes when the solution is obtained.

An iterative parameter α is employed to implement the partial cancellation of the influence of the additional terms of $\mathbf{A}'$. This is done by writing the modified equations $\mathbf{A}'$ for the nine-point scheme in the form

$$\begin{aligned} &A^s_{i,j}T_{i,j-1} + A^w_{i,j}T_{i-1,j} + A^p_{i,j}T_{i,j} + A^n_{i,j}T_{i,j+1} + A^e_{i,j}T_{i+1,j} + A^{sw}_{i,j}T_{i-1,j-1} \\ &\quad + A^{nw}_{i,j}T_{i-1,j+1} + A^{ne}_{i,j}T_{i+1,j+1} + A^{se}_{i,j}T_{i+1,j-1} + \phi^1_{i,j}[T_{i+2,j-1} \\ &\quad - \alpha(-2T_{i,j} + 2T_{i+1,j} + T_{i,j-1})] + \phi^2_{i,j}[T_{i-2,j} - \alpha(-T_{i,j} + 2T_{i-1,j})] \\ &\quad + \phi^3_{i,j}[T_{i+2,j} - \alpha(-T_{i,j} + 2T_{i+1,j})] + \phi^4_{i,j}[T_{i-2,j+1} - \alpha(-2T_{i,j} \\ &\quad + 2T_{i-1,j} + T_{i,j+1})] = q_{i,j} \end{aligned} \tag{12}$$

which is more closely representative of the original system of equations for which a solution is sought. The equations, Eqs. (9) and (10), are modified to simultaneously include the influence of partial cancellation present in Eq. (12) prior to effecting the solution. The equations to be used to determine the coefficients of **L** and **U** such that the original nine coefficients in $\mathbf{A}'$ remain unchanged from those given by Eq. (12) are given, after rearrangement to permit their explicit evaluation, by the following equations.

$$a_{i,j} = A^{sw}_{i,j} \tag{13a}$$

$$b_{i,j} = \frac{A^s_{i,j} - a_{i,j}f_{i-1,j-1} - \alpha A^{se}_{i,j}f_{i+1,j-1}}{1 - \alpha f_{i,j-1}f_{i+1,j-1}} \tag{13b}$$

$$c_{i,j} = A^{se}_{i,j} - b_{i,j}f_{i,j-1} \tag{13c}$$

$$d_{i,j} = \frac{A^w_{i,j} - a_{i,j}h_{i-1,j-1} - b_{i,j}g_{i,j-1} - 2\alpha a_{i,j}g_{i-1,j-1}}{1 + 2\alpha g_{i-1,j}} \tag{13d}$$

$$\begin{aligned} e_{i,j} = {} & A^p_{i,j} - a_{i,j}u_{i-1,j-1} - b_{i,j}h_{i,j-1} - c_{i,j}g_{i+1,j-1} - d_{i,j}f_{i-1,j} \\ & + \alpha(2\phi^1_{i,j} + \phi^2_{i,j} + \phi^3_{i,j} + 2\phi^4_{i,j}) \end{aligned} \tag{13e}$$

$$f_{i,j} = \frac{A^e_{i,j} - b_{i,j}u_{i,j-1} - c_{i,j}h_{i+1,j-1} - 2\alpha(\phi^1_{i,j} + \phi^3_{i,j})}{e_{i,j}} \tag{13f}$$

$$g_{i,j} = \frac{A^{nw}_{i,j} - d_{i,j}h_{i-1,j}}{e_{i,j}} \tag{13g}$$

$$h_{i,j} = \frac{A^n_{i,j} - d_{i,j}u_{i-1,j} - \alpha\phi^4_{i,j}}{e_{i,j}} \tag{13h}$$

$$u_{i,j} = \frac{A^{ne}_{i,j}}{e_{i,j}} \tag{13i}$$

where the $\phi^1_{i,j}$, $\phi^2_{i,j}$, $\phi^3_{i,j}$, and $\phi^4_{i,j}$ used above are evaluated by using Eq. (10) and where the values of a, c, d, f, g, and u in Eq. (10) are those obtained through application of Eq. (13). Note that the ϕ's are required, and therefore should be evaluated, as soon as the evaluation of $d_{i,j}$ is complete

In the special case of application to a five-point formulation we have

$$A_{i,j}^{nw} = A_{i,j}^{sw} = A_{i,j}^{se} = A_{i,j}^{ne} = 0 \tag{14}$$

which leads to the result that

$$a_{i,j} = u_{i,j} = \phi_{i,j}^{2} = \phi_{i,j}^{3} = 0 \tag{15}$$

The resulting numerical molecule corresponding to the modified coefficient matrix $\mathbf{A}'$ for the five-point scheme is shown schematically in Fig. 4*b*, and illustrates that the influence of the additional terms of the **LU** product is further removed from the (i, j) location than are those in Stone's original method [2].

ITERATIVE PROCEDURE

Following the same procedure as outlined by Stone for the SIP procedure [2], we assign $\omega = 1$ in Eq. (4). Thus we form the iteration equation

$$[\mathbf{A} + \mathbf{B}]\,\mathbf{T}^{n+1} = [\mathbf{A} + \mathbf{B}]\,\mathbf{T}^{n} - ([\mathbf{A}]\,\{\mathbf{T}\}^{n} - \{\mathbf{q}\}) \tag{16}$$

Defining a difference vector and a residual vector according to the relations

$$\{\delta\}^{n+1} \equiv \{\mathbf{T}\}^{n+1} - \{\mathbf{T}\}^{n} \tag{17a}$$

$$\{\mathbf{R}\}^{n} \equiv \{\mathbf{q}\} - [\mathbf{A}]\,\{\mathbf{T}\}^{n} \tag{17b}$$

the iteration equation, Eq. (16), becomes

$$[\mathbf{A} + \mathbf{B}]\,\{\delta\}^{n+1} = \{\mathbf{R}\}^{n} \tag{18}$$

wherein the elements of **B** for the five-point scheme are simply the ϕ^1 and ϕ^4 values as determined from Eq. (10). Replacing $[\mathbf{A} + \mathbf{B}]$ by the **LU** product results in

$$[\mathbf{L}]\,[\mathbf{U}]\,\{\delta\}^{n+1} = \{\mathbf{R}\}^{n} \tag{19}$$

Defining an intermediate vector **V** by

$$\{\mathbf{V}\}^{n+1} \equiv [\mathbf{U}]\,\{\delta\}^{n+1} \tag{20}$$

the solution procedure takes on the familiar two-step process given by

$$[\mathbf{L}]\,\{\mathbf{V}\}^{n+1} = \{\mathbf{R}\}^{n} \tag{21a}$$

and

$$[\mathbf{U}]\,\{\delta\}^{n+1} = \{\mathbf{V}\}^{n+1} \tag{21b}$$

The process represented by Eq. (21) consists simply of a forward substitution followed by a backward substitution. Since the coefficients remain unchanged for the process, each iteration requires simply an evaluation of the new residual vector by a forward and backward triangular substitution procedure.

TESTING OF THE PROCEDURE

It is essential in the presentation of new methods that the procedures be thoroughly evaluated to determine their sensitivities, advantages, disadvantages, and relative merits when compared to the existing procedures. This section of the paper addresses this need by first examining the sensitivities of the method to mesh and relaxation parameters for a relatively simple problem. Subsequently, the method is applied to more complicated problems and its performance compared with that of existing procedures. Finally, the influence of different boundary-condition specifications is examined.

The first problem examined is that illustrated in Fig. 5*a*, for which the solution is a simple one-dimensional temperature distribution for steady-state heat conduction with no heat generation. Although the problem is one-dimensional, the two-dimensional five-point formulation was employed to determine the numerical solution. The reasons for selecting this problem for preliminary examination are threefold:

1. To check the computer code operation.
2. To gain familiarity with the method.
3. To determine the method's sensitivity to the iteration parameter, control volume aspect ratio, number of control volumes employed, problem orientation, and relaxation parameter.

Comparisons are provided for this problem with the SIP procedure. For all cases considered the SIP method was used in the form that entails the grid renumbering scheme

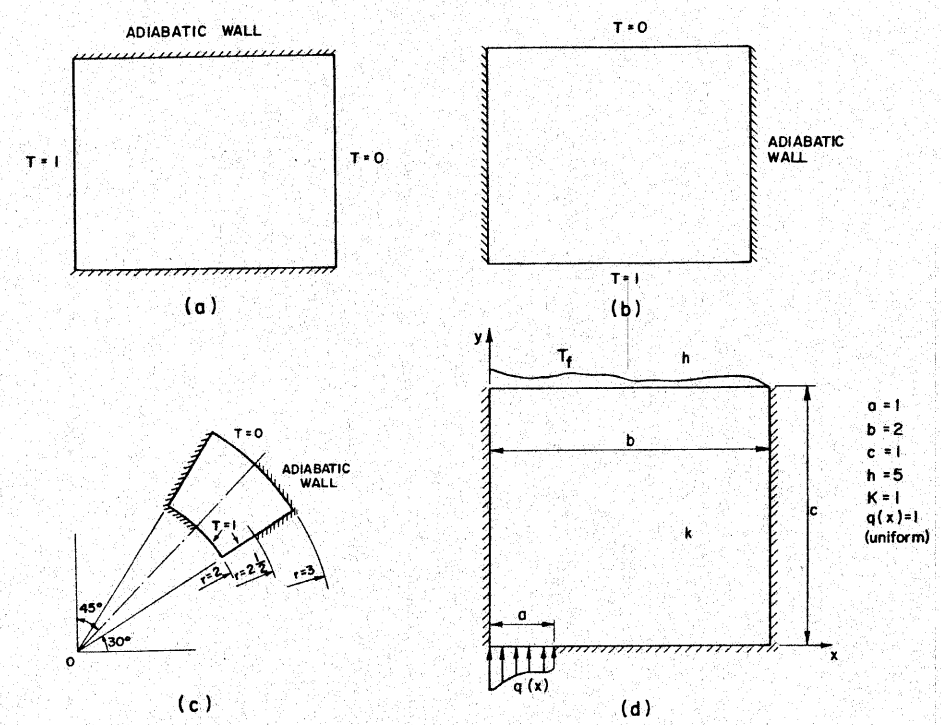


Fig. 5 Test problems: (*a*) model for sensitivity studies, (*b*) model of (*a*) rotated through 90°, (*c*) circular cylinder constriction problem, and (*d*) constriction with convective and Neuman boundaries.

suggested by Stone [2], but comparable results were realized when reordering was not used. The iterative parameter α, when used with the SIP method, has the same meaning as implied in Eq. (12), although different nodal locations will be involved. See [2] for details. The results presented are selected from a large number of tests performed.

The impact that the partial cancellation factor α and the grid aspect ratio $\Delta x/\Delta y$ has on the MSI performance was examined first. The influence of these parameters is presented in Fig. 6. In Fig. 6*a* the sensitivity of the convergence to α is presented for a uniform 7 X 7 mesh and compared with the SIP method. It can be seen from the figure that the MSI procedure is somewhat less sensitive to α than is the SIP method, and moreover that the average residual over the domain is significantly smaller for the MSI procedure. The curves presented are for a fixed computing time of 0.8 s, using the WATFIV compiler on an IBM 370/158 installation. It is also noted that the SIP procedure diverges for α greater than 0.8, while the MSI procedure does not exhibit this divergence.

Figure 6*b* indicates the sensitivity of the MSI convergence characteristics to the partially canceling parameter α for values of the control volume aspect ratio ranging from 0.1 to 10.0. In general, although it is not shown in the figure, the sensitivity of the MSI

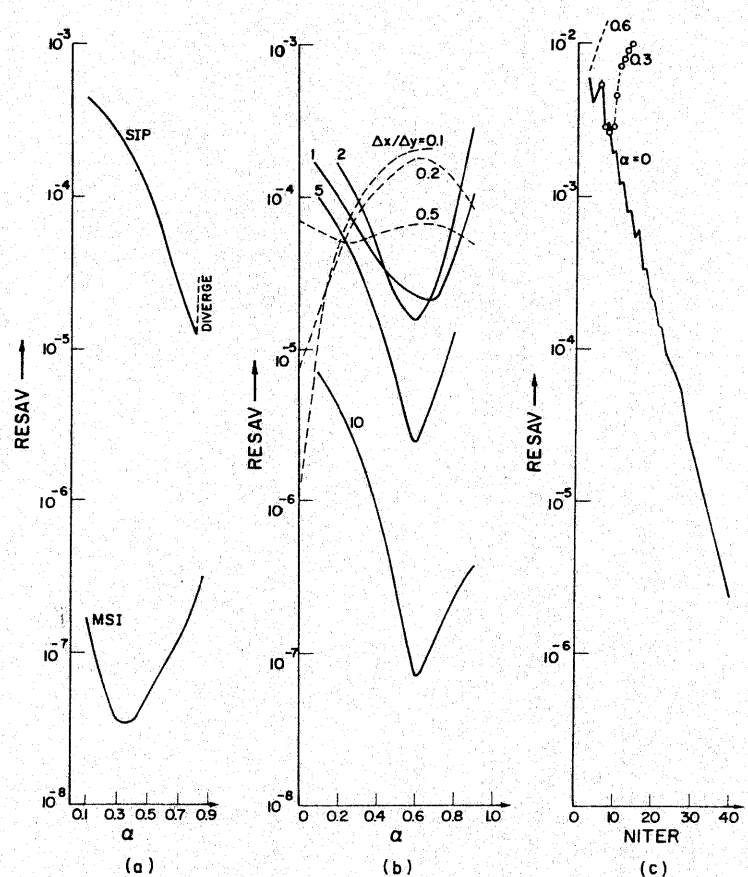


Fig. 6 Dependence of MSI and SIP on α and $\Delta x/\Delta y$.

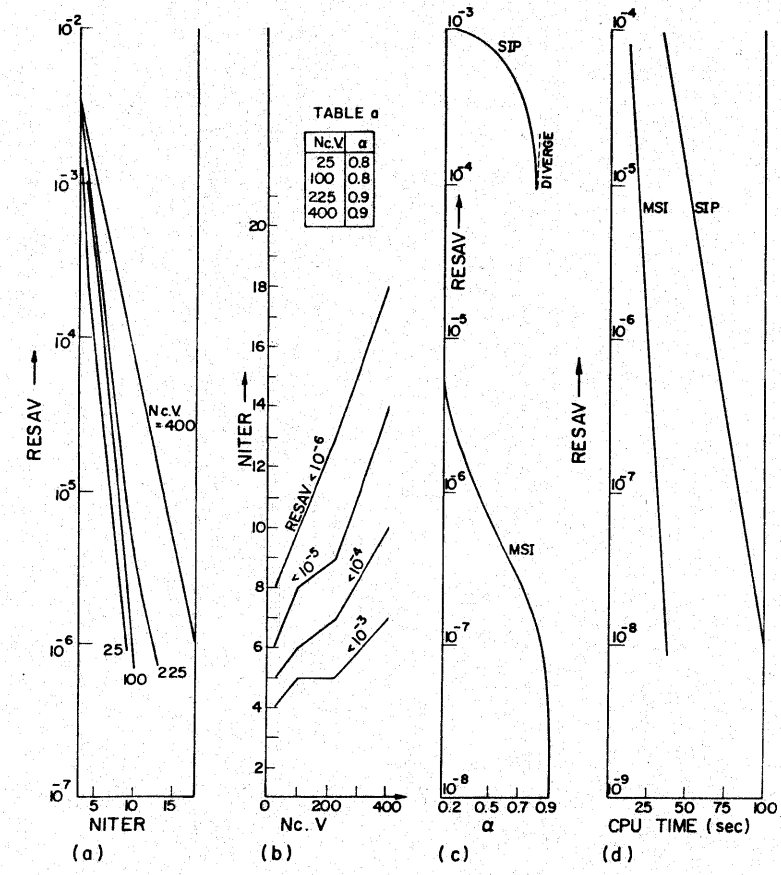


Fig. 7 Convergence characteristics of MSI and SIP.

procedure to $\Delta x/\Delta y$ is less than that of SIP over the entire range of α. All curves shown are for a 7 X 7 mesh with the size of the domain adjusted to obtain the indicated $\Delta x/\Delta y$ ratio. Identical results were observed, however, when the domain was fixed and the discretization level changed in the x and y directions to yield the indicated $\Delta x/\Delta y$ ratio. It is observed that the optimal value of α is relatively high and stable except for $\Delta x/\Delta y < 1$. Figure 6*c* indicates that the SIP procedure diverges at an aspect ratio of 10 for all except very small values of α. Indeed, even where the SIP procedure does converge, the computational time required for solution is approximately eight times that required by the MSI procedure.

Figure 7 indicates the influence of the number of control volumes on the convergence characteristics of the MSI method and provides selected comparisons with the SIP procedure. As expected, for the larger number of control volumes the convergence rate is lower than for the smaller number of control volumes. However, the sensitivity is not extreme. These trends are displayed in Fig. 7, *a* and *b*. In Fig. 7*c* a comparison is made with the SIP procedure for 400 control volumes and the average residual over the domain for a fixed computational time is presented for both methods. It is seen that residuals for the MSI procedure are at least an order of magnitude smaller than for the SIP procedure. Figure 7*d* provides a dramatic demonstration for 400 control volumes

of the rapid convergence of the MSI procedure when compared to the SIP procedure. The abscissa in this figure is the actual (WATFIV) computational time rather than the number of iterations.

The influence of problem orientation has been examined through testing of the method in a manner similar to that described above for the problem orientation shown in Fig. 5*b*. The influence of the relaxation factor ω is illustrated in Fig. 8 for a 7 X 7 grid. In Fig. 8*a* results are presented for $\alpha = 0.4$ and the value of ω near unity appears to provide the most rapid convergence. This is further exemplified in Fig. 8*b*, where $\omega = 1.0$ provides the lowest residual for seven iterations for near-optimal values of α. It would appear, therefore, that the use of a relaxation factor other than unity is of little, if any, benefit to the procedure.

Figure 9 presents results for the same test problem but with the full nine-point difference scheme employed in the MSI procedure. Comparisons with the SIP method cannot be made in this case, since the SIP method is incapable of utilizing the complete nine-point formulation. The figure illustrates that even with the use of the full nine-point formulation, the attributes of the MSI procedure are similar to those already presented. It is noteworthy, here, that the inclusion of the complete nine-point formulation capability is in itself a significant advance in the application of strongly implicit methods.

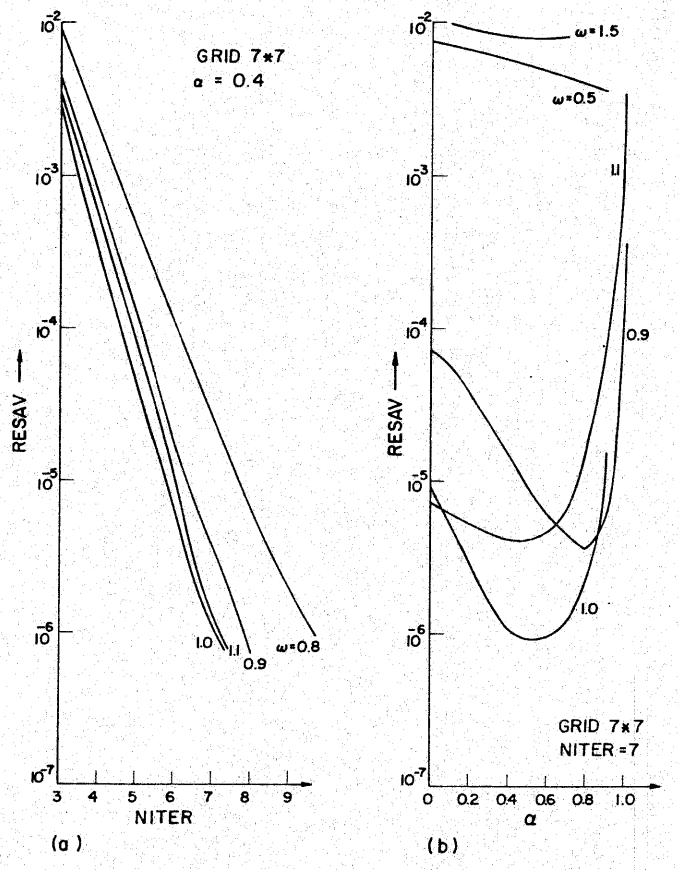


Fig. 8 Influence of relaxation parameter for MSI procedure.

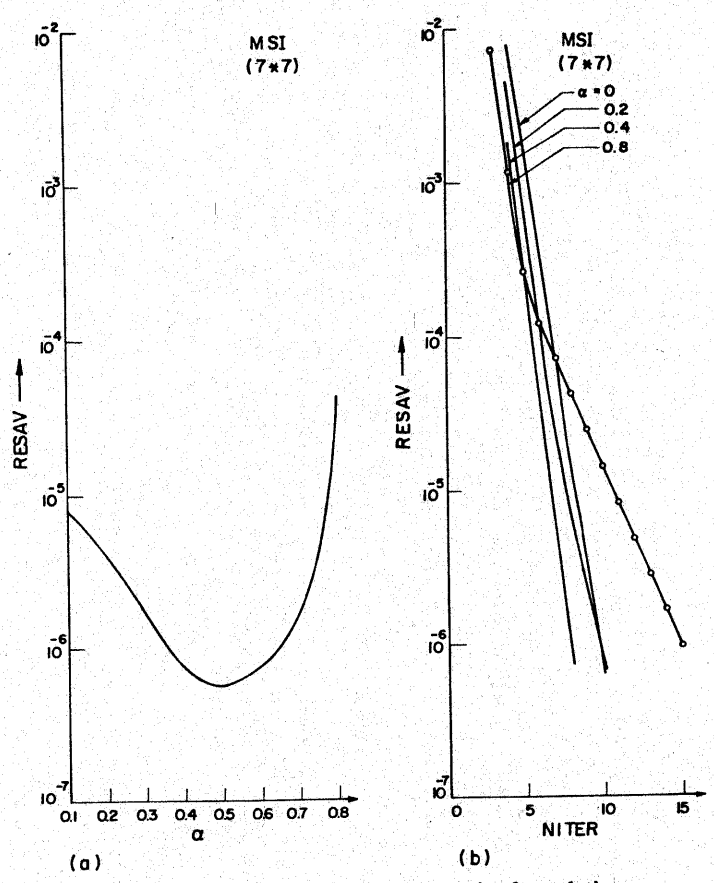


Fig. 9 Influence of α on MSI for nine-point formulation

APPLICATION TO TWO SPECIFIC PROBLEMS

Having gained familiarity with the characteristics and attributes of the proposed MSI method, it is instructive to examine its application to problems for which the heat flow pattern is more complex than that of the previously described problem. This is done for two specific problems in this section of the paper. The first problem, as illustrated in Fig. 5*c*, is that of two-dimensional, steady-state heat conduction with no heat sources or sinks and is posed in a circular cylinder coordinate system. There is an isothermal inflow boundary and an isothermal outflow boundary, with the remaining surface being adiabatic. The motivation for examining this problem is twofold: (1) to examine the MSI performance for a problem having a more complex heat flow pattern and (2) to verify its usage in circular cylinder coordinates.

The analytical solution to this problem, in terms of the thermal resistance, has recently been obtained by Schneider and Yovanovich [7]. The first phase of the comparison was in the form of a convergence study to determine the numerically asymptotic value of the thermal resistance. The value obtained numerically agrees well with the analytical solution obtained by Schneider and Yovanovich [7]. The second phase, and more relevant to this work, was to compare the MSI procedure with the common alter-

nate methods, namely the SIP, ADI, and SOR methods. The comparison is based on the average relative error from the fully converged numerical solution for each grid size considered. The relative error at each node in the grid network is defined by

$$\text{Error} \equiv \frac{T^n_{i,j} - T^e_{i,j}}{T^e_{i,j}} \tag{22}$$

where $T^e_{i,j}$ is the exact value of the numerical solution at location (i, j) and $T^n_{i,j}$ is the corresponding value after the nth iteration.

For a finite-difference mesh of 24 X 24 control volumes, the computational times required to achieve an average relative error of less than 10^{-4} are presented in Table 1 for the various methods considered. It is seen from the table that the MSI procedure achieves this accuracy with a computational time requirement 2.426 times smaller than that of its nearest competitor for this problem, the SIP method. It is seen from Fig. 10, where the computational time required to achieve a specified accuracy is plotted, that the time savings increase as the desired accuracy is increased (error is reduced). Figure 10*a* corresponds to an 8 X 8 grid network, while Fig. 10*b* corresponds to a 24 X 24 grid network. By comparing Fig. 10*a* with Fig. 10*b* it is also seen that the computational savings increase as the problem size (in terms of the number of control volumes) increases.

The second problem examined, as illustrated in Fig. 5*d*, is a plate of rectangular cross section, one of whose surfaces is convectively coupled to a uniform fluid environment while the opposite surface has a flux prescribed over a portion of its boundary. The remaining surfaces are adiabatic. The motivation for examining this problem was to examine the performance of the method on a problem for which both convective and specified heat flux boundary conditions are applied and to examine its performance on a second problem with a somewhat more complex heat flow pattern. Performing a convergence study on this problem to determine the numerically asymptotic value of the thermal resistance yielded an asymptote that is in good agreement with the analytical solution obtained by Schneider et al. [8]. Further testing was then directed at comparing the MSI procedure with the SIP, ADI, and SOR methods for this problem. Results of the comparison are presented in Table 2 and Fig. 11.

The results presented in Table 2 are the computational times required to achieve an average relative error in the domain of less than 10^{-5} for two mesh sizes, 8 X 8 and 24 X 24. It is seen that the computational time requirements for this problem are lower by a factor in excess of 2.8 for both mesh sizes. Moreover, it is also observed for the 8 X 8 case that the ADI procedure provides a more economical solution than does the SIP procedure. For a larger number of nodes, however, the relative performance of SIP

Table 1 Comparison of Solution Methods for Circular Cylinder Problem

Method	CPU time (s)	Ratio
MSI	7.689	1
SIP	18.653	2.426
ADI	30.722	3.996
SOR	38.756	5.04

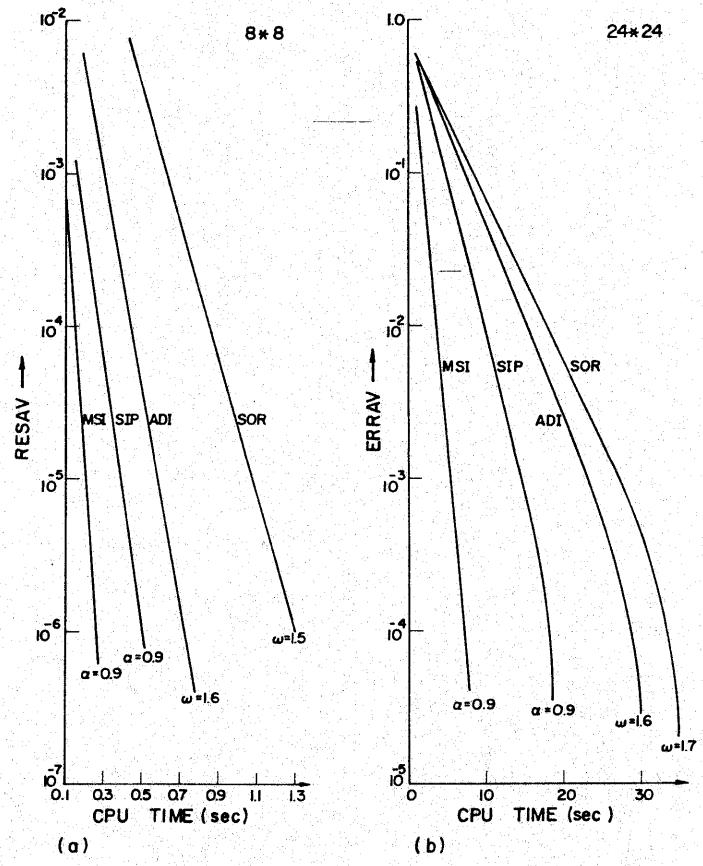


Fig. 10 Comparison of computational methods for circular cylinder problem.

and ADI reverts to that previously observed. The computational time requirements for each of the methods for a specified average relative error are presented graphically for the two mesh arrangements in Fig. 11. Again it is observed that the computational cost savings available through the use of the proposed MSI procedure increase as higher accuracy of the solution to the algebraic equation set is demanded. Further, as higher accuracy of the solution to the physical problem (through the use of a larger number of control volumes) is demanded, the economic advantage of the proposed MSI procedure becomes even more pronounced.

Table 2 Comparison of Solution Methods for Convective/Neuman Boundary Problem

Method	CPU time (8 X 8, s)	Ratio	CPU time (24 X 24, s)	Ratio
MSI	0.3466	1	11.399	1
SIP	1.3433	3.876	45.42	3.985
ADI	1.0000	2.885	88.24	7.741
SOR	2.43328	7.020	119.638	10.50

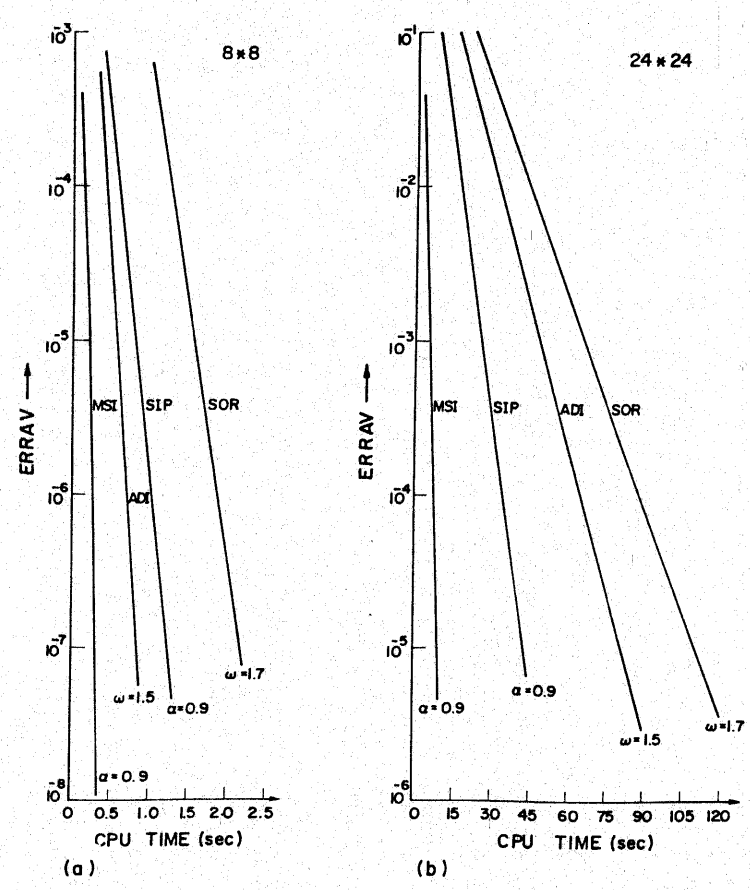


Fig. 11 Comparison of computational methods for convection/Neuman problem.

DISCUSSION AND CONCLUSIONS

A new procedure for solving the algebraic equations that arise in the finite-difference representation of field problems has been presented. The motivation for development of this MSI procedure has been to lower the computational cost from that currently required, to enable the application of strongly implicit methods to nine-point difference formulations, and to reduce the dependence of the convergence characteristics of the strongly implicit method on the partial cancellation parameter α. All of these objectives have been realized by the work presented in this paper.

The new MSI procedure has been extensively examined through its application to the test problem illustrated in Fig. 5, *a* and *b*. By application to this relatively simple problem, it was possible to examine in detail the many important attributes of an equation solver and to make comparisons with the previously available strongly implicit method. Having established the characteristics of the method, it was then possible to examine two different problems for which the heat flow pattern is far more complex than that for the first test problem. The method performed extremely well on both of these problems in comparison to the currently employed methods.

On the basis of the testing and experience gained with the new MSI procedure and the comparisons made with the alternate procedures, the following conclusions can be advanced regarding the procedure.

1. The MSI procedure is less sensitive to the parameter α than is the SIP method.
2. The MSI procedure can be confidently employed for a very wide range of $\Delta x/\Delta y$.
3. The optimum value of α is relatively insensitive to problem parameters.
4. The MSI procedure is insensitive to boundary-condition specification.
5. Relaxation is not useful for improving the convergence of the procedure.
6. Renumbering the grid network after each iteration is not required.
7. The computational cost for solution varies from 25 to 50% of that of the most economical alternative, based on the problems examined in this paper.
8. The MSI procedure is capable of use with nine-point formulations with no increase in computational cost.

In view of the above conclusions, and their relevance to the state of the art of solving field problems, we feel that this work provides a significant contribution to those involved in the solution of field problems.

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