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On the Robustness of a Multiple Semi-coarsened Grid Method

*Es wird die Robustheit einer Multiple Semi-coarsened Grid-Methode (*MSG – *mehrfach halbgrobes Gitter) genannten nichtstandard Mehrgittermethode untersucht. Zur Glättung wird eine gedämpfte Punkt-Jacobi-Relaxations-Methode verwendet. Drei verschiedene Grobgitter-Korrektur-Strategien werden implementiert und verglichen. Die Robustheit wird über die Lösung einer gedrehten anisotropen Diffusionsgleichung und der Konvektions-Diffusionsgleichung untersucht.*

The robustness of a non-standard multigrid method called Multiple Semi-coarsened Grids (MSG) *is investigated. The smoother is a damped point-Jacobi relaxation method. Three different coarse grid correction strategies are implemented and compared. Robustness is investigated by solving the rotated anisotropic diffusion equation and the convection diffusion equation.*

MSC (1991): 65N55, 65F10, 65Y99, 58G20.

1. Introduction

In 1989 W. MULDER published a paper ([4]) on a non-standard multigrid method, "a new multigrid approach to convection problems". In this method several grids exist on coarse grid levels, but contrary to semi-coarsened multigrid ([1]), the work to solve problems with N unknowns is still $O(N)$ when an F-cycle is used. The method, not given a name in [4], is called MSG (as in [5]) here, due to the fact that Multiple Semi-coarsened Grids are used. The method is still $O(N)$, because a coarse grid with $n_3 \times n_4$-unknowns on a grid level is related to an $n_1 \times n_4$ and an $n_3 \times n_2$ grid on a finer level, where $n_1 = 2 \times n_3$, and $n_2 = 2 \times n_4$. It is interesting to investigate MSG with a point smoother for robustness: Due to the robust coarse grid correction it is possible to use relatively simple smoothing methods, and still obtain a robust multigrid method. Simple smoothing methods, like damped point-Jacobi ([6]), possess good parallelization properties on massive parallel computers, and do not need much communication with other processors.

Here MSG is investigated for robustness by solving the two reference problems proposed and investigated for standard multigrid in [7], the rotated anisotropic diffusion equation and the convection-diffusion equation. In [7] it appeared that robustness was obtained for a standard multigrid method with ILU-type smoothers and alternating line variants of basic iterative methods, like alternating damped line-Jacobi (damping parameter 0.7). Then, both equations are solved accurately. However, these robust smoothers are fairly expensive. It is not easy to implement them efficiently on massively parallel machines. Furthermore, in three-dimensional problems line-smoothers will change to plane smoothers, which are even more expensive.

Here a damped point-Jacobi smoother is implemented in an MSG-code and the rotated anisotropic diffusion equation and the convection-diffusion equation are investigated for robustness. Three different coarse grid correction strategies are implemented and compared. Results are presented in Section 3.

The original nonlinear FAS-type multigrid algorithm in [4] is translated to a linear MSG correction scheme. A variant of MSG with matrix-dependent prolongations is investigated in [5]. Other publications on non-standard multigrid methods, in which coarse grid corrections are made more robust are for example presented in [2] and [3].

2. The Multiple Semi-coarsened Grids (MSG) method

In a "standard" semi-coarsened multigrid method, for example found in [1], for two-dimensional problems every grid is semi-coarsened into two coarser grids. Problems then arise, because the number of grid points on every coarse grid level remains the same. The storage and the amount of work for the multigrid V-cycle is $O(N \log^2 N)$, where N represents the number of grid points. Other cycles are even more expensive. Furthermore, for three-dimensional problems the number of points on coarser levels increases, so problems with storage requirements and the amount of work can be expected.

The modification in [4] is as follows: Suppose the finest grid is, for example a 63×63-grid. The first coarse grid level is identical for standard semi-coarsening methods: a 31×63- and 63×31-grid are obtained. On the next coarser level only three grids (instead of four) are generated: a 15×63-, 31×31- and 63×15-grid. The information for the intermediate 31×31-grid is coming from both finer grids. This procedure is continued on all grid levels, resulting in a sequence of coarse grid levels with several grids per level. For a finest grid of 63×63 the sequence is presented in Figure 1.

Grids $G^{m,n}$ are defined as follows,

$$G^{m,n} = \{(x_i, y_j): x_i = i \cdot \Delta x, y_j = j \cdot \Delta y, i = 0, 1, \ldots, \alpha_x \cdot 2^m, j = 0, 1, \ldots, \alpha_y \cdot 2^n, \\ \Delta x = 2^{-m}/\alpha_x, \Delta y = 2^{-n}/\alpha_y\} \tag{1}$$

with: $m, n := 1, 2, \ldots, k$.

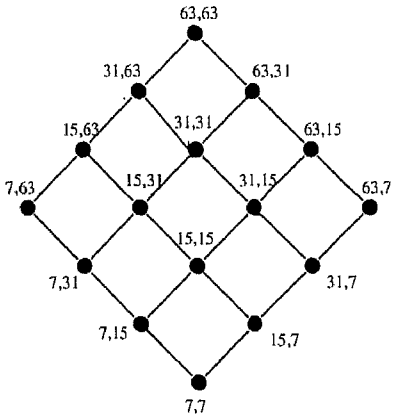


Fig. 1. A sequence of semi-coarsened grids $G^{m,n}$ for the MSG-method. The finest grid consists of 63×63 points

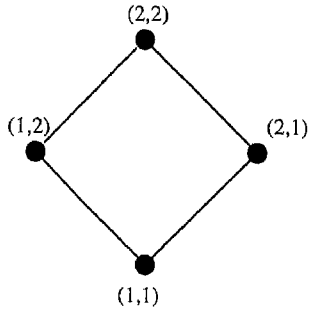


Fig. 2. A sequence of three grid levels with grid indices (m, n) for the MSG-method

The finest grid is $G^{k,k}$, the coarsest is $G^{3,3}$ containing 7×7 interior grid points. The grids on level l are all grids $G^{m,n}$ with $m + n = l$.

Between levels l and $l + 1$ the following transfer operators are defined:

$$\begin{aligned} U^{m,n} &= \{u^{m,n} : G^{m,n} \Rightarrow \mathbb{R}\} \\ \text{Prolongations:} \quad & P_x^{m,n} : U^{m-1,n} \Rightarrow U^{m,n};\ P_y^{m,n} : U^{m,n-1} \Rightarrow U^{m,n} \\ \text{Restrictions:} \quad & R_x^{m,n} : U^{m+1,n} \Rightarrow U^{m,n};\ R_y^{m,n} : U^{m,n+1} \Rightarrow U^{m,n}. \end{aligned} \tag{2}$$

Transfer operators exist between all grids which are connected.

In [4] and [5] it has been shown that the storage and the amount of work for the F-cycle is still $O(N)$ with MSG, so a typical multigrid property is retained. Because only linear equations are solved here, a linear correction variant of the nonlinear FAS scheme in [4] is implemented. In the correction scheme several coarse grid correction strategies are implemented and compared. The linear MSG algorithm will be illustrated with 3 grid levels containing 4 grids (see Figure 2): The three-level correction scheme is presented below.

Three-level MSG correction algorithm: (to solve $T^{2,2}u^{2,2} = f^{2,2}$)

A:

begin

- Apply ν_1 pre-smoothing iterations on $G^{2,2}$
- Compute the fine grid residual:

$$r^{2,2} = f^{2,2} - T^{2,2}u^{2,2} \tag{3}$$

B_1:

- Apply x-Restriction ($R_x^{1,2}$) to the fine grid residual:

$$f^{1,2} = R_x^{1,2} r^{2,2} \tag{4}$$

- $u^{1,2} = 0$
- Apply ν_1 pre-smoothing iterations on $G^{1,2}$
- Compute residual:

$$r^{1,2} = f^{1,2} - T^{1,2}u^{1,2} \tag{5}$$

B_2:

- Apply y-Restriction ($R_y^{2,1}$) to the fine grid residual:

$$f^{2,1} = R_y^{2,1} r^{2,2} \tag{6}$$

- $u^{2,1} = 0$
- Apply ν_1 pre-smoothing iterations on $G^{2,1}$
- Compute residual:

$$r^{2,1} = f^{2,1} - T^{2,1}u^{2,1} \tag{7}$$

C:

- Apply x-Restriction ($R_x^{1,1}$) and y-Restriction ($R_y^{1,1}$):

$$f^{1,1} = \tfrac{1}{2} R_x^{1,1} r^{2,1} + \tfrac{1}{2} R_y^{1,1} r^{1,2} \tag{8}$$

- Solve the coarse grid equation for $u^{1,1}$

$$T^{1,1} u^{1,1} = f^{1,1} \tag{9}$$

D_1:

- Apply Prolongation $P_y^{1,2}$

$$u^{1,2} = u^{1,2} + s^{1,2} P_y^{1,2} u^{1,1} \tag{10}$$

- Apply ν_2 post-smoothing iterations on $G^{1,2}$

D_2:

- Apply Prolongation $P_x^{2,1}$

$$u^{2,1} = u^{2,1} + s^{2,1} P_x^{2,1} u^{1,1} \tag{11}$$

- Apply ν_2 post-smoothing iterations on $G^{2,1}$

E:

- Apply Prolongation $P^{2,2}$

$$u^{2,2} = u^{2,2} + \mathcal{L}(P_x^{2,2} u^{1,2}, P_y^{2,2} u^{2,1}, s^{2,2}) \tag{12}$$

- Apply ν_2 post-smoothing iterations on $G^{2,2}$

end Three-level MSG correction algorithm.

$R_x^{m,n}$ and $R_y^{m,n}$ are standard one-dimensional restriction operators in x- and y-direction, respectively; $P_x^{m,n}$ and $P_y^{m,n}$ are standard one-dimensional prolongation operators in x- and y-direction, respectively.

Contrary to standard linear multigrid a parameter $s^{m,n}$ appears in the coarse grid corrections (10), (11) and (12). A small (constant) overrelaxation parameter was found to improve the convergence behaviour of the overall algorithm.

Note that MSG possesses good vectorization and parallelization possibilities. All points on a grid level can be smoothed in parallel. On each level all coarse grid correction and restriction operators can be performed in parallel; in the three-level algorithm the parts $\mathbf{B}_1$ & $\mathbf{B}_2$ and $\mathbf{D}_1$ & $\mathbf{D}_2$ can be done independently.

The smoothing method implemented is a damped point-Jacobi relaxation method ([6]). Damping parameter 0.8 was found to give good and level-independent convergence rates. With different coarse grid correction strategies (12) different algorithms result.

Three different coarse grid correction strategies are investigated:

A standard approach is the transfer of corrections with equal weights $\frac{1}{2}$:

$$u^{2,2} = u^{2,2} + \tfrac{1}{2} P_x^{2,2} u^{1,2} + \tfrac{1}{2} P_y^{2,2} u^{2,1} . \tag{13}$$

This correction strategy is labeled Strategy **A**.

This strategy has also been investigated in [5], and it was found to be not attractive. As already mentioned in [5] matrix-dependent prolongation operators are proposed.

Strategy **B** is a straightforward implementation of the proposal for (12) in [4] in a linear algorithm:

$$u^{2,2} = u^{2,2} + P_x^{2,2} u^{1,2} + P_y^{2,2} u^{2,1} - P_y^{2,2} R_y^{2,2} P_x^{2,2} u^{1,2} . \tag{14}$$

Equation (14) results in an asymmetry in the algorithm with respect to one of the coordinate directions: Low frequency components $P_y^{2,2} R_y^{2,2}$ are subtracted in this correction, while the low frequency components in x-direction are not subtracted. As in [4] this asymmetry is reduced by applying the low frequency correction in alternating directions, so in the next MSG-iteration the correction is given by:

$$u^{2,2} = u^{2,2} + P_x^{2,2} u^{1,2} + P_y^{2,2} u^{2,1} - P_x^{2,2} R_x^{2,2} P_y^{2,2} u^{2,1} . \tag{15}$$

Therefore, Strategy **B** for two multigrid iterations consists of correction (14) in the first of the two iterations, and correction

Strategy C is a symmetric coarse grid correction obtained from (14) and (15), as follows:

$$u^{2,2} = u^{2,2} + P_x^{2,2}u^{1,2} + P_y^{2,2}u^{2,1} - \tfrac{1}{2}P_y^{2,2}R_y^{2,2}P_x^{2,2}u^{1,2} - \tfrac{1}{2}P_x^{2,2}R_x^{2,2}P_y^{2,2}u^{2,1}\,. \tag{16}$$

These three strategies are implemented and compared in the next section.

3. Results

The equations investigated are the rotated anisotropic diffusion equation and the convection-diffusion equation. They are investigated for robustness in a standard multigrid method in [7].

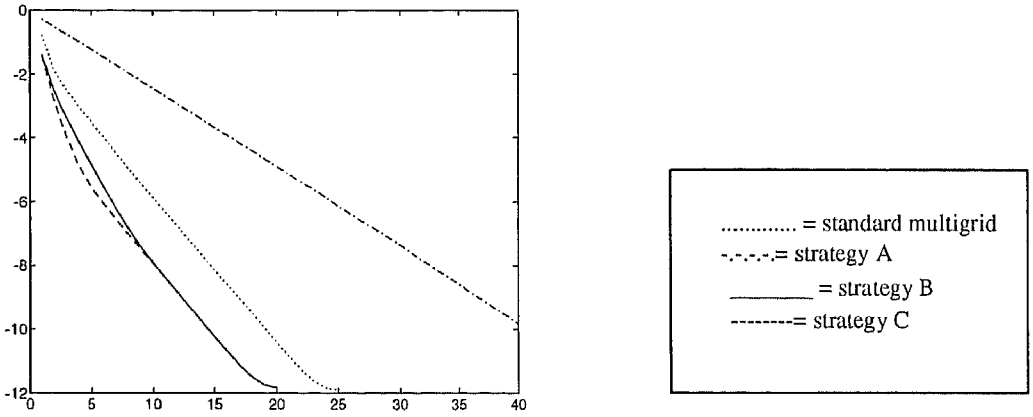
The *rotated anisotropic diffusion equation* is given by:

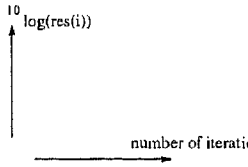
$$-(\varepsilon\cos^2\beta + \sin^2\beta)\frac{\partial^2 u}{\partial x^2} - 2(\varepsilon - 1)\cos\beta\sin\beta\frac{\partial^2 u}{\partial x\,\partial y} - (\varepsilon\sin^2\beta + \cos^2\beta)\frac{\partial^2 u}{\partial y^2} = f\,,$$
$$u|_\Gamma = 0\,. \tag{17}$$

Two constants are to be varied, namely $\varepsilon(>0)$ and angle β.

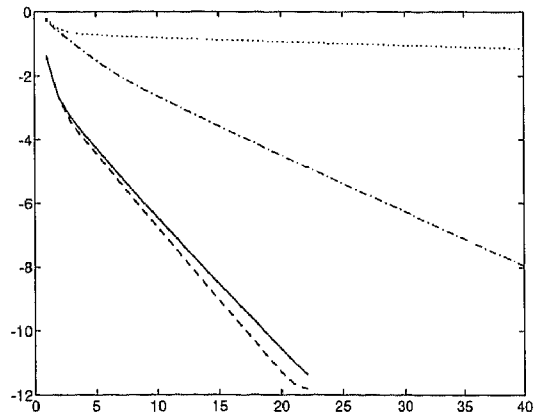
Equation (17) also models the effect of a large spatial discretization for the Poisson equation in one space-direction only:

Choosing $\beta = 0$, $\varepsilon \gg 0$ and $\Delta x = \Delta y$ leads to a similar discretization as choosing: $\beta = 0$, $\varepsilon = 1$ and $\Delta x \ll \Delta y$.





(a)



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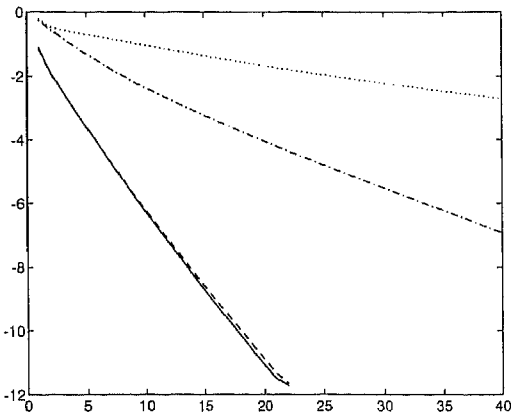

(c)

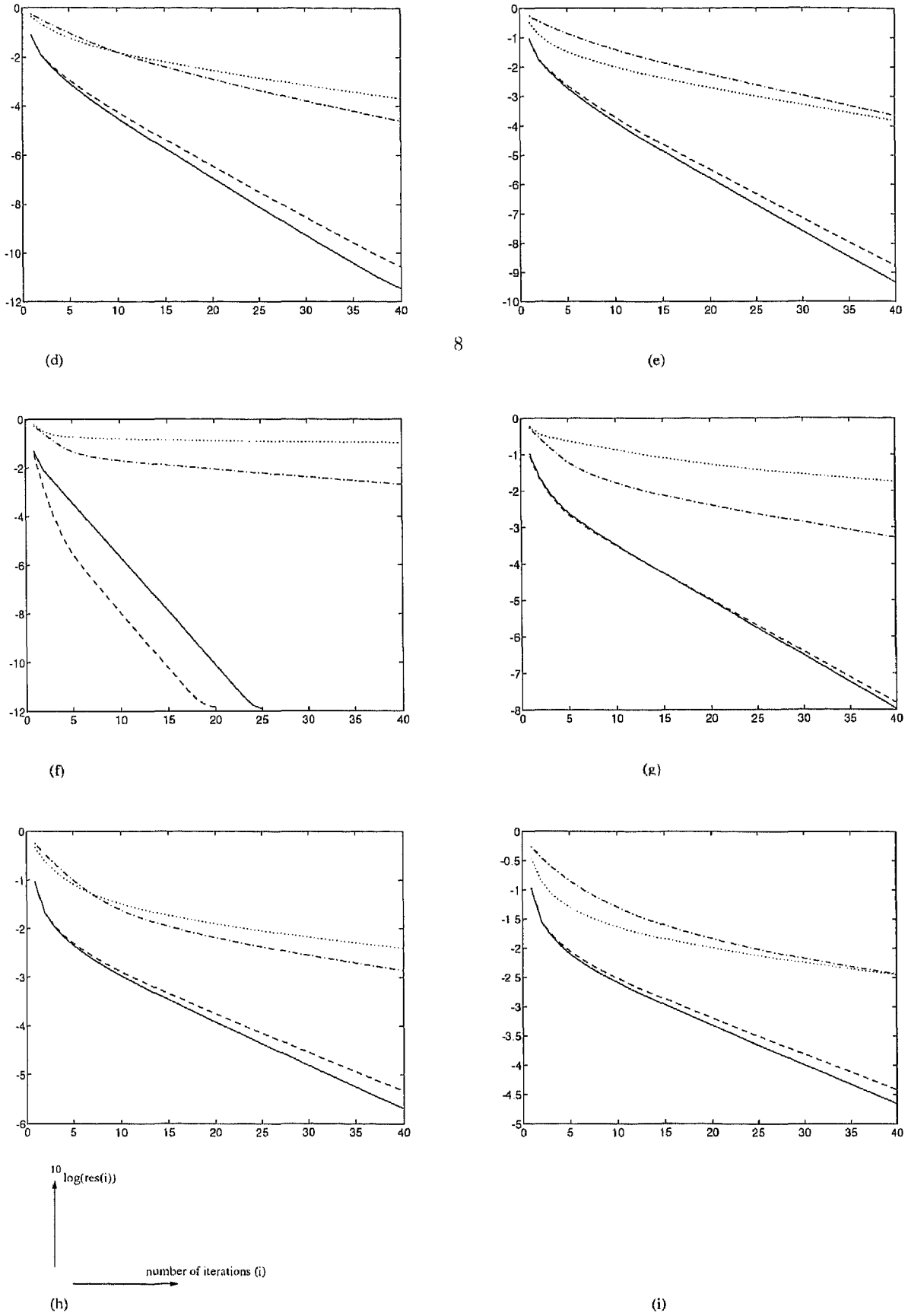

Fig. 3. The l_2-norm of the residual (logarithmic scale) versus the number of iterations for a 256×256 mesh; rotated anisotropic diffusion equation. (a): $\varepsilon = 1$, ($\beta = 0°$); (b): $\varepsilon = 1 \times 10^{-2}$, $\beta = 0°$; (c): $\varepsilon = 1 \times 10^{-2}$, $\beta = 15°$; (d): $\varepsilon = 1 \times 10^{-2}$, $\beta = 30°$; (e): $\varepsilon = 1 \times 10^{-2}$, $\beta = 45°$; (f): $\varepsilon = 1 \times 10^{-8}$, $\beta = 0°$; (g): $\varepsilon = 1 \times 10^{-8}$; $\beta = 15°$; (h): $\varepsilon = 1 \times 10^{-8}$, $\beta = 30°$; (i): $\varepsilon = 1 \times 10^{-8}$, $\beta = 45°$

The stencil of a symmetric vertex-centred finite difference discretization of (17) is given by:

$$[A] = \frac{\varepsilon c^2 + s^2}{\Delta x^2}[-1 \quad 2 \quad -1] + \frac{(\varepsilon - 1)\,cs}{2\Delta x \Delta y}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \frac{\varepsilon s^2 + c^2}{\Delta y^2}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \tag{18}$$

where c and s stand for cos β and sin β, respectively.

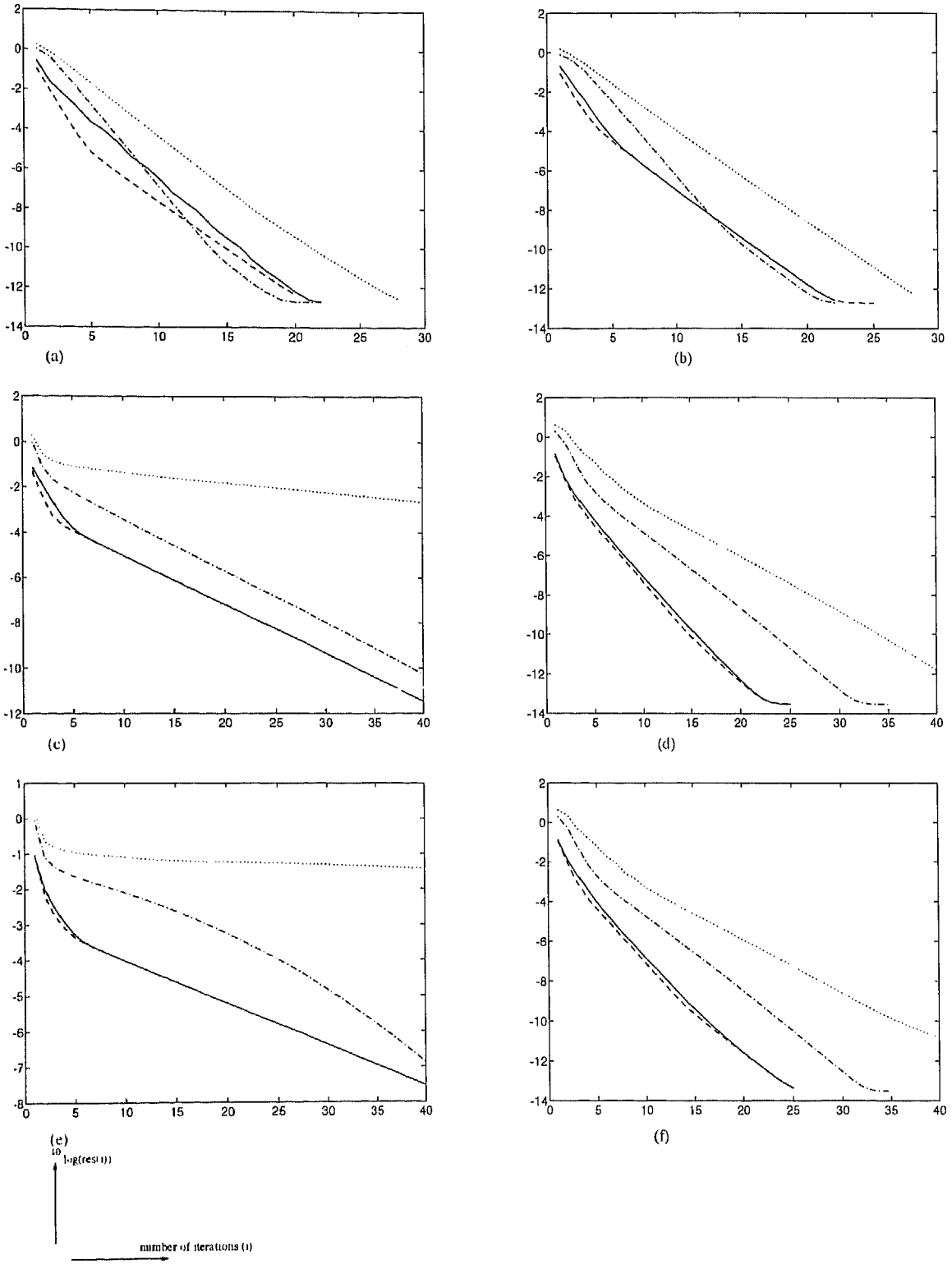

Fig. 4. The l_2-norm of the residual (logarithmic scale) versus the number of iterations for a 256×256 mesh; convection-diffusion equation. The legenda is as in Figure 3. (a): $\varepsilon = 1 \times 10^{-2}$, $\beta = 0°$; (b): $\varepsilon = 1 \times 10^{-2}$, $\beta = 45°$; (c): $\varepsilon = 1 \times 10^{-4}$, $\beta = 0$; (d): $\varepsilon = 1 \times 10^{-4}$, $\beta = 45°$; (e): $\varepsilon = 1 \times 10^{-8}$, $\beta = 0°$; (f): $\varepsilon = 1 \times 10^{-8}$, $\beta = 45°$

The multigrid results shown are the convergence results obtained on a 256×256-grid. Then, in MSG 10 grid levels are visited with 36 grids. The results obtained for this very fine grid can be considered as level-independent (worst case) results.

As already mentioned the smoothing method is damped point-Jacobi with damping parameter 0.8. The results shown are obtained with the F-cycle, in which 1 pre- and 1 post-smoothing iteration are performed. On the coarsest (7×7) grid the equations are solved "exactly" by applying 10 smoothing iterations. The prolongation overrelaxation parameters $s^{m,n}$ are chosen 1.08. Parameter ε is choosen 1, 1×10^{-8} and an intermediate value: 1×10^{-2}. Angle β is chosen 0°, in which case the well-known anisotropic diffusion equation arises, to 45° with steps of 15°. These results are representative for all β-values.

The case $\beta = 45°$ is in fact the worst case. A change in (18) to a difference scheme with a non-symmetric 7-points stencil, like in [7], did not lead to better convergence results. The results are presented in Figure 3. The MSG results are also compared to results obtained with a standard multigrid method and the damped point-Jacobi smoother. Strategies **B** and **C** show the best performances in all cases considered. Strategy **B** is best in the worst case: $\beta = 45°$, $\varepsilon = 1 \times 10^{-8}$, and leads to a convergence rate of 0.83.

The second equation investigated is the *convection-diffusion equation*, which is given by:

$$\cos\beta \frac{\partial u}{\partial x} + \sin\beta \frac{\partial u}{\partial y} - \varepsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f\,, \qquad u|_\Gamma = 0\,. \tag{19}$$

Again two constants, namely ε $(0 < \varepsilon \ll 1)$ and angle β are to be varied.

The main problem investigated is the almost hyperbolic, convection-dominated behaviour of (19) for very small values of ε $(\varepsilon \ll 1)$. This feature is commonly found in, for example, computational fluid dynamics.

Using a first order vertex-centred upwind discretization scheme for the convection terms results in a so-called K-matrix ([6]), which is favourable for multigrid solution techniques. The stencil for (19) then looks like:

$$[A] = \frac{1}{2\Delta x}[-c - |c| \quad 2|c| \quad c - |c|] + \frac{1}{2\Delta y}\begin{bmatrix} s - |s| \\ 2|s| \\ -s - |s| \end{bmatrix} + \varepsilon\begin{bmatrix} 0 & -\frac{1}{\Delta y_2} & 0 \\ -\frac{1}{\Delta x^2} & \frac{2}{\Delta x^2} + \frac{2}{\Delta y^2} & -\frac{1}{\Delta x^2} \\ 0 & -\frac{1}{\Delta y^2} & 0 \end{bmatrix}. \tag{20}$$

The MSG algorithm is tested on the 256×256-grid with three values of ε; $\varepsilon = 1 \times 10^{-2}$, 1×10^{-4} and 1×10^{-8}. With increments of 15° the worst value of β $(0 \leqq \beta < 0)$ was found to be 0°. Representative for the other β-values is $\beta = 45°$, which is also presented in Figure 4.

Again all results of strategies **B** and **C** are very good. They are best and for many cases equal. However, contrary to the results for (17) the results for strategy **A** are also satisfactory for the convection-diffusion equation.

4. Conclusions

The MSG method with the point smoother damped point-Jacobi is a robust multigrid method, when a good coarse grid correction strategy is chosen. The strategies **B** and **C** both are satisfactory strategies. Good convergence results are obtained for the reference problems, the rotated anisotropic diffusion equation and the convection-diffusion equation, solved with MSG on a very fine mesh. Good convergence results are obtained. Interesting future developments will be the investigation for three-dimensional problems and the investigation of the performance on a massive parallel computer.

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