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Short note

One-sided finite-difference approximations suitable for use with Richardson extrapolation

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Abstract

New expressions for one-sided finite-difference approximations are proposed. In these approximations the odd-order error terms are eliminated while the even-order terms are left to be taken care of by Richardson extrapolation. The effective local truncation error is shown to be less than for higher-order one-sided finite-difference approximations but the solutions for a test problem are shown to have comparable accuracy for both approximations.

Keywords: Finite-difference method; One-sided difference approximation; Order of accuracy; Richardson extrapolation

1. Introduction

A finite-difference approximation is one of the commonly used methods for numerical solution of ordinary and partial differential equations. The approximations most often used have second-order accuracy. The order of accuracy can be increased either by using higher-order finite difference approximations or by Richardson extrapolation [2,3]. For internal grid points, if central difference approximations are used, the truncation error has the form $A_2h^2 + A_4h^4 + A_6h^6 + \cdots$, where h is the distance between adjacent grid points and $A_2, A_4, A_6, \ldots$ are constants. Doing the computation with two or three different grid sizes and carrying out one or two extrapolations to eliminate the error terms of second order or of second and fourth order we obtain a solution with fourth- or sixth-order accuracy. Since each of the computations uses a second-order-accurate finite-difference approximation there is no difficulty with the boundary conditions for a Dirichlet problem. However, a problem arises when the boundary condition involves a derivative and we use a one-sided finite-difference approximation for the derivative. By using three grid points, including one on the boundary, we can obtain a one-sided approximation which is second-order-accurate [1]. The truncation error has the form $B_2h^2 + B_3h^3 + B_4h^4 + \cdots$. So if we carry out one extrapolation to eliminate the second-order error term the result would have third-order rather than fourth-order accuracy. One way of obtaining fourth-order accuracy

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is by using a one-sided difference approximation which has a higher order of accuracy. Such higher-order-accurate one-sided finite-difference approximations are known [1]. To obtain fourth-order accuracy, one could use a one-sided finite-difference approximation which has fourth-order accuracy. This would involve five grid points, including one on the boundary. The additional two points are required in order to eliminate the terms of the order of h^2 and h^3 in the truncation error. However, if we are going to use Richardson extrapolation it is not necessary to eliminate the order h^2 term in the truncation error for the one-sided finite-difference approximations. So we can use just one more point to obtain a one-sided difference approximation which has a truncation error of the form $B_2h^2 + B_4h^4 + B_5h^5 + \cdots$. After one extrapolation, this would give a result which has fourth-order accuracy. Since this approximation involves smaller number of grid points compared to the fourth-order approximation which involves points farther away from the boundary, although both the methods have fourth-order accuracy, we expect the coefficient of h^4 in the truncation error to be smaller in the first case. Continuing in this manner, if we require sixth-order accuracy, we can use one more point to obtain a one-sided finite-difference approximation which has a truncation error of the form of $B_2h^2 + B_4h^4 + B_6h^6 + B_7h^7 + \cdots$.

In this study, we derive one-sided finite-difference approximations with odd-order terms in the truncation error eliminated and the even-order terms left to be taken care of by Richardson extrapolation. We compare the local truncation error with that of higher-order one-sided finite-difference approximations. We then apply these approximations to a test problem and check the accuracy of the solutions obtained.

2. One-sided finite-difference approximations

Let us consider a function $\phi(x)$ and define a set of uniformly spaced grid points x_i, $i = 0, 1, \ldots, N$, with the grid spacing $h = x_{i+1} - x_i$. We represent $\phi(x_i)$ by ϕ_i. Similarly $\phi'(x_i), \phi''(x_i), \ldots$, are represented by $\phi'_i, \phi''_i, \ldots$, where the primes denote differentiation. We can write a Taylor series expansion for ϕ around x_i

$$\phi_{i\pm k} = \phi_i \pm kh\phi'_i + \frac{k^2h^2}{2}\phi''_i \pm \frac{k^3h^3}{6}\phi'''_i + \frac{k^4h^4}{24}\phi_i^{(iv)} \pm \frac{k^5h^5}{120}\phi_i^{(v)} + \frac{k^6h^6}{720}\phi_i^{(vi)} \pm \cdots. \tag{1}$$

From Eq. (1) we readily obtain first-order-accurate one-sided finite-difference approximations [1]

$$\phi'_i = \frac{\mp(\phi_i - \phi_{i\pm1})}{h} + \mathrm{O}(h), \tag{2}$$

where the upper and lower signs are to be used to the right and to the left of x_i. Writing Eq. (1) for $k = 1$ and 2 and eliminating ϕ''_i between them, we obtain second-order-accurate one-sided finite-difference approximations [1]

$$\phi'_i = \frac{\mp(3\phi_i - 4\phi_{i\pm1} + \phi_{i\pm2})}{2h} + \mathrm{O}(h^2). \tag{3}$$

We now derive an approximation for use with Richardson extrapolation which has truncation error of the form $B_2h^2 + B_4h^4 + B_5h^5 + \cdots$. Writing Eq. (1) for $k = 1$, 2 and 3 and eliminating ϕ''_i and $\phi_i^{(iv)}$ between them we obtain the one-sided approximations

$$\phi'_i = \frac{\mp(10\phi_i - 15\phi_{i\pm1} + 6\phi_{i\pm2} - \phi_{i\pm3})}{6h} + h^2\frac{1}{6}\phi'''_i - h^4\frac{11}{120}\phi_i^{(v)} + \cdots. \tag{4}$$

Alternatively, we can use a fourth-order-accurate one-sided finite-difference approximation obtained by writing Eq. (1) for $k = 1$, 2, 3, and 4 and eliminating ϕ''_i, ϕ'''_i and $\phi_i^{(iv)}$, to obtain

$$\phi'_i = \frac{\mp(25\phi_i - 48\phi_{i\pm1} + 36\phi_{i\pm2} - 16\phi_{i\pm3} + 3\phi_{i\pm4})}{12h} + h^4\frac{24}{120}\phi_i^{(v)} + \cdots. \tag{5}$$

When using the approximations given by Eq. (4), if we use Richardson extrapolation to eliminate the second-order error term then the leading order error is due to the fourth-order term. Effectively the local truncation error for this approximation becomes $(11/120)h^4\phi_i^{(v)}$. If instead of Eq. (4) we use Eq. (5) the local truncation error is $(24/120)h^4\phi_i^{(v)}$. Thus the one-sided finite-difference approximations we have proposed, given by Eq. (4), when used together with Richardson extrapolation, effectively give a local truncation error which is

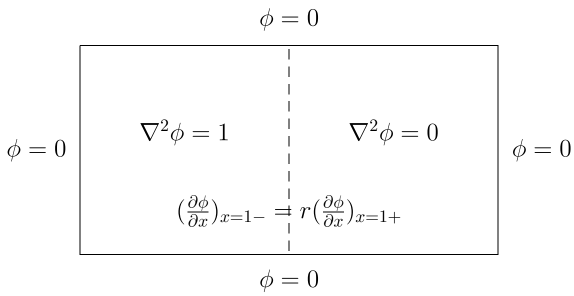

Fig. 1. The test problem.

slightly less than half the local truncation error when using higher-order one-sided approximations given by Eq. (5). Continuing the logic used in deriving Eq. (4) we can derive one-sided finite-difference approximations which have a truncation error of the form of $B_2h^2 + B_4h^4 + B_6h^6 + B_7h^7 + \cdots$. These are

$$\phi'_i = \frac{\mp(35\phi_i - 56\phi_{i\pm1} + 28\phi_{i\pm2} - 8\phi_{i\pm3} + \phi_{i\pm4})}{20h}. \tag{6}$$

When used together with central difference approximations for internal grid points, if we carry out two Richardson extrapolations to eliminate the error terms of order h^2 and h^4, we can obtain a solution with sixth-order accuracy.

Table 1
Solution of the test problem using a first-order-accurate one-sided finite-difference approximation at the interface

N	ϕ_N	$\phi_N - \phi_{N/2}$	$\frac{\phi_{N/2}-\phi_{N/4}}{\phi_N-\phi_{N/2}}$
$x = 0.25$, $y = 0.375$			
8	56.490197067	–	–
16	56.993268539	0.503071472	–
32	57.126816809	0.133548270	3.767
64	57.163421776	0.036604967	3.648
128	57.174149692	0.010727916	3.412
256	57.177619246	0.003469554	3.092
$x = 0.75$, $y = 0.375$			
8	69.833661737	–	–
16	70.500906337	0.667244600	–
32	70.704841909	0.203935572	3.272
64	70.774614378	0.069772469	2.923
128	70.801514255	0.026899877	2.594
256	70.812977553	0.011463298	2.347
$x = 1.25$, $y = 0.375$			
8	16.065279385	–	–
16	16.223815006	0.158535621	–
32	16.298348196	0.074533190	2.127
64	16.335630665	0.037282469	1.999
128	16.354399344	0.018768679	1.986
256	16.363829300	0.009429956	1.990
$x = 1.75$, $y = 0.375$			
8	2.721814715	–	–
16	2.716177209	−0.005637506	–
32	2.720323097	0.004145888	−1.360
64	2.724438065	0.004114968	1.008
128	2.727034781	0.002596716	1.585
256	2.728470993	0.001436212	1.808

3. Application to a test problem

In this section, we use the finite-difference approximations derived in the previous section to compute the numerical solution for a test problem and check the order of accuracy obtained. The test problem is defined as

$$
\begin{aligned}
&\nabla^2\phi = -1000 \quad \text{for } 0 < x < 1,\ 0 < y < 1,\\
&\nabla^2\phi = 0 \quad \text{for } 1 < x < 2,\ 0 < y < 1,\\
&\left(\frac{\partial\phi}{\partial x}\right)_{x=1-} = r\left(\frac{\partial\phi}{\partial x}\right)_{x=1+} \quad \text{for } 0 < y < 1,\\
&\phi = 0 \quad \text{for } x = 0 \text{ or } 2,\ 0 \leqslant y \leqslant 1,\\
&\phi = 0 \quad \text{for } 0 \leqslant x \leqslant 2,\ y = 0 \text{ or } 1.
\end{aligned}
\tag{7}
$$

This is shown in Fig. 1. For most of our numerical calculations we use $r = 2$. We define a set of grid points (x_j, y_k) where $x_j = jh$ for $0 \leqslant x \leqslant 2N$, $y_k = kh$ for $0 \leqslant x \leqslant N$ and $h = 1/N$. For the internal grid points, except the grid points at the interface $x = 1$, we use a second-order central difference approximation

$$(\nabla^2\phi)_{j,k} = \frac{\phi_{j,k-1} + \phi_{j-1,k} - 4\phi_{j,k} + \phi_{j+1,k} + \phi_{j,k+1}}{h^2}. \tag{8}$$

The truncation error for this approximation has the form $A_2h^2 + A_4h^4 + A_6h^6 + \cdots$. The boundary condition $\phi = 0$ on the domain boundary is easily taken into account. For the condition at the interface, $x = 1$, we study the effect of using the different one-sided finite-difference approximations derived in the previous section. To

Table 2
Solution of the test problem using a second-order-accurate one-sided finite-difference approximation at the interface

N	ϕ_N	$\phi_N - \phi_{N/2}$	$\frac{\phi_{N/2}-\phi_{N/4}}{\phi_N-\phi_{N/2}}$	$\phi_{N,N/2}$	$\phi_{N,N/2} - \phi_{N/2,N/4}$	$\frac{\phi_{N/2,N/4}-\phi_{N/4,N/8}}{\phi_{N,N/2}-\phi_{N/2,N/4}}$
$x = 0.25,\ y = 0.375$						
8	56.559208580	–	–	–	–	–
16	57.021969345	0.462760765	–	57.176222933	–	–
32	57.140164802	0.118195457	3.915	57.179563288	0.003340355	–
64	57.169894850	0.029730048	3.976	57.179804866	0.000241578	13.827
128	57.177341913	0.007447063	3.992	57.179824267	0.000019401	12.452
256	57.179204997	0.001863084	3.997	57.179826025	0.000001758	11.036
$x = 0.75,\ y = 0.375$						
8	70.242735390	–	–	–	–	–
16	70.672971048	0.430235658	–	70.816382934	–	–
32	70.785095839	0.112124791	3.837	70.822470769	0.006087835	–
64	70.813561534	0.028465695	3.939	70.823050099	0.000579330	10.508
128	70.820724716	0.007163182	3.974	70.823112443	0.000062344	9.292
256	70.822520916	0.001796200	3.988	70.823119649	0.000007206	8.652
$x = 1.25,\ y = 0.375$						
8	16.474353037	–	–	–	–	–
16	16.395879717	−0.078473320	–	16.369721944	–	–
32	16.378602127	−0.017277590	4.542	16.372842930	0.003120986	–
64	16.374577822	−0.004024305	4.293	16.373236387	0.000393457	7.932
128	16.373609809	−0.000968013	4.157	16.373287138	0.000050751	7.753
256	16.373372671	−0.000237138	4.082	16.373293625	0.000006487	7.823
$x = 1.75,\ y = 0.375$						
8	2.790826227	–	–	–	–	–
16	2.744878015	−0.045948212	–	2.729561944	–	–
32	2.733671090	−0.011206925	4.100	2.729935448	0.000373504	–
64	2.730911139	−0.002759951	4.061	2.729991155	0.000055707	6.705
128	2.730227009	−0.000684130	4.034	2.729998966	0.000007811	7.132
256	2.730056755	−0.000170254	4.018	2.730000004	0.000001038	7.525

begin with we use the first-order-accurate one-sided approximations given by Eq. (2). The interface condition in finite-difference approximation becomes

$$\frac{\phi_{N,j} - \phi_{N-1,j}}{h} = r\frac{\phi_{N+1,j} - \phi_{N,j}}{h}. \tag{9}$$

The system of equations given by Eqs. (8) and (9) are solved numerically. The solutions at $x = 0.25$, 0.75, 1.25 and 1.75 and $y = 0.375$ for different number of grid points are shown in Table 1, where ϕ_N represents the solution obtained using N grid intervals along each coordinate axis on each unit square. We observe that at $x = 0.25$ and $y = 0.375$ for small number of grid points it almost appears that we have second-order accuracy but as we increase the number of grid points the order of accuracy is seen to decrease. Near the interface $x = 0.75$ and 1.25, $y = 0.375$ we observe close to first-order accuracy. Further at $x = 1.75$, $y = 0.375$ we observe that the accuracy is even worse but, with increase in number of grid points, appears to tend to first order. This probably occurs because the solution in the square on the right is driven by the source term in the square on the left through the interface condition. So an inaccurate modelling of the interface condition leads to inaccurate solution in the entire square on the right, though why the order of accuracy at $x = 1.75$ seems worse than at $x = 1.25$ is not clear. Thus we find that although the approximations for the internal grid points is second-order-accurate, since we use a first-order-accurate approximation for the condition at the interface the solution is only first-order-accurate.

We repeat the calculation using second-order-accurate one-sided approximations given by Eq. (3) for the condition at the interface. The results are shown in Table 2 and we clearly see second-order accuracy. We next carry out Richardson extrapolation to eliminate the h^2 term in the error. The extrapolated values are shown in

Table 3
Solution of the test problem using at the interface a second-order-accurate one-sided finite-difference approximation with the third-order error term eliminated

N	ϕ_N	$\phi_{N,N/2}$	$\phi_{N,N/2} - \phi_{N/2,N/4}$	$\frac{\phi_{N/2,N/4}-\phi_{N/4,N/8}}{\phi_{N,N/2}-\phi_{N/2,N/4}}$
$x = 0.25$, $y = 0.375$				
8	56.554744860	–	–	–
16	57.021465119	57.177038539	–	–
32	57.140103470	57.179649587	0.002611048	–
64	57.169887237	57.179815159	0.000165572	15.770
128	57.177340963	57.179825538	0.000010379	15.953
256	57.179204877	57.179826182	0.000000644	16.116
$x = 0.75$, $y = 0.375$				
8	70.216290624	–	–	–
16	70.669948778	70.821168163	–	–
32	70.784727144	70.822986599	0.001818436	–
64	70.813515737	70.823111935	0.000125336	14.508
128	70.820719000	70.823120088	0.000008153	15.373
256	70.822520201	70.823120601	0.000000513	15.893
$x = 1.25$, $y = 0.375$				
8	16.447908271	–	–	–
16	16.392857448	16.374507174	–	–
32	16.378233431	16.373358759	−0.001148415	–
64	16.374532025	16.373298223	−0.000060536	18.971
128	16.373604094	16.373294784	−0.000003439	17.603
256	16.373371956	16.373294577	−0.000000207	16.614
$x = 1.75$, $y = 0.375$				
8	2.786362507	–	–	–
16	2.744373789	2.730377550	–	–
32	2.733609758	2.730021748	−0.000355802	–
64	2.730903526	2.730001449	−0.000020299	17.528
128	2.730226059	2.730000237	−0.000001212	16.748
256	2.730056636	2.730000162	−0.00000075	16.160

Table 4
Solution of the test problem using a fourth-order-accurate one-sided finite-difference approximation at the interface

N	ϕ_N	$\phi_{N,N/2}$	$\phi_{N,N/2} - \phi_{N/2,N/4}$	$\frac{\phi_{N/2,N/4}-\phi_{N/4,N/8}}{\phi_{N,N/2}-\phi_{N/2,N/4}}$
$x = 0.25$, $y = 0.375$				
8	56.555836789	–	–	–
16	57.021497060	57.176717150	–	–
32	57.140104452	57.179640249	0.002923099	–
64	57.169887268	57.179814873	0.000174624	16.739
128	57.177340964	57.179825529	0.000010656	16.387
256	57.179204877	57.179826181	0.000000652	16.344
$x = 0.75$, $y = 0.375$				
8	70.222733447	–	–	–
16	70.670139896	70.819275379	–	–
32	70.784733036	70.822930749	0.003655370	–
64	70.813515920	70.823110215	0.000179466	20.368
128	70.820719004	70.823120032	0.000009817	18.281
256	70.822520198	70.823120596	0.000000564	17.406
$x = 1.25$, $y = 0.375$				
8	16.454351095	–	–	–
16	16.393048566	16.372614390	–	–
32	16.378239323	16.373302909	0.000688519	–
64	16.374532209	16.373296504	−0.000006405	−107.497
128	16.373604099	16.373294729	−0.000001775	3.608
256	16.373371955	16.373294574	−0.000000155	11.452
$x = 1.75$, $y = 0.375$				
8	2.787454436	–	–	–
16	2.744405730	2.730056161	–	–
32	2.733610740	2.730012410	−0.000043751	–
64	2.730903557	2.730001163	−0.000011247	3.890
128	2.730226060	2.730000228	−0.000000935	12.029
256	2.730056636	2.730000161	−0.000000067	13.955

the table as $\phi_{N,N/2}$. Here $\phi_{N,N/2}$ denotes the value obtained by the extrapolation using ϕ_N and $\phi_{N/2}$. The ratio of the differences appear close to 8 indicating third-order accuracy. If the interface condition were not present we should have got fourth-order accuracy after one extrapolation.

In order to obtain fourth-order accuracy after one extrapolation we use one-sided finite-difference approximations given by Eq. (4). The results are shown in Table 3. We observe that after one extrapolation we clearly obtain fourth-order accuracy. Next we use the approximations given by Eq. (5) and the results are shown in Table 4. Again after one extrapolation we obtain fourth-order accuracy. Comparing Tables 3 and 4 we find that although the local truncation error is smaller for the approximations in Eq. (4) the accuracy of the solution for the test problem is similar to that obtained using Eq. (5). However, Eq. (4) still has the advantage that it is easier to derive.

Next we use the approximation given by Eq. (6) and the results are shown in Table 5. Here $\phi_{N,N/2,N/4}$ is the value obtained by using Richardson extrapolation to eliminate the order h^4 error term between $\phi_{N,N/2}$ and $\phi_{N/2,N/4}$. After two extrapolations the differences are very small and consequently, it is difficult to show that we obtain sixth-order accuracy. However, we do observe that agreement to 9 significant figures is obtained with reasonable number of grid points. Alternatively if we were to use sixth-order-accurate one-sided finite difference approximations the algebra involved in deriving the expressions would be extremely tedious. Thus the advantage of Eq. (6) is that, when used together with Richardson extrapolation, it provides high accuracy without requiring very tedious algebra.

So far the calculations have been carried out for $r = 2$. We now check whether the one-sided approximations work well for larger values of r. The results for the test problem with $r = 10$ computed using Eq. (4) are shown in Table 6. Again we find that this provides fourth-order accuracy after one Richardson extrapolation, as expected. The magnitude of the errors are also comparable with those for $r = 2$.

Table 5
Solution of the test problem using at the interface a second-order-accurate one-sided finite-difference approximation with the third- and fifth-order error terms eliminated

N	ϕ_N	$\phi_{N,N/2}$	$\phi_{N,N/2,N/4}$
$x = 0.25$, $y = 0.375$			
8	56.554967006	–	–
16	57.021471538	57.176973049	–
32	57.140103667	57.179647710	57.179826021
64	57.169887243	57.179815102	57.179826261
128	57.177340963	57.179825536	57.179826232
256	57.179204877	57.179826182	57.179826225
$x = 0.75$, $y = 0.375$			
8	70.217600995	–	–
16	70.669987186	70.820782583	–
32	70.784728324	70.822975370	70.823121556
64	70.813515774	70.823111591	70.823120672
128	70.820719001	70.823120077	70.823120643
256	70.822520200	70.823120600	70.823120635
$x = 1.25$, $y = 0.375$			
8	16.449218642	–	–
16	16.392895856	16.374121594	–
32	16.378234611	16.373347529	16.373295925
64	16.374532062	16.373297879	16.373294569
128	16.373604095	16.373294773	16.373294566
256	16.373371956	16.373294576	16.373294563
$x = 1.75$, $y = 0.375$			
8	2.786584653	–	–
16	2.744380208	2.730312060	–
32	2.733609955	2.730019871	2.730000392
64	2.730903532	2.730001391	2.730000159
128	2.730226059	2.730000235	2.730000158
256	2.730056636	2.730000162	2.730000157

Table 6
Solution of the test problem using at the interface a second-order-accurate one-sided finite-difference approximation with the third-order error term eliminated and with $r = 10$

N	ϕ_N	$\phi_{N,N/2}$	$\phi_{N,N/2} - \phi_{N/2,N/4}$	$\frac{\phi_{N/2,N/4}-\phi_{N/4,N/8}}{\phi_{N,N/2}-\phi_{N/2,N/4}}$
$x = 0.25$, $y = 0.375$				
8	54.528299401	–	–	–
16	55.025556909	55.191309412	–	–
32	55.152023647	55.194179226	0.002869814	–
64	55.183775584	55.194359563	0.000180337	15.914
128	55.191722015	55.194370825	0.000011262	16.013
256	55.193709150	55.194371528	0.000000703	16.020
$x = 0.75$, $y = 0.375$				
8	58.254175518	–	–	–
16	58.747870635	58.912435674	–	–
32	58.873284649	58.915089320	0.002653646	–
64	58.904765175	58.915258684	0.000169364	15.668
128	58.912643299	58.915269340	0.000010656	15.894
256	58.914613331	58.915270008	0.000000668	15.952

(*continued on next page*)

Table 6 (*continued*)

N	ϕ_N	$\phi_{N,N/2}$	$\phi_{N,N/2} - \phi_{N/2,N/4}$	$\frac{\phi_{N/2,N/4}-\phi_{N/4,N/8}}{\phi_{N,N/2}-\phi_{N/2,N/4}}$
$x = 1.25$, $y = 0.375$				
8	4.485793165	–	–	–
16	4.470779304	4.465774684	–	–
32	4.466790936	4.465461480	−0.000313204	–
64	4.465781462	4.465444971	−0.000016509	18.972
128	4.465528390	4.465444033	−0.000000938	17.600
256	4.465465080	4.465443977	−0.000000056	16.750
$x = 1.75$, $y = 0.375$				
8	0.759917048	–	–	–
16	0.748465579	0.744648423	–	–
32	0.745529934	0.744551386	−0.000097037	–
64	0.744791871	0.744545850	−0.000005536	17.528
128	0.744607107	0.744545519	−0.000000331	16.725
256	0.744560901	0.744545499	−0.000000020	16.550

4. Conclusion

In this study, we have proposed some one-sided finite-difference approximations for use with Richardson extrapolation. The essential logic is to use extra grid points to eliminate the odd-order terms in the truncation error but leave the even-order terms to be eliminated by Richardson extrapolation. Using a test problem we have demonstrated that the computed results have the order of accuracy we would expect. These one-sided finite-difference approximations, when used together with Richardson extrapolation are shown to have smaller local truncation error than the conventional higher-order one-sided finite-difference approximations. However, for the test problem for both approximations the solutions have comparable accuracy. The one-sided difference approximations proposed in this manuscript involve smaller number of grid points and are easier to derive than the conventional one-sided approximations which provide same order of accuracy.

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