distribution of flaws in the filament, fracture of several of the filaments occurs at isolated locations at loads below that for gross failure. At the end of the fractured filament, a stress concentration exists which under subsequent loading may initiate a small localized crack in the matrix.

Under continued loading, other filament-matrix cracks are formed at various sites in the composite. As gross failure of the composite is approached, a sufficient number of matrix cracks will have formed so that adjacent ones coalesce by shearing of the matrix. This process of filament failure, local matrix cracking, and subsequent shearing coalescence of neighbors continues until gross failure occurs.

One important feature of the gross failure process appears to depend upon the relationship between applied stress and the rate of growth of crack density such as shown experimentally by Rosen[1] for parallel glass fibers in an epoxy matrix. Cracking was first observed to occur as a result of fracture of the glass fibers at loads considerably below the ultimate of the system. As the load was increased, the crack density also increased.

If it is assumed that the stress-dependent rate of increase of crack density, $dc/d\sigma$, is proportional to the instantaneous value of crack density, this would lead to an exponential relationship between crack density and applied stress in the form

$$c/c_u = \exp(-a\sigma_u)(1 - \sigma/\sigma_u) \qquad (1)$$

The test data of Rosen were utilized to test this hypothesis. As shown in Fig. 1, the data behave in a somewhat random fashion at low values of σ/σ_u. At a value of 0.6, however, the experimental data reveal an excellent fit to a straight line on the semilogarithmic plot of Fig. 1, indicating the potential value of Eq. (1). Should the validity of Eq. (1) be established, the significance to nondestructive testing may also be considerable.

Reference

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Detection of Oblique Shocks in a Conical Nozzle with a Circular-Arc Throat

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Nomenclature

M = Mach number
p = static pressure
p_t' = pitot pressure
p_t = stagnation pressure at nozzle inlet
r = nozzle radius
r_{th} = nozzle throat radius
r_c = nozzle throat radius of curvature
T_{to} = stagnation temperature
z = axial distance from nozzle inlet
γ = specific heat ratio

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Introduction

Method of characteristics predictions of supersonic flow through conical nozzles[1–3] reveal shock formation where characteristics originating just downstream of the tangency between the conical divergent section and throat curvature section approach the nozzle axis. This shock is not associated with overexpanded nozzle operation.

This note presents some measurements that confirm the predicted shock formation. The pitot tube measurements were made along and slightly off the axis of a nozzle that has a convergent half-angle of 45°, a divergent half-angle of 15°, and a relatively small throat radius of curvature so that the ratio of throat radius of curvature to throat radius r_c/r_{th} is 0.625. These measurements were made with dried air at a stagnation temperature of 530°R and a stagnation pressure of 151 psia. The nozzle discharged into the atmosphere.

The pitot tube, whose 0.042-in. diam was small compared with the nozzle throat diameter of 1.6 in., was motor-driven along the nozzle axis from the exit, and a helipot was used to determine probe location. The local pitot pressure and the stagnation pressure measured at the nozzle inlet were read on Heise gages, except in the region just downstream of the throat where the small difference between pitot and stagnation pressure necessitated using a differential mercury manometer. Wall static pressures were also measured. These are described in Ref. 4. The nozzle inlet boundary layer was turbulent, as determined from boundary-layer measurements, and the thickness was about 2% of the 2.5-in. inlet radius.

Experimental Results and Discussion

The ratio of local pitot pressure to stagnation pressure at the nozzle inlet is shown in Fig. 1. Proceeding along the nozzle axis from the throat, one sees that the pitot pressure, downstream of the pitot shock, decreases as the flow accelerates and the Mach number increases. A rather abrupt rise in pitot pressure then occurs. The rise results from the flow passing through oblique shocks upstream of the pitot shock. Compared with the shock-free flow, the entropy increase is less, and the pitot reads nearer the stagnation pressure. The pitot pressure decreases again downstream of the rise as the flow is again accelerated. The pitot pressure rise across the oblique shock structure is seen to be appreciable, amounting to about 0.3 of the stagnation pressure, and thus is easily detectable.

From these measurements, and the pitot tube measurements 0.20 in. radially from the nozzle axis, as shown in Fig. 1, the oblique shock structure shown in the vicinity of the nozzle axis could be determined. The upstream leg extends in a direction toward the wall region just downstream of the

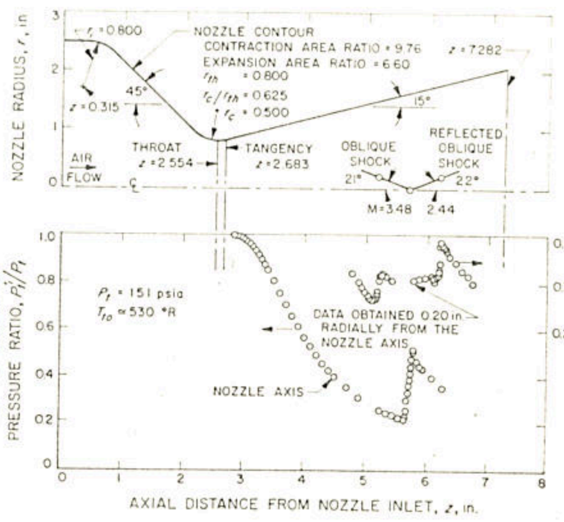

Fig. 1 Ratio of pitot to stagnation pressure.

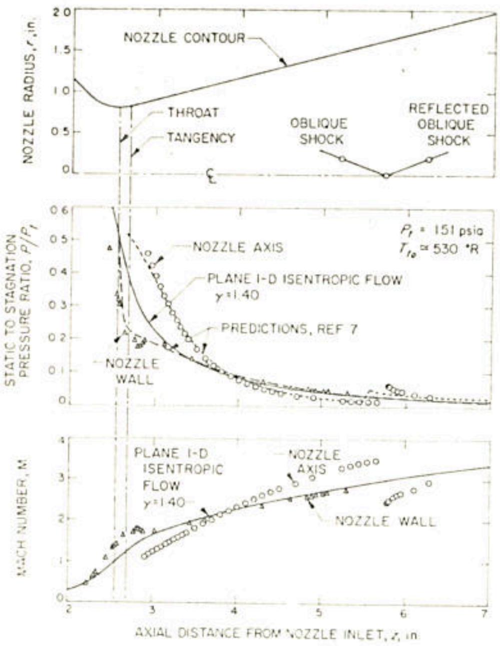

Fig. 2 Ratio of static to stagnation pressure and Mach number along the nozzle axis and wall.

tangency between the circular-arc throat and the conical divergent section. The flow is turned toward the axis by this shock and then turned back again, parallel to the axis, by the reflected oblique shock which extends downstream. The oblique shocks are weak, so that the calculated stagnation pressure decrease across the shock structure is about 1%, and the compression process is nearly isentropic.

From the pitot pressure to stagnation pressure ratio, the static pressure and Mach number distributions along the axis were calculated. These distributions are shown in Fig. 2 along with wall pressure measurements from which Mach numbers were calculated for isentropic flow. Reference to the one-dimensional isentropic flow prediction indicates the magnitude of deviations from one-dimensional flow. The wall static pressure rise (adverse pressure gradient) just downstream of the tangency is believed to be associated with the change in direction of the momentum of the gas in flowing from the circular-arc throat, where the relatively small throat radius of curvature induces a strong angular motion in the flow, and then into the conical section, where the flow eventually becomes conical. This static pressure rise, not observable in Ref. 4, was detected by additional pressure taps installed in the region of interest following the earlier tests. The persistence of the strong angular motion in the flow can lead to an overturning of the flow through the throat region, so that streamlines near the wall are inclined to the conical wall downstream, as indicated by calculations.[3] For the flow then to become parallel to the conical wall, a compressive turning of the flow is necessary that can lead to shock formation. Referring again to Fig. 2, one sees that the flow along the axis is rapidly accelerated, relative to the flow along the wall, up to the oblique shock structure across which the static pressure rises by a factor of 4.6. Downstream of the shock structure, the Mach number of the flow along the axis begins to approach the wall value, so that the exit flow may be fairly uniform. Since the nozzle was operated at an overexpanded condition, wall pressure measurements are not shown in the shock-induced flow separation region. The flow separation shock was inclined to the axis and was located downstream of the oblique shock structure at the axis.

Since insertion of the pitot tube, drive motor, and helipot in the flow dictated relatively low stagnation temperature operation, condensation or freezing of water vapor in the air undoubtedly occurred just downstream of the throat (e.g., see Ref. 5). To minimize condensation or freezing effects, the air supplied from the Jet Propulsion Laboratory (JPL) hypersonic wind-tunnel facility was dried to a dew-point temperature of about −70°F at atmospheric pressure, so that the mass fraction of water vapor to air was as low, 10^{-5}. No condensation or freezing effects were detectable, the wall static pressure rise just downstream of the throat (seen in Fig. 2) being also observable at higher stagnation temperature operation which precluded condensation or freezing anywhere in the nozzle. This was expected because of the small amount of water vapor in the air. In fact, if all the water vapor froze, the increase in total enthalpy would amount to only 0.01%. For this small increase, the flow can be considered isentropic.

Shock Formation in a Rectangular Nozzle

Shock formation is not peculiar to axisymmetric flow through a conical nozzle with a circular-arc throat. The counterpart in a nozzle with a rectangular cross section has a straight-walled divergence section and a circular-arc throat. A schlieren photograph by McKenney (Ref. 6, Fig. 6) of dry nitrogen flowing through a nozzle with a 15° divergent half-angle, and a ratio of throat radius of curvature to throat half-height of 3, reveals an oblique shock structure along the axis. The upstream leg extends toward the wall region just downstream of the tangency, apparently reaching near the thin boundary layer. The downstream leg, reflected from the axis, is also observable although it is diminished in strength as it approaches the wall. The reflection of the downstream leg from the wall is not discernible in the schlieren photograph. Unfortunately, wall static pressure measurements were not made in the tangency region.

Prediction of Shock Formation

The extent to which flow in the supersonic region is predictable is limited by throat configuration (a solution in the transonic region is needed to initiate the method of characteristics). Provided that the ratio of throat radius of curvature to throat radius r_c/r_{th} exceeds unity, two-dimensional isentropic flow predictions in the transonic region were found to be in agreement with measurements[4] and essentially independent of inlet configuration. For r_c/r_{th} less than unity, available predictions are inadequate as indicated by comparison with the measurements[4] in the 45°–15° nozzle with r_c/r_{th} = 0.625. In the absence of a transonic solution for the 45°–15° nozzle, predictions[7] are shown in Fig. 2 from the analysis by Migdal[8] in which the irrotational method of characteristics (γ = 1.4) was employed. The prediction assumes uniform and parallel flow to the axis along the nearly sonic surface (M = 1.01) that is taken to be a cone extending from the nozzle wall at the throat to the vertex 0.14 throat radii downstream. The penalty for initiating the prediction in this manner is higher predicted static pressures along the wall and lower predicted static pressures along the axis just downstream of the throat. The measured wall static pressure rise is not predicted. Of note, however, is the rapid coincidence to the measurements further downstream. Shock formation is predicted along the axis slightly upstream of its actual location. The predicted magnitude of the static pressure rise across the shock structure is less than that deduced from the measurements. Surprisingly, the analysis of Ref. 3 reveals that shock formation would not occur in a nozzle with a rectangular cross section, contrary to the measurements of McKenney, previously mentioned. A more realistic initiation of the method of characteristics in the transonic region may resolve this discrepancy. Boundary-layer displacement effects, not accounted for in the theory, may also be important, especially in the region just downstream of the tangency where the measured wall pressure rise is not predicted. The local adverse pressure gradient can affect the boundary layer and, thus, heat transfer from a hot gas to a cooled wall; consequently, its prediction is important.

With reference to the wall static pressure rise and shock formation, mention should be made of the Oswatitsch and Rothstein[9] prediction that indicates a step rise in the wall static pressure at the tangency between a circular-arc throat and either a straight-walled divergent section for a rectangular nozzle or a conical section for an axisymmetric nozzle where the second derivative of the nozzle contour is discontinuous. At the tangency, there is also a predicted step decrease in the flow velocity along the wall (ignoring the boundary layer), which in a supersonic flow can nearly be provided by an oblique shock wave. Although the prediction is unrealistic in regions where the nozzle curvature changes abruptly, such as the tangency, the prediction does indicate that the nozzle contour should have continuous second derivatives. Such a criterion may be a necessary condition, but not sufficient to avoid shock formation. For example, in contour nozzles with large wall curvatures, but with continuous second derivatives, shocks could exist downstream of the inflection point if the turning of the flow to become eventually uniform and parallel at the exit is not fairly gradual.

Some Consequences of Shock Formation

The effects of shock formation depend on the intended use of the nozzle. In the applied field, for instance in a rocket engine, the performance characteristics such as flow and thrust coefficients and heat transfer to the wall are important. Since the mass flow rate depends primarily on throat configuration (e.g., see Ref. 4), shock formation downstream of the throat should not influence the flow coefficient. It would influence the thrust and, consequently, the specific impulse to an extent not determinable from testing a single nozzle in which shock formation is found. To at least indicate the performance characteristics, values of the coefficients were shown for the nozzle in Ref. 4. The heat-transfer distribution in the conical divergent section, just downstream of the adverse pressure gradient, is influenced in a way that is dependent on boundary-layer structure. (These results will be subsequently published).

In those applications where either flow uniformity is sought, as in wind tunnels, or other phenomena are to be studied, such as nonequilibrium flow effects, elimination of shock formation is desirable. To eliminate shock formation, both Refs. 2 and 3 suggest attaching the conical divergence section at the inflection point of a conventionally designed contour nozzle. Since at the inflection point the nozzle slope is a maximum and the second derivative vanishes, this suggestion is tantamount to requiring continuity of the second derivative—a condition also found from the analysis of Ref. 9.

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Adjoint Systems in Nonconservative Problems of Elastic Stability

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In treating dissipative, dynamic systems, which are governed by nonself-adjoint linear operators, it is often found convenient to introduce the adjoint system (or field) and to consider formally a conservative process.[1,2] The original field contains an energy sink, and in the adjoint field an energy source of the same strength is incorporated in order to make the combined field conservative.

It is of interest to note that the notion of the adjoint field can be introduced also in treating nondissipative, nonconservative systems, i.e., dynamic systems subjected to circulatory forces. In particular, in structural systems subjected to follower forces, the consideration of adjoint force fields leads to interesting consequences. Indeed, for this class of nonconservative systems, both the original field and its adjoint force field are associated with energy sources,[3] and yet the combination of these two fields results in a conservative one.

As an example, consider the Beck problem,[4] i.e., a cantilevered elastic bar subjected at its free end to a compressive follower force (see Fig. 1). The equation of motion and the boundary conditions are

$$\left.\begin{aligned}&\frac{\partial^4 y}{\partial x^4} + F\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} = 0\\ &y = \frac{\partial y}{\partial x} = 0 \qquad \text{at} \qquad x = 0\\ &\frac{\partial^2 y}{\partial x^2} = \frac{\partial^3 y}{\partial x^3} = 0 \qquad \text{at} \qquad x = 1\end{aligned}\right\} \tag{1}$$

where dimensionless quantities are employed. We now construct the adjoint boundary-value problem by considering a function $z = z(x, t)$, defined for $0 \leq x \leq 1$ and $t \geq 0$, such that the following equation of motion and boundary conditions at $x = 0$ are satisfied identically:

$$\left.\begin{aligned}&\frac{\partial^4 z}{\partial x^4} + F\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0\\ &z = \frac{\partial z}{\partial x} = 0 \qquad \text{at} \qquad x = 0\end{aligned}\right\} \tag{2}$$

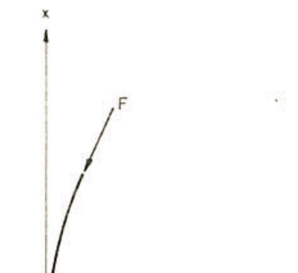

Fig. 1 A cantilever under a follower force (the Beck problem).

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