

# A CALCULATION PROCEDURE FOR HEAT, MASS AND MOMENTUM TRANSFER IN THREE-DIMENSIONAL PARABOLIC FLOWS

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**Abstract**—A general, numerical, marching procedure is presented for the calculation of the transport processes in three-dimensional flows characterised by the presence of one coordinate in which physical influences are exerted in only one direction. Such flows give rise to parabolic differential equations and so can be called three-dimensional parabolic flows. The procedure can be regarded as a boundary-layer method, provided it is recognised that, unlike earlier published methods with this name, it takes full account of the cross-stream diffusion of momentum, etc., and of the pressure variation in the cross-stream plane. The pressure field is determined by: first calculating an intermediate velocity field based on an estimated pressure field; and then obtaining appropriate correction so as to satisfy the continuity equation. To illustrate the procedure, calculations are presented for the developing laminar flow and heat transfer in a square duct with a laterally-moving wall.

## NOMENCLATURE

$A, B, C,$  } coefficients in the finite-difference equations;  
 $d,$  dimension of the duct cross-section (Fig. 6);  
 $D,$  coefficient of the pressure-gradient term;  
 $F,$  a body force, equation (2.2) etc.;  
 $F_U,$  forward flow at upstream station, equation (3.3);  
 $F_D,$  forward flow at downstream station, equation (3.3);  
 $J,$  diffusion flux, equation (2.5);  
 $L^y, L^z,$  } lateral flows defined by equation (3.3);  
 $m_p,$  a mass source defined by equation (2.13);  
 $\dot{m},$  mass-flow rate through the duct;  
 $p,$  pressure in the cross-stream momentum equations;  
 $\bar{p},$  pressure in the main-direction momentum equation;  
 $Pr,$  the Prandtl number;

$Re,$  a Reynolds number based on the duct side  $d$ ;  
 $S,$  the source term in equation (2.5);  
 $S_U, S_P,$  } finite-difference expressions representing the source term, equation (3.3);  
 $t,$  temperature;  
 $\bar{t},$  the bulk temperature;  
 $T^y, T^z,$  } transport coefficients defined by equation (3.3);  
 $\tilde{T},$  a modified form of  $T$  defined by equation (3.6);  
 $u,$  velocity component in the  $x$  direction;  
 $v,$  velocity component in the  $y$  direction;  
 $w,$  velocity component in the  $z$  direction;  
 $x,$  distance in the main-flow direction;  
 $y, z,$  } the cross-stream co-ordinates;  
 $\delta y, \delta z,$  } distances between neighbouring grid points (Fig. 5);  
 $\Delta x,$  size of the forward step (Fig. 4);  
 $\Delta y, \Delta z,$  } cross-stream dimensions of the control volume (Fig. 5);  
 $\Gamma,$  transport property in equation (3.1);

- $\rho$ , density;  
 $\tau$ , shear stress;  
 $\phi$ , a general dependent variable.

#### Superscripts

- $p$ , pressure;  
 $u$ ,  
 $v$ , } corresponding velocity components;  
 $w$ , }  
 $\phi$ , the dependent variable  $\phi$ ;  
 $*$ , an estimated pressure;  
 $'$ , the correction;  
 I, first phase of the double sweep;  
 II, second phase of the double sweep.

#### Subscripts

- $D$ , downstream station;  
 $e$ , the point  $e$  in Fig. 5;  
 $E$ , the point  $E$  in Fig. 3 or 5;  
 $in$ , inlet to the duct;  
 $max$ , maximum over the duct cross-section;  
 $n$ ,  
 $N$ , } the corresponding points in Figs. 3 or 5;  
 $P$ , }  
 $s$ , }  
 $S$ , }  
 $u$ , the velocity  $u$ ;  
 $U$ , upstream station;  
 $v$ , the velocity  $v$ ;  
 $w$ , the velocity  $w$ , or the point  $w$  in Fig. 5;  
 $W$ , the point  $W$  in Fig. 3 or 5;  
 $wall$ , the moving wall;  
 $xy$ , } the corresponding co-ordinate planes;  
 $xz$ , }  
 $\phi$ , the dependent variable  $\phi$ .

## 1. INTRODUCTION

### 1.1 *The purpose of the present paper*

BOUNDARY-LAYER theory is one of the most advanced and popular of all the branches of fluid mechanics. Text-books describe it; research workers add daily to its repertoire of methods and store of experimental knowledge; and students and their teachers find it an un-failing source of educational exercises and of

subjects for minor publications. Yet, from the point of view of engineering practice, the fruits of boundary-layer theory must be judged disappointing; despite the decades of development, the flow in an engine intake or over an aircraft fuselage, for example, must be determined, if at all, from experiment rather than calculation.

The reason is that almost all practically important boundary layers are three-dimensional. Even in the laboratory, the efforts of skilled experimenters fail to achieve sufficient two-dimensionality to allow adequate comparison with two-dimensional prediction procedures. (Extensive evidence of this is to be found, for example, in the proceedings of the 1968 Stanford Conference [1].) General techniques that are currently available for predicting boundary-layer phenomena on the other hand are exclusively two-dimensional in character.

It is true that a few techniques exist which may be applied to some special three-dimensional flows, and that these may have a limited success in predicting phenomena of the relevant class. However, the engineer needs general and flexible techniques, to which arbitrary initial and boundary conditions can be supplied in a straightforward way; and which yield predictions of velocities, of temperatures, of concentrations, and of the corresponding fluxes, without fuss or the supply of special insight.

It is in the nature of the problem that such techniques can be only of the finite-difference variety. Such numerical techniques exist for two-dimensional flows (e.g. Patankar and Spalding [2]); it is the purpose of the present paper to report the development of, and to describe and illustrate, a numerical procedure for predicting boundary-layer phenomena which are three-dimensional.

### 1.2 *Statement of the problem*

*Definition.* Here it must be made clear that we use the term "boundary layer" in a more general sense than is usual in the literature. We apply the term to all the flows which can be adequately

described by differential equations that are parabolic in one distance co-ordinate. Thus, we call a flow a boundary layer,

(a) if there exists a predominant direction of flow (i.e. there is no reverse flow in that direction),

(b) if the diffusion of momentum, heat, mass, etc. is negligible in that direction, and

(c) if the downstream pressure field has little influence on the upstream flow conditions. When these conditions are satisfied, the co-ordinate in the main flow direction becomes a "one-way" co-ordinate; i.e. the upstream conditions can determine the downstream flow properties, but not vice versa. It is this convenient behaviour of the boundary-layer flows that enables us to employ a marching integration from an upstream station to a downstream one.

Some readers may feel that this extension of the term "boundary layer" is inconvenient or unwarranted. It is for this reason that we use, in the title of the paper, the more precise but unfamiliar term "parabolic flows".

*Example.* In order to appreciate the main features of these flows, it is useful to consider the situation illustrated in Fig. 1. Air flows steadily through a duct of rectangular cross-section; through the floor of the duct there penetrates a jet of a different fluid, say steam, which is blown obliquely along the wall. Downstream of the injection plane, the steam mixes with the air; and the interchanges of momentum between the two streams co-operate with the pressure gradient along the duct and the friction on the walls to produce in the mixture a swirling motion which decreases in intensity with longitudinal distance. The task of our three-dimensional boundary-layer theory is to predict this process, and all that is connected with it.

Figure 2 clarifies the matter further, by exemplifying some of the quantities which the prediction procedure must supply. Figure 2a shows how the mixing of the steam and air produces variations of the steam concentration that would be detected by analysis of the mixture clinging to the floor of the duct. This lateral

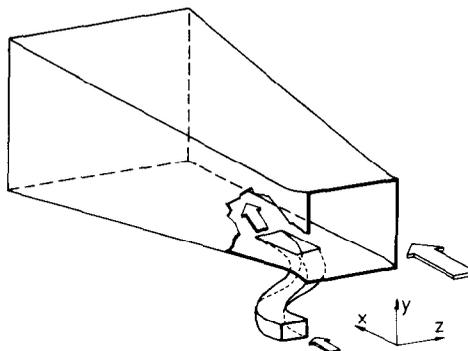


FIG. 1. Illustration of a three-dimensional parabolic flow.

spread of steam is the result of both convection and diffusion in the  $z$  direction; so we must be sure that both these processes are represented in the equations which are solved.

Figure 2b sketches the variation with longitudinal distance  $x$  of the space-average pressure across the duct,  $\bar{p}$ . This quantity is indicated as rising at first, in response to the injector-like action of the jet; thereafter it falls, as a consequence of friction on the walls. The sketch reminds us that  $\bar{p}$  must be calculated; we do not know, as we do in some external boundary-layer situations, the pressure variation before the start of the computation.

Also to be computed is the variation of the longitudinal velocity component,  $u$ . Figure 2c illustrates, by a contour diagram, the probable form of this variation at the outlet section; the highest value of  $u$  appears in the bottom right-hand corner, because of the oblique injection of the fast-moving steam jet. The  $u$  variation is influenced by the gradient of the longitudinal momentum, by the shear stresses on the  $xy$  and  $xz$  planes, and by the convection of momentum from upstream. The differential equation governing  $u$  must express these influences individually and in simultaneous action.

Finally, Fig. 2d represents, by way of a set of vectors, the motion of the fluid in the plane of the duct outlet; it shows a large general vortex, in the sense resulting from the oblique injection

of steam, with minor vortices of opposite sense in two of the corners. The values of the velocity components,  $v$  and  $w$ , are the result of the interaction with the convected momenta of the shear stresses and the normal stresses on the  $xy$  and  $xz$  planes. It is therefore necessary to take these

Nash [7], Krause *et al.* [8], and Wang [9] have developed calculation procedures for the three-dimensional boundary layer outside of a solid body.

Although satisfactory, no doubt, for the particular purposes which their authors had in

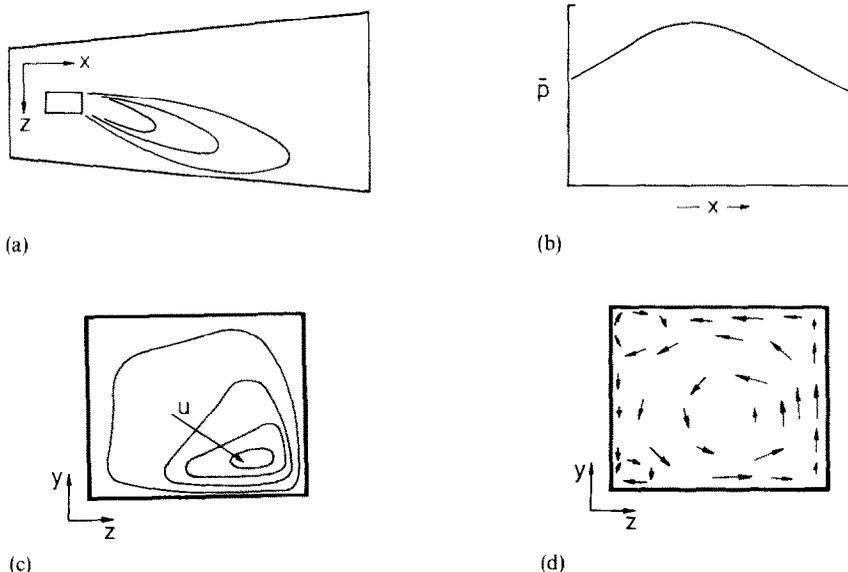


FIG. 2. Illustration of some of the quantities to be predicted by the calculation procedure. (a) Contours of steam concentration at the floor of the duct. (b) Variation of mean pressure with longitudinal distance. (c) Contours of longitudinal velocity at the outlet plane. (d) Velocity vectors in the outlet plane.

stresses into account in the computation; and the calculations of the pressure gradients will, it is clear, play a crucial part in the procedure.

### 1.3 Some remarks about previous work

There are a few papers in the literature which report finite-difference procedures for solving the three-dimensional boundary-layer equations. The first appears to be that of Raetz [3]; but no use of the method has been reported. In recent years, Hall [4], Dwyer [5], Fanelop [6],

mind, none of these methods will solve our general problem. The reasons are: firstly, the procedures neglect the stresses and diffusion fluxes across either the  $xy$  or  $xz$  plane; and secondly, they do not take full account of the pressure variations in the  $yz$  plane. These omissions rob the model of precisely those agents which, in many circumstances, have the most significant effect. Moreover, since all these procedures have been applied to only *external* boundary layers, they do not provide any means

of calculating the unknown pressure gradient in a confined flow.

Miller [10] has described a procedure which would indeed solve our general problem; he has applied it to the developing flow in ducts of arbitrary cross-section. His procedure, however, does not take advantage of the boundary-layer character of the flow, but treats the equations as elliptic in all the three space co-ordinates. Thus, Miller needs three-dimensional computer storage, the downstream boundary conditions, and excessive computer time. While looking for a method for *boundary-layer* flows, we should regard Miller's method as unnecessarily complex and hence unsuitable for our purposes.

When the available procedures in the special field of our enquiry are so seriously restricted (or complicated), it is helpful to look for guidance in related fields. Specifically, since steady three-dimensional flows and unsteady two-dimensional ones have several mathematical features in common, it is useful to enquire as to what methods have been employed for the latter brand of parabolic differential equations. There is a large literature on this subject, usefully digested by Harlow [11]. The papers most relevant to our present subject are those of Harlow and Welch [12], Amsden and Harlow [13], and Chorin [14]. These authors all use finite-difference procedures in which the dependent variables are the velocity components and the pressure (or a closely related quantity); the pressure is deduced from an equation which is obtained by the combination of the continuity equation and the momentum equations; and the idea is present of a first approximation to the solution, followed by a succeeding correction. It will later be seen that the method of the present paper shares these features.

It is appropriate to mention also some earlier work by the authors and their colleagues. Their two-dimensional boundary-layer procedure [2], when used for flows confined in ducts, involved calculating the pressure from the continuity equation by a non-iterative

self-correcting process. This feature, not wholly unlike that of methods in the previous paragraph, will be employed below. Secondly, two procedures for three-dimensional boundary layers have just recently been developed (Caretto, Curr and Spalding [15]); one of these solves the same equations as the present method, *albeit* in a different manner; the other suppresses the pressure as a main variable, in favour of the  $x$ -direction vorticity. The present method is a rival to these two recent methods, and, it now appears, a successful one.

#### 1.4 *Outline of the present paper*

The description of a numerical procedure for solving simultaneous equations can have two distinct aims, which it is seldom possible to accomplish simultaneously. The first aim is to convey to the reader the main principles, and the crucial tricks, and to leave him with the feeling that he could work out the rest for himself; the second is to present the particular equations, and to list the steps needed for their solution, with sufficient precision of detail to enable a computer programmer to begin his work.

Because the latter aim requires the equations to be written out in full, and because this entails a proliferation of subscripts that impede smooth reading and inhibit understanding, its fulfilment is deferred to a later section (Section 3); and even there the treatment is curtailed.

In Section 2 however, an attempt will be made to fulfil the first aim. Just sufficient of the details will be presented to convey the essential ideas; and the inessential features will be suppressed.

The second aim is difficult to fulfil within the normal length of a paper. Advantage is, therefore, taken of the fact that the procedure to be described here has many details in common with the present authors' two-dimensional procedure, which has been more completely reported [2]. Thus, the details given in Section 3 are by way of examples, and should be generalized and completed by reference to [2].

Section 4 describes an application of the

procedure to the flow in a square duct with a laterally-moving wall.

## 2. MAIN FEATURES OF THE CALCULATION PROCEDURE

### 2.1 The differential equations

*The equations.* We can now express the problem described in Section 1.2 as that of solving the following equations, written with reference to the Cartesian co-ordinates  $x, y, z$ :

Continuity:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad (2.1)$$

Momentum:

$$\begin{aligned} \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho wu) \\ = \frac{\partial \tau_{u,xz}}{\partial y} + \frac{\partial \tau_{u,xy}}{\partial z} - \frac{\partial \bar{p}}{\partial x} + F_u, \end{aligned} \quad (2.2)$$

$$\begin{aligned} \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2) + \frac{\partial}{\partial z}(\rho wv) \\ = \frac{\partial \tau_{v,xz}}{\partial y} + \frac{\partial \tau_{v,xy}}{\partial z} - \frac{\partial p}{\partial y} + F_v, \end{aligned} \quad (2.3)$$

$$\begin{aligned} \frac{\partial}{\partial x}(\rho uw) + \frac{\partial}{\partial y}(\rho vw) + \frac{\partial}{\partial z}(\rho w^2) \\ = \frac{\partial \tau_{w,xz}}{\partial y} + \frac{\partial \tau_{w,xy}}{\partial z} - \frac{\partial p}{\partial z} + F_w. \end{aligned} \quad (2.4)$$

Other conservation equations (general form):

$$\begin{aligned} \frac{\partial}{\partial x}(\rho u\phi) + \frac{\partial}{\partial y}(\rho v\phi) + \frac{\partial}{\partial z}(\rho w\phi) \\ = -\frac{\partial}{\partial y}(J_{\phi,xz}) - \frac{\partial}{\partial z}(J_{\phi,xy}) + S_\phi. \end{aligned} \quad (2.5)$$

In regard to these equations, it is necessary to explain both what is included and what is omitted. As to symbols,  $\rho$  stands for density,  $\tau$  for shear stress,  $J$  for diffusion flux,  $F$  for a body force; the symbol  $\phi$  can stand for any property which can be convected and diffused, for example, stagnation enthalpy, chemical-species

concentration, and turbulence energy;  $S_\phi$  is the corresponding volumetric source rate. The subscripts  $u, v$  and  $w$  indicate which component of the momentum is in question; the subscripts  $xy$  and  $xz$  denote the planes on which the stresses or fluxes act.

The omissions from the equations are the shear stresses and diffusion fluxes acting on the  $yz$  plane. These omissions accord with our definition of a boundary layer and with the consequent necessity to ensure that no influence from downstream can penetrate upstream; stresses and fluxes on the  $yz$  plane would allow such an influence.

*The uncoupling of longitudinal and lateral pressure gradients.* A further point to note is that the symbol  $\bar{p}$  used for the pressure in the  $x$ -momentum equation (2.2) is different from the symbol  $p$  in the two other momentum equations. This is a reminder of the fact that in our calculation procedure an inconsistency is deliberately introduced into the treatment of pressure, and that the quantities  $\bar{p}$  and  $p$  are calculated differently. The pressure  $\bar{p}$  can be thought of as a form of space-averaged pressure over a cross-section, and the gradient  $\partial\bar{p}/\partial x$  is supposed to be known (or calculated) before we proceed to get the lateral pressure gradients  $\partial p/\partial y$  and  $\partial p/\partial z$ . (The reader may find this point difficult to understand and appreciate at first; it should become clearer after perusal of Section 2.4 below.)

This practice is implicit in two-dimensional boundary-layer theories also; but it escapes notice because there is no necessity to solve the momentum equation for the cross-stream direction. Here we have two cross-stream directions; and we must solve the momentum equations for both of them, in order to find out how the fluid distributes itself between these two directions.

The practice is a necessary consequence of our intention to exploit the boundary-layer nature of the flow; it is the final step to be made in preventing downstream influences from propagating upstream. If the step is omitted, the result

is not increased in accuracy, as one might naively expect; it is often a solution which is wholly unrealistic physically. The inconsistency in the treatment of pressure, it may be said, is one part of the price we pay for making the equations parabolic; the gain is the freedom to employ marching integration, and to use two-dimensional computer storage, even though the flow is three-dimensional and the full equations are elliptic.

*Auxiliary information.* The differential equations do not alone specify the problem; we need additional information of two kinds: initial and boundary conditions for all the dependent variables ( $u, v, w, p, \phi$ ); and auxiliary equations allowing the density, shear stresses, diffusion fluxes, body forces and sources to be computed in terms of the dependent variables at each point in the field. Since this information is of the same kind as is needed for two-dimensional boundary layers, we shall treat it as well known, and allow it to be exemplified without preface in the subsequent discussion.

2.2 *The finite-difference equations*

*The "staggered grid".* Figure 3 shows how the points are arrayed in the  $yz$  plane at which are stored the variables  $u, v, w, p$  and  $\phi$ . The boomerang-shaped envelopes enclose the triads of points denoted by a single letter,  $N, S, E, W$ , or  $P$ . This arrangement, which is similar to the one used by Harlow *et al.*, has the convenient feature

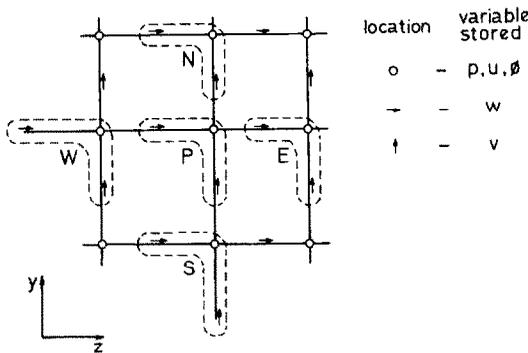


FIG. 3. The staggered grid.

that the cross-stream velocities  $v$  and  $w$  are stored at just the points at which they are needed for the calculation of the convective contribution to the balances of  $u$  and  $\phi$ ; and the pressures are stored so as to make it easy to calculate the pressure gradients which affect  $v$  and  $w$ .

*The finite-difference equations.* The differential equations of Section 2.1 can be expressed in the following finite-difference form:

$$C^v\{(\rho v)_N - (\rho v)_P\} + C^w\{(\rho w)_E - (\rho w)_P\} = C^u\{(\rho u)_{P,U} - (\rho u)_{P,D}\}. \quad (2.6)$$

$$u_P = A_N^u u_N + A_S^u u_S + A_E^u u_E + A_W^u u_W + B^u + D^u (\partial \bar{p} / \partial x), \quad (2.7)$$

$$v_P = A_N^v v_N + A_S^v v_S + A_E^v v_E + A_W^v v_W + B^v + D^v (p_P - p_S). \quad (2.8)$$

$$w_P = A_N^w w_N + A_S^w w_S + A_E^w w_E + A_W^w w_W + B^w + D^w (p_P - p_W). \quad (2.9)$$

$$\phi_P = A_N^\phi \phi_N + A_S^\phi \phi_S + A_E^\phi \phi_E + A_W^\phi \phi_W + B^\phi. \quad (2.10)$$

Here the  $A$  coefficients contain mass fluxes, viscosities, diffusion coefficients, etc.; the  $B$  coefficients express the effects of convection from the upstream  $x$  station, and of source terms (including body forces); the  $C$ 's are areas across which the fluid flows; and the  $D$ 's involve areas, mass flow rates, and other quantities. Subscripts  $D$  and  $U$  in equation (2.6) distinguish downstream (larger- $x$ ) values from upstream (smaller- $x$ ) ones; but, where neither is subscribed to a variable, the downstream value is meant.

The problem is to solve equations (2.6)-(2.10) simultaneously for all the  $u$ 's,  $v$ 's,  $w$ 's,  $p$ 's and  $\phi$ 's at the downstream  $x$  station; the  $A$ 's,  $B$ 's,  $C$ 's and  $D$ 's can be taken as known, because they can be evaluated with sufficient accuracy from values prevailing at the upstream station. We seek if possible a non-iterative means of solution.

2.3 *An outline of the solution procedure*

*The central idea.* If the pressures were known,

there would be little difficulty; for then the momentum equations would be uncoupled, and could be solved individually. They are *not* known in advance, but we can guess the pressures, obtain a first approximation to the velocity field, and then make corrections to the pressure field in such a sense as to bring the velocity field into conformity with the continuity equation.

The confined-flow procedure of the authors' two-dimensional boundary-layer theory has this guess-and-correct feature; but the correction is applied at the next step downstream. This deferred-correction technique could be adopted here; instead however, influenced by the examples of Chorin [14] and Amsden and Harlow [13], we have preferred to make the correction before proceeding to the next step.

*The cross-stream pressure and velocities.* Let us for the time being assume that we know  $(\partial\bar{p}/\partial x)$  and that we have solved equation (2.7) to get the downstream values of  $u$ . Now, the next step in our procedure is to obtain a preliminary set of  $v$  and  $w$  from:

$$v_p^* = A_N^v v_N^* + A_S^v v_S^* + A_E^v v_E^* + A_W^v v_W^* + B^v + D^v (p_p^* - p_s^*), \quad (2.11)$$

$$w_p^* = A_N^w w_N^* + A_S^w w_S^* + A_E^w w_E^* + A_W^w w_W^* + B^w + D^w (p_p^* - p_w^*), \quad (2.12)$$

where the superscript \* given to  $v$  and  $w$  denotes that these are based on an estimated pressure field  $p^*$ ; usually the upstream values of  $p$  are a good estimate.

The starred velocities  $v^*$  and  $w^*$  will in general not satisfy the continuity equation (2.6), but will produce a net mass source  $m_p$  for the point  $P$ . This is defined by:

$$m_p \equiv C^v \{(\rho v^*)_N - (\rho v^*)_P\} + C^w \{(\rho w^*)_E - (\rho w^*)_P\} + C^u \{(\rho u)_{P,D} - (\rho u)_{P,U}\}. \quad (2.13)$$

Now our aim is to correct the pressure and velocities so as to annihilate this mass source. For this, we write:

$$p = p^* + p', \quad (2.14)$$

where  $p'$  is the pressure correction. The velocity corrections then follow:

$$v_p = v_p^* + D^v (p'_P - p'_S), \quad (2.15)$$

$$w_p = w_p^* + D^w (p'_P - p'_W). \quad (2.16)$$

It should be noted that the last two are *not* rigorously derived from equations (2.8) and (2.9); we are using approximate forms\* of the momentum equations to give us our pressure corrections, just as we did in the two-dimensional confined-flow procedure; and we may expect the practice to suffice here, just as it did before.

The substitution of equations (2.15) and (2.16) into (2.6) gives:

$$p'_P = A_N^p p'_N + A_S^p p'_S + A_E^p p'_E + A_W^p p'_W + B^p. \quad (2.17)$$

where † the  $A$ 's involve  $C$ 's,  $D$ 's and  $\rho$ 's, and the mass source  $m_p$  has been incorporated into  $B^p$ . This equation can now be solved to yield the  $p'$ 's. Thereupon the  $p$ 's,  $v$ 's and  $w$ 's are computed from equations (2.14)–(2.16).

*The longitudinal pressure gradient.* The foregoing procedure for the calculation of  $p$ ,  $v$  and  $w$  was based on the assumption that we knew  $(\partial\bar{p}/\partial x)$  and could solve equation (2.7) for  $u$ . Here we disclose how  $(\partial\bar{p}/\partial x)$  can be obtained. For this purpose, we need to distinguish between external and confined flows. In external flows,  $(\partial\bar{p}/\partial x)$  is taken to be the same as the longitudinal pressure gradient prevailing in the irrotational free stream adjacent to the boundary layer. Then the solution of equation (2.7) is straightforward. In confined flow, we regard  $(\partial\bar{p}/\partial x)$  as uniform over a cross-section and obtain it from the integral mass-conservation equation in the following manner.

\* A correct implication of equation (2.8) would be:

$$v_p = v_p^* + D^v (p'_P - p'_S) + A_N^v (v_N - v_N^*) + A_S^v (v_S - v_S^*) + A_E^v (v_E - v_E^*) + A_W^v (v_W - v_W^*).$$

By dropping the last four terms on the right-hand side of this equation, we get equation (2.15).

† If there are appreciable compressibility effects, care is needed in calculating the densities. This point will not however be elaborated here.

At first, we make an estimate of  $(\partial\bar{p}/\partial x)$ , which is denoted by  $(\partial\bar{p}/\partial x)^*$ . This enables us to compute a  $u^*$  field from

$$u_p^* = A_N^u u_N^* + A_S^u u_S^* + A_E^u u_E^* + A_W^u u_W^* + B^u + D^u (\partial\bar{p}/\partial x)^*. \quad (2.18)$$

This preliminary velocity field will imply a total mass-flow rate  $\Sigma \rho u^* \Delta y \Delta z$  (taken over the duct cross-section) which will in general be different from the true mass-flow rate through the duct,  $\dot{m}$ , which can be computed directly from the inlet and boundary conditions. The difference can be used to lead us to the correct values of  $(\partial\bar{p}/\partial x)$ . For this, we write:

$$(\partial\bar{p}/\partial x) = (\partial\bar{p}/\partial x)^* + (\partial\bar{p}/\partial x)', \quad (2.19)$$

$$u_p = u_p^* + D^u (\partial\bar{p}/\partial x)'. \quad (2.20)$$

Since we want

$$\Sigma \rho u \Delta y \Delta z = \dot{m}, \quad (2.21)$$

we get, by the substitution of equation (2.20) into (2.21),

$$(\partial\bar{p}/\partial x)' = \frac{\dot{m} - \Sigma \rho u^* \Delta y \Delta z}{\Sigma \rho D^u \Delta y \Delta z}. \quad (2.22)$$

This gives us the required correction to the longitudinal pressure gradient; so now it is a simple matter to obtain  $(\partial\bar{p}/\partial x)$  and  $u$  from equations (2.19) and (2.20). The similarity between the equation set (2.11), (2.14), (2.15), and the set (2.18), (2.19), (2.20) should be very obvious. The important difference, however, is that, whereas  $p'$  is obtained from the *local* continuity equation,  $(\partial\bar{p}/\partial x)'$  is the outcome of the *overall* continuity equation.

*Other dependent variables.* So far, we have looked at equations (2.6)–(2.9) and obtained the three velocity components and pressure. The equation (2.10) for any other dependent variable  $\phi$  (such as stagnation enthalpy, chemical-species concentration etc.) does not offer any particular difficulty and can be solved straightaway. This completes one forward step.

*Solution of the finite-difference equations.* In the above description, we referred to “solving”

finite-difference equations like equation (2.10). The actual method of solution that we use can be summarized as follows: we employ two sweeps, one in the  $y$  and one in the  $z$  direction, of the standard tri-diagonal matrix algorithm (TDMA), which is used in the two-dimensional procedure [2] also. Thus, for equation (2.10),  $\phi_E$  and  $\phi_W$  are taken as constants when the sweep is in the  $y$  direction, and  $\phi_N$  and  $\phi_S$  are held constant for the sweep in the  $z$  direction. More details of this method will be given in Section 3.3.

#### 2.4 Some general remarks

*The Poisson equation for pressure.* At this stage, it will be clear that we obtain the velocity and pressure fields by the solution of the three momentum equations and of the equation (2.17) for the pressure correction  $p'$ , which is derived from the continuity equation. This equation for  $p'$  is just a new form of what is known in the literature as the Poisson equation for pressure. This interpretation may enable the reader to see more clearly why we must treat  $(\partial\bar{p}/\partial x)$  differently from  $(\partial p/\partial y)$  and  $(\partial p/\partial z)$ . A general Poisson equation will be elliptic in all the three space co-ordinates and will not allow solution by a marching technique. To be able to march in the  $x$  direction, we must treat the term  $(\partial^2 p/\partial x^2)$  as known and regard the equation as elliptic in only the  $y$  and  $z$  co-ordinates. This is precisely why we obtain  $(\partial\bar{p}/\partial x)$  *before* the Poisson equation for  $p'$  is solved.

*The boundary conditions.* One of the less obvious but important features of the present method is the ease with which the hydrodynamic boundary conditions can be applied. When we solve for the starred velocity field we can use the actual boundary conditions for velocity, as the starred velocities are expected to be very close to the true velocities. After this is done, the boundary conditions for the pressure correction are also simple: at a wall boundary for example, there will be no velocity correction at the boundary, and so the gradient of  $p'$  normal to that boundary must be zero; at a boundary

adjacent to a free stream on the other hand, the pressure is known, and if  $p^*$  is set equal to this pressure, the correction  $p'$  at the boundary must be zero. In contrast to the present procedure, the methods that use vorticity as a variable require complicated derivations of the boundary conditions [15].

*The non-iterative nature of the procedure.* Numerical procedures for solving the partial differential equations in fluid dynamics tend to be iterative for three main reasons: (a) the equations are non-linear; (b) the pressure renders the continuity and momentum equations strongly linked; and (c) a direct solution of the implicit finite-difference equations, even when they are linear, is time-consuming. We have attempted to make the present procedure non-iterative by: (a) the calculation of the  $A$ ,  $B$ ,  $C$  and  $D$  coefficients in the finite-difference equations from values at the upstream station; (Thus, we "force" the equations to be linear); (b) the use of approximate forms of momentum equations (equations (2.15), (2.16) and (2.20)); and (c) the solution of the finite-difference equations by the two sweeps of the TDMA. It is true that these three "tricks" introduce some errors in our solution compared to a solution produced by a fully iterative procedure. But, firstly, these errors are of the same kind as the "truncation" errors in any finite-difference procedure and hence can be reduced to an acceptable level by the use of small forward steps; and secondly, it is possible for us, at the end of each forward step, to calculate the error in satisfying each conservation equation (these can be considered as mass or momentum sources which our numerical approximations have introduced), and then to make a corresponding correction at the next step downstream. Thus, by leaving errors which can be detected and, if necessary, corrected for, we enjoy the benefits of a non-iterative procedure without serious penalty.

We hope by now to have conveyed to the reader the essential features of our calculation procedure. The actual algebraic details remain

to be given. It is to this matter that we now turn.

### 3. SOME DETAILS OF THE CALCULATION PROCEDURE

#### 3.1 Restrictions

The general calculation procedure described so far is restricted only by those conditions which define parabolic flows, and which are described in Section 1.2. However, the algebraic details of the general procedure with various types of boundary conditions, grid systems, auxiliary information, etc. will be quite lengthy and tedious to report here. For this reason, we shall present the equations for a uniform-property laminar flow and give only the important details. The remaining details are either so straightforward that the reader could work them out himself, or are similar to the corresponding features of our two-dimensional procedure [2]. We shall use a Cartesian coordinate system  $xyz$ .

#### 3.2 The finite-difference equations

*The differential equation considered.* For a laminar uniform-property flow, equation (2.5) takes the form:

$$\frac{\partial}{\partial x}(\rho u \phi) + \frac{\partial}{\partial y}(\rho v \phi) + \frac{\partial}{\partial z}(\rho w \phi) = \Gamma \frac{\partial^2 \phi}{\partial y^2} + \Gamma \frac{\partial^2 \phi}{\partial z^2} + S_\phi, \quad (3.1)$$

where  $\Gamma$  is the transport property such as viscosity. When  $\phi$  stands for a velocity component, the differential equation has the same form except that a pressure-gradient term appears on the right-hand side. (This term should be written separately, and not included in  $S_\phi$ , as we treat the pressure as an unknown.) Therefore, it will be sufficient to describe here how equation (3.1) is transformed into a finite-difference equation.

*Some basic decisions.* We transform equation (3.1) into a finite-difference equation by inte-

grating it over the control volume shown in Fig. 4 by dotted lines. Figure 5 gives more details of the  $yz$  face of the control volume. The points  $n, s, e, w$  are the midpoints of the lines  $PN, PS, PE$  and  $PW$  respectively. (The "boomerangs" in Fig. 3 have disappeared in Figs. 4 and 5; there the points  $n, s, e, w$  have been introduced.

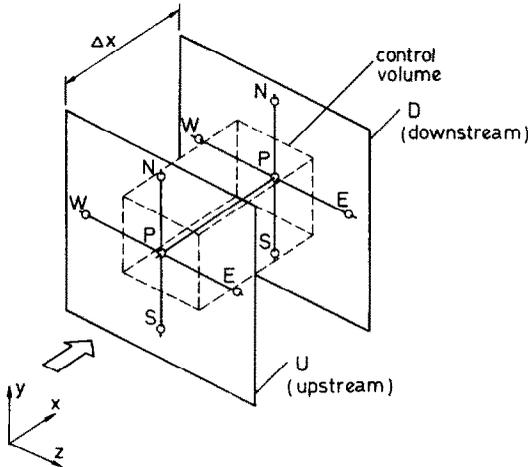


FIG. 4. The control volume used to obtain the finite-difference equation.

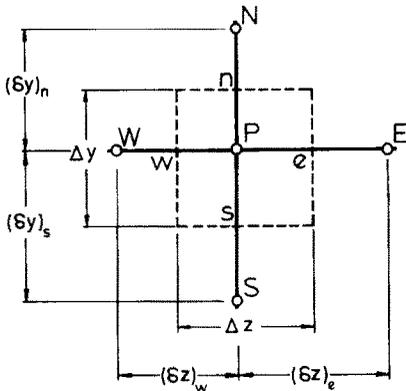


FIG. 5. The  $yz$  face of the control volume.

which enable us to present the algebraic details more precisely.) We make the following assumptions about the variation of  $\phi$  between the grid points:

(a) In the  $x$  direction  $\phi$  varies in a stepwise

manner; i.e. the downstream ( $x = x_D$ ) values of  $\phi$  are supposed to prevail over the interval from  $x_U$  to  $x_D$  except at  $x_U$ . This makes our finite-difference scheme a fully-implicit one.

(b) For the calculation of the  $x$ -direction convection and of source terms that may depend on  $\phi$ , the variation of  $\phi$  in the  $yz$  plane is also taken to be stepwise. Thus, in the  $yz$  plane the value of  $\phi$  is assumed to remain uniform and equal to  $\phi_P$  over the dotted rectangle (Fig. 5) surrounding the point  $P$  and to change sharply to  $\phi_N, \phi_S, \phi_E$  or  $\phi_W$  outside the rectangle.

(c) For the cross-stream convection from the  $xy$  and  $xz$  faces of the control volume, the value of  $\phi$  convected is taken to be the arithmetic mean of the  $\phi$  values on either side of that face, except when this practice is altered by the "high-lateral-flux modification" mentioned below. Thus we use a convenient combination of the central-difference and upwind-difference formulae for the first-order derivatives.

(d) For diffusion across the  $xy$  and  $xz$  faces of the control volume, we assume that  $\phi$  varies linearly between grid points, except when the high-lateral-flux modification dictates otherwise.

*Details of the main finite-difference equation.* When the above-mentioned decisions are taken, it is a simple matter to obtain the finite-difference equation by integration of equation (3.1) over the control volume. We get:

$$\begin{aligned}
 & \underbrace{F_D \phi_P - F_U \phi_{P,U}}_{(1)} + \underbrace{L_n^y(\phi_N + \phi_P) - L_s^y(\phi_S + \phi_P)}_{(2)} \\
 & + \underbrace{L_e^z(\phi_E + \phi_P) - L_w^z(\phi_W + \phi_P)}_{(3)} \\
 & = \underbrace{T_n^y(\phi_N - \phi_P) - T_s^y(\phi_P - \phi_S)}_{(4)} \\
 & + \underbrace{T_e^z(\phi_E - \phi_P) - T_w^z(\phi_P - \phi_W)}_{(5)} \\
 & \quad \quad \quad \underbrace{+ S_U + S_P \phi_P}_{(6)} \quad (3.2)
 \end{aligned}$$

where the numbers in the parentheses indicate the corresponding terms in equation (3.1), and the new symbols are defined as follows:

$$\left. \begin{aligned}
 F_U &\equiv \frac{(\Delta y)(\Delta z)}{\Delta x} (\rho u)_{P,U} \\
 L^y &\equiv \frac{(\Delta z)}{2} (\rho v)_U \\
 L^z &\equiv \frac{(\Delta y)}{2} (\rho w)_U \\
 F_D &\equiv F_U - 2L_n^y + 2L_s^y - 2L_e^z + 2L_w^z \\
 T^y &\equiv \frac{\Gamma(\Delta z)}{\delta y} \\
 T^z &\equiv \frac{\Gamma(\Delta y)}{\delta z} \\
 (S_U + S_P \phi_P) &\equiv S_{\phi, P}(\Delta y)(\Delta z)
 \end{aligned} \right\} (3.3)$$

Rearranging the terms, we get :

$$\phi_P = A_N \phi_N + A_S \phi_S + A_E \phi_E + A_W \phi_W + B, \tag{3.4}$$

where

$$\left. \begin{aligned}
 A_N &\equiv A'_N/A'_P, \\
 A_S &\equiv A'_S/A'_P, \\
 A_E &\equiv A'_E/A'_P, \\
 A_W &\equiv A'_W/A'_P, \\
 B &\equiv B'/A'_P; \\
 A'_N &\equiv T_n^y - L_n^y, \\
 A'_S &\equiv T_s^y + L_s^y, \\
 A'_E &\equiv T_e^z - L_e^z, \\
 A'_W &\equiv T_w^z + L_w^z, \\
 B' &\equiv F_U \phi_{P,U} + S_U, \\
 A'_P &\equiv A'_N + A'_S + A'_E + A'_W + F_U - S_P.
 \end{aligned} \right\} (3.5)$$

*The high-lateral-flux modification.* When the lateral flow (denoted by the symbol  $L$ ) is large, some of the coefficients  $A_N, A_S, A_E, A_W$  can become negative; this event leads to physically unrealistic results. The cure is a simple one and is discussed at length in [2], where it is given the name "the high-lateral-flux modification".

Here we merely state that the modification consists of replacing all the  $T$ 's by  $\tilde{T}$ 's defined by:

$$\tilde{T} \equiv \left(\frac{1}{2}\right) \{T + |L| + |T - |L||\}, \tag{3.6}$$

where the  $T$  and  $L$  should be the corresponding ones (e.g.  $T_n^y$  with  $L_n^y$ ). It should be noted that this modification becomes "active" only when  $|L| > T$ ;  $T$  itself is always positive.

*Finite-difference equations for velocity components.* As mentioned earlier, the difference equations for  $u, v$  and  $w$  will be similar to equation (3.4) except for an additional pressure term. In deriving the equations for the cross-stream velocities  $v$  and  $w$ , we must note that, since  $v$  and  $w$  have "staggered" storage locations, they require different control volumes. The actual details, however, will not be given here.

*Finite-difference equation for pressure correction.* If we write equations (2.15) and (2.16) as:

$$\left. \begin{aligned}
 v_n &= v_n^* + D_n^v (p'_N - p'_P), \\
 v_s &= v_s^* + D_s^v (p'_P - p'_S), \\
 w_e &= w_e^* + D_e^w (p'_E - p'_P), \\
 w_w &= w_w^* + D_w^w (p'_P - p'_W),
 \end{aligned} \right\} (3.7)$$

the continuity equation written for the control volume shown in Fig. 4 becomes:

$$\begin{aligned}
 F_U \left\{ \frac{u_{P,D}}{u_{P,U}} - 1 \right\} + 2L_n^{y*} - 2L_s^{y*} + 2L_e^{z*} \\
 - 2L_w^{z*} + \rho \Delta z D_n^v (p'_N - p'_P) + \rho \Delta z D_s^v (p'_S \\
 - p'_P) + \rho \Delta y D_e^w (p'_E - p'_P) \\
 + \rho \Delta y D_w^w (p'_W - p'_P) = 0. \tag{3.8}
 \end{aligned}$$

The superscript \* on the  $L$ 's denotes that these are calculated from the starred velocity components. Now it is a mere matter of rearrangement to get equation (2.17).

### 3.3 Solution of the finite-difference equations

*The double sweep.* The finite-difference equations like (3.4) can be solved by the successive use of the TDMA in the  $y$  and  $z$  directions. For the  $y$ -direction sweep, we write:

$$\phi_P^I = A_N \phi_N^I + A_S \phi_S^I + (A_E \phi_{E,U} + A_W \phi_{W,U} + B), \quad (3.9)$$

where the expression in the parentheses is known and the TDMA can be applied. The superscript I denotes the values obtained from this first phase of solution. The second phase, namely the  $z$ -direction sweep, is the solution of:

$$\phi_P^{II} = A_E \phi_E^{II} + A_W \phi_W^{II} + (A_N \phi_N^I + A_S \phi_S^I + B) \quad (3.10)$$

in a similar manner.

*Remarks.* It is true that the above procedure does not give us an exact solution of the finite-difference equations; but its use is advocated on the following considerations:

- (1) It can be easily seen that, when the  $y$ -direction coefficients  $A_N$ ,  $A_S$  are much smaller or much larger in magnitude than  $A_E$ ,  $A_W$ , the above procedure does give a nearly correct solution.
- (2) When the forward step  $\Delta x$  is small, the equation is dominated by  $B$  which contains the upstream value  $\phi_{P,U}$ ; then the use of slightly approximate values of  $\phi_N$ ,  $\phi_S$ ,  $\phi_E$ ,  $\phi_W$  introduce a very small error in  $\phi_P$ .
- (3) The last remark applies to all the finite-difference equations except the one for the pressure correction, which does not have an "upstream convection" term. For the pressure-correction equation, therefore, it may be worthwhile obtaining greater accuracy by *repeating* the double sweep a few times. Usually about three executions of the double sweep are sufficient.
- (4) Thus, to reduce the error resulting from the TDMA-double-sweep method of solution, we can use one or more of the following devices:
  - (i) use smaller forward step;
  - (ii) repeat the double sweep a small number of times;
  - (iii) calculate the error at the end of the forward step and correct for it during the next step.

### 3.4 Some miscellaneous matters

Many details of the calculation procedure

still remain to be reported. Here we merely draw attention to a few points and state that these can be handled by means similar to those in the authors' two-dimensional procedure [2].

*The turbulent boundary layer.* When the flow is turbulent rather than laminar, the same calculation procedure is to be used except that the laminar viscosity and other transport properties are to be replaced by "effective" transport coefficients given by a "turbulence model".

*The specified-flux boundary.* When at a wall boundary the heat flux (rather than the temperature) is specified, the finite-difference equation for a control volume adjacent to that boundary must be rewritten in such a way that the coefficient of the boundary temperature is zero.

*The wall functions.* Often the variations of the dependent variables are quite steep near a wall boundary and therefore the diffusion flux at the wall cannot be accurately obtained from a linear-profile assumption for  $\phi$ . In such cases, one can employ a function (called the wall function in [2]) for the flux at the wall; this takes into account the non-linearity of the  $\phi$  profile resulting from pressure gradient, mass transfer, transport-property variation etc.

*Adjusting grids.* In this paper, we have used a Cartesian coordinate system throughout; but it is possible to employ other coordinate systems which may be convenient for particular problems. For example, the flow in a duct of elliptic cross-section can be conveniently calculated on a curvilinear orthogonal coordinate system in the cross-stream plane. For external boundary layers, it is profitable to use a grid (as in [2]), which expands or contracts as the boundary-layer thickness increases or decreases.

## 4. AN APPLICATION OF THE CALCULATION PROCEDURE

### 4.1 Statement of the problem

Here we illustrate the use of the present method by applying it to the developing flow

and heat transfer in a duct of square cross-section with a laterally-moving wall, as shown in Fig. 6. This flow situation is found in screw extruders, bearing lubricators, membrane oxygenators etc. Further, in regions of fully-developed flow, the cross-stream velocity and pressure fields are identical to those in a steady

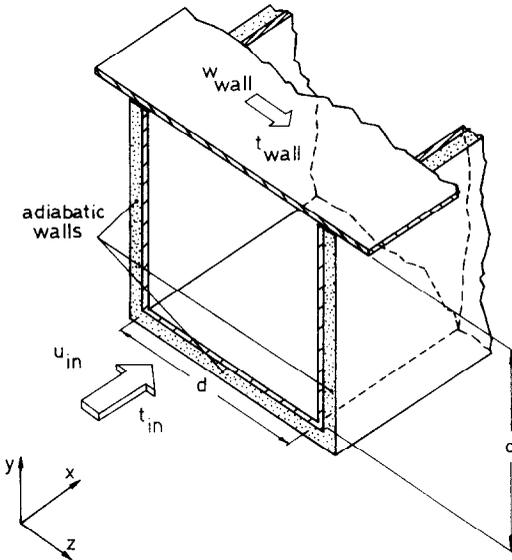


FIG. 6. The flow situation considered.

two-dimensional flow in a square cavity with a moving wall.\* The latter problem has been analysed by many authors (for example, [16–18]), and we have their solutions for comparison.

The flow is regarded as laminar, and the fluid properties as uniform. At the inlet, the velocity, pressure and temperature are taken to be uniform over the cross-section. The temperature of the moving wall is held at a fixed value, whereas the other three walls are considered adiabatic. Buoyancy effects are neglected.

\* This can be easily understood if we note that, in the fully-developed region, all the velocity components cease to vary with  $x$ . Then the cross-stream velocities  $v$  and  $w$  are governed by precisely the same equations as those for a two-dimensional flow.

#### 4.2 Details of the computation

The computations were performed on an IBM 7094 computer. A uniform rectangular grid of  $16 \times 16$  nodes was used for all runs except those which were made to examine the effect of the grid size. Each forward step took 2 s of computer time. About 100 forward steps were necessary to attain the fully-developed situation.

#### 4.3 Results

*The mean pressure.* Figure 7 shows the variation of the mean pressure with the distance along the duct for various velocities of the moving wall. For the case of the stationary wall, our predictions are compared with the experimental data of [19]; the agreement can be seen to be very good.

*The effect of the grid size.* In Fig. 8 are plotted the predictions of mean-pressure variation for various grid sizes. As can be expected, the successive refinement of the grid takes us asymptotically towards the correct solution. We can conclude that a  $16 \times 16$  grid gives us a sufficiently accurate solution for this problem.

*The velocity field.* Figure 9 shows the variation of the maximum longitudinal velocity with the distance along the duct. Once again, the predictions for zero wall velocity are compared with the experimental data of [20] and the agreement is good. Figures 10 and 11 refer to the fully-developed (large- $x$ ) region of the flow. In Fig. 10 are presented the contours of the longitudinal velocity for various velocities of the moving wall. Figure 11 compares the variation of a cross-stream velocity (along a centre-line of the cross-section) with the numerical results of Burggraf [16], who solved the steady two-dimensional square-cavity problem. Once again the agreement is very satisfactory.

*The temperature field.* Figure 12 shows how the bulk temperature of the fluid rises with the longitudinal distance. As can be expected, the higher the wall velocity, the faster is the rise of the bulk temperature. Figure 13 shows the effect of the Prandtl number on the bulk-

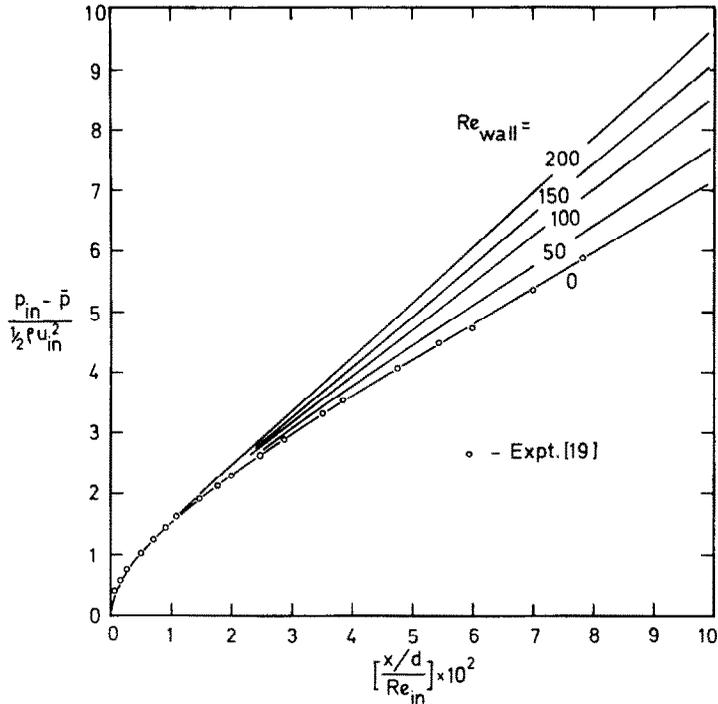


FIG. 7. Variation of mean pressure with longitudinal distance.

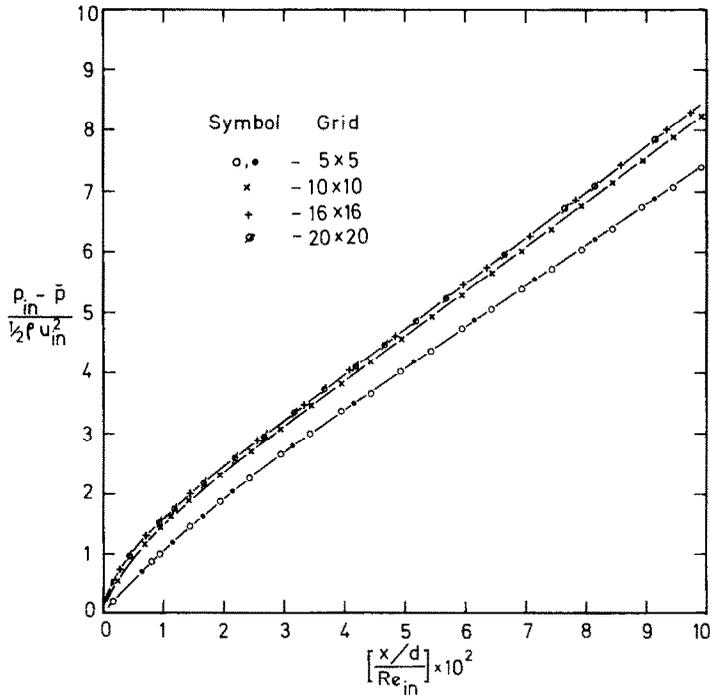


FIG. 8. Effect of the grid size. (Here  $Re_{wall} = 100$ ; the two symbols used for the  $5 \times 5$  grid refer to two different values of the forward step  $\Delta x$ .)

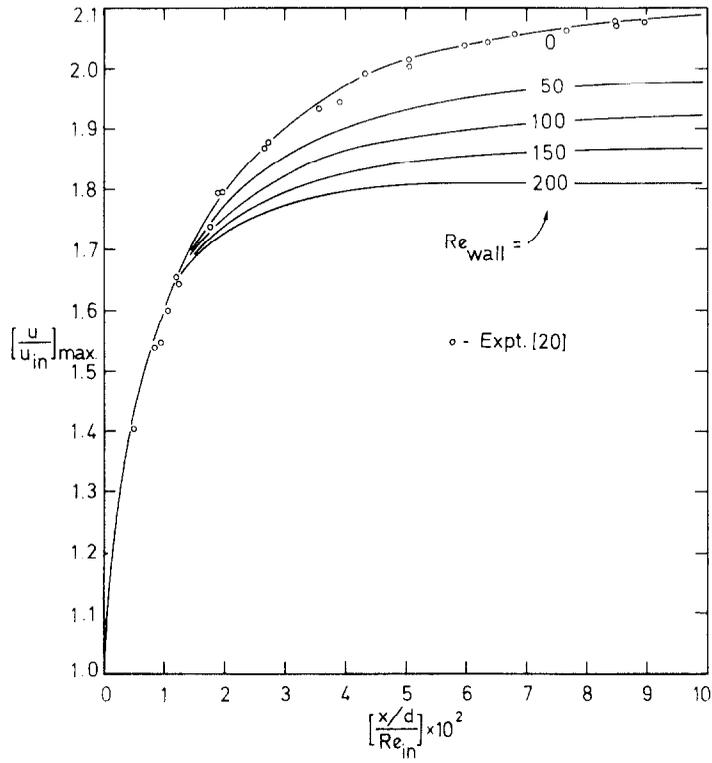


FIG. 9. Variation of maximum longitudinal velocity.

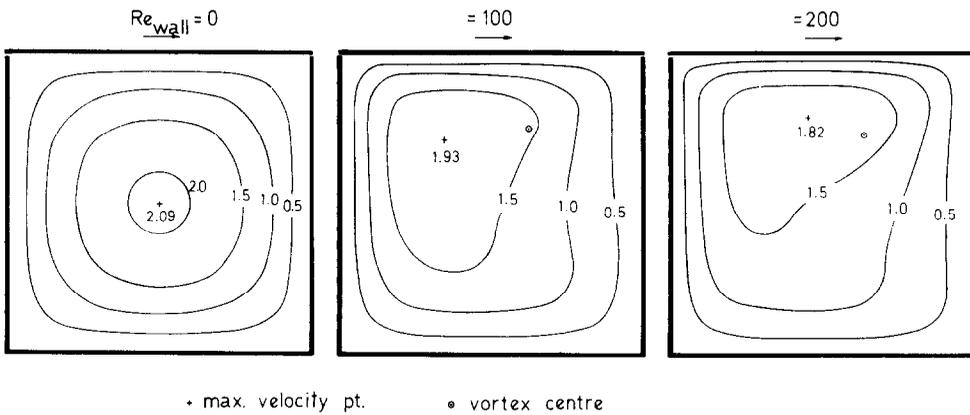


FIG. 10. Longitudinal-velocity contours in the fully-developed region.

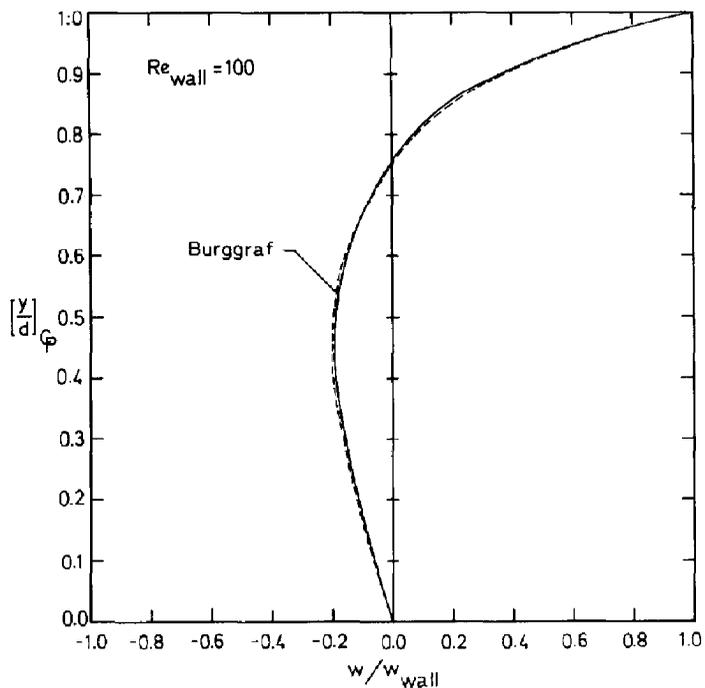


FIG. 11. Profile of a cross-stream velocity in the fully-developed region.

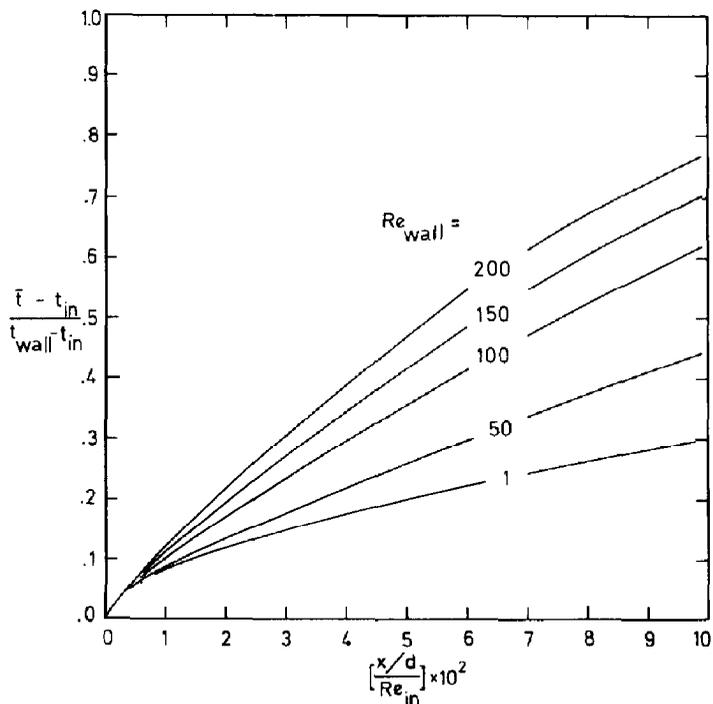


FIG. 12. Variation of bulk temperature for various wall velocities ( $Pr = 1$ ).

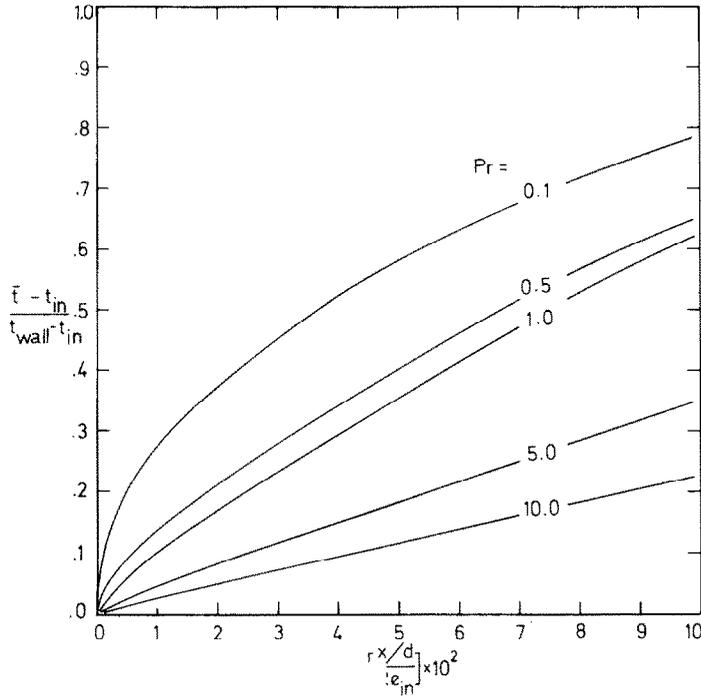


FIG. 13. Variation of bulk temperature for various Prandtl numbers ( $Re_{wall} = 100$ ).

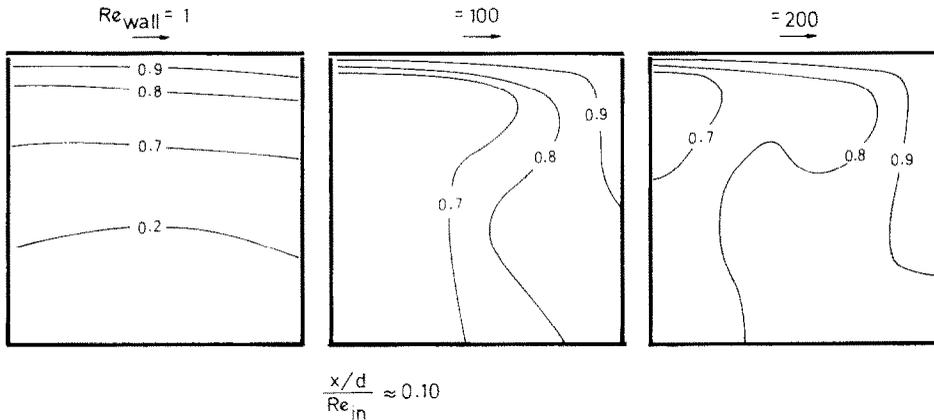


FIG. 14. Contours of  $(t - t_{in}) / (t_{wall} - t_{in})$  for various wall velocities ( $Pr = 1$ ).

temperature development. The temperature distribution in the cross-stream plane is presented in Fig. 14 in the form of contours; it can be seen how the swirl induced by the moving wall distorts the temperature field.

**5. CONCLUDING REMARKS**

- (1) The present paper has described a generally applicable, accurate and economical method for calculating heat, mass and momentum transfer in three-dimensional parabolic flows.

- (2) The uncoupling of the longitudinal and cross-stream pressure gradients is an important feature of the method; it is essential for making the equations parabolic.
- (3) The non-iterative nature of the method derives from the use of upstream convection fluxes, from the explicit corrections of pressure and velocity, and from the double-sweep-IDMA solution of the finite-difference equations.
- (4) The procedure described here shares many useful features with the present authors' two-dimensional procedure [2].
- (5) Various applications of the present procedure are in progress, and will be reported elsewhere. Further advances in the prediction of three-dimensional parabolic flows would come from the development of the models for turbulence, radiation and chemical reaction.

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### UNE METHODE DE CALCUL DU TRANSFERT DE CHALEUR, DE MASSE ET DE QUANTITE DE MOUVEMENT DANS LES ECOULEMENTS PARABOLIQUES TRIDIMENSIONNELS

**Résumé**—Une méthode numérique générale est présentée pour le calcul des processus de transport dans des écoulements tridimensionnels caractérisés par la présence d'une coordonnée pour laquelle les influences physiques sont sensibles dans une seule direction. De tels écoulements donnent lieu à des équations aux dérivées partielles paraboliques et ainsi peuvent être appelés écoulements paraboliques tridimensionnels. La procédure peut être considérée comme une méthode de couche limite mais on remarque que, contrairement à des méthodes antérieurement publiées sous ce nom, elle tient entièrement compte de la diffusion transversale de quantité de mouvement etc. et de la variation de pression dans le plan perpendiculaire à l'écoulement. Le champ de pression est déterminé par : premièrement le calcul d'un champ de vitesse intermédiaire basé sur un champ de pression estimé; ensuite par l'obtention de corrections appropriées de façon à satisfaire l'équation de continuité. Pour illustrer la méthode, des calculs sont présentés pour le développement d'un écoulement laminaire et du transfert thermique dans un conduit carré avec une paroi mobile latéralement.

### EIN RECHENVERFAHREN FÜR WÄRME-, STOFF- UND IMPULSÜBERTRAGUNG IN DREIDIMENSIONALEN PARABOLISCHEN STRÖMUNGEN

**Zusammenfassung**—Es wird ein allgemeines numerisches Verfahren gezeigt für die Berechnung der Transportvorgänge in dreidimensionalen Strömungen, die dadurch gekennzeichnet sind, dass in einer Koordinate physikalische Einflüsse nur in einer Richtung wirken. Solche Strömungen führen zu parabolischen Differentialgleichungen, daher kann man von dreidimensionalen parabolischen Strömungen sprechen. Das Rechenverfahren kann als Grenzschichtmethode betrachtet werden, wobei allerdings im Unterschied zu früher veröffentlichten Methoden mit diesem Namen der Impulstransport u.ä. senkrecht zur Strömung und auch die Druckänderung in der Ebene senkrecht zur Strömung voll berücksichtigt sind. Das Druckfeld wird wie folgt bestimmt: Die erste Berechnung eines mittleren Geschwindigkeitsfeldes beruht auf einem angenommenen Druckfeld; dann werden passende Korrekturen ermittelt, so dass die Kontinuitätsgleichung erfüllt wird. Um das Rechenverfahren zu demonstrieren, werden Berechnungen für den Einlaufvorgang einer laminaren Strömung mit Wärmeübergang in einem quadratischen Kanal mit einer querbewegten Wand angeführt.

### РАСЧЕТ ПЕРЕНОСА ТЕПЛА, МАССЫ И ИМПУЛЬСА В ТРЕХМЕРНЫХ ПАРАБОЛИЧЕСКИХ ПОТОКАХ

**Аннотация**—Представлен общий конечно-разностный метод расчета процессов переноса в трехмерных течениях, характеризующихся наличием одного преимущественного направления изменения, на котором физические эффекты сказываются только в характеристике переноса. Эти течения описываются параболическими дифференциальными уравнениями и могут быть названы трехмерными параболическими течениями. Представленный метод можно рассматривать как метод расчета пограничного слоя, который в отличие от известных ранее методов полностью учитывает поперечный перенос количества движения и т.д., а также изменение давления в плоскости, нормальной к потоку. Предварительно поле давления определяется путем расчета поля скорости по наперед заданному полю давления с последующим введением соответствующих поправок с тем, чтобы удовлетворить уравнению непрерывности. Для иллюстрации приводятся расчеты развивающегося ламинарного течения и переноса тепла в трубе квадратного сечения с движущейся горизонтальной стенкой.