Application of Richardson Extrapolation to the Numerical Solution of Partial Differential Equations

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Richardson extrapolation is a methodology for improving the order of accuracy of numerical solutions that involve the use of a discretization size h. By combining the results from numerical solutions using a sequence of related discretization sizes, the leading order error terms can be methodically removed, resulting in higher order accurate results. Richardson extrapolation is commonly used within the numerical approximation of partial differential equations to improve certain predictive quantities such as the drag or lift of an airfoil, once these quantities are calculated on a sequence of meshes, but it is not widely used to determine the numerical solution of partial differential equations. Within this article, Richardson extrapolation is applied directly to the solution algorithm used within existing numerical solvers of partial differential equations to increase the order of accuracy of the numerical result without referring to the details of the methodology or its implementation within the numerical code. Only the order of accuracy of the existing solver and certain interpolations required to pass information between the mesh levels are needed to improve the order of accuracy and the overall solution accuracy. Using the proposed methodology, Richardson extrapolation is used to increase the order of accuracy of numerical solutions of the linear heat and wave equations and of the nonlinear St. Venant equations in one-dimension. Numer Methods Partial Differential Eq 25: 810–832, 2009

Keywords: Richardson extrapolation; order of accuracy; higher order methods; defect correction method

I. INTRODUCTION

Richardson extrapolation is a methodology for increasing the order of accuracy of numerical solutions involving a discretization size h. In most introductory numerical analysis textbooks (e.g., [1]), this methodology is taught in the chapter on numerical differentiation as a way to increase the order of accuracy of stencils for approximating various derivatives. This methodology also forms the basis for the Romberg algorithm used within numerical integration, which relates the trapezoid rule, Simpson's rule, and Boole's rule for approximating the value of a definite integral. It can also be used within numerical solutions to ordinary differential equations, although it is not competitive in terms of computational costs with other higher-order accurate methods, such as the

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Runge-Kutta method, and hence it is not taught widely. Within various applications of numerical solutions to partial differential equations, it has been used to increase the accuracy of various integral outputs such as the calculation of the lift or drag coefficient for flow past an airfoil.

Richardson extrapolation relies on an observation about the shape of the error terms in a numerical approximation. Assume $N(h)$ is a numerical approximation of order p to an exact result $N(0)$. The algorithm is consistent when the exact result is obtained as h goes to 0. Using these assumptions, the numerical approximation can be expanded as

$$N(h) = N(0) + Ah^p + O(h^{p+1}) \tag{1}$$

Ignoring the higher order terms and using three solutions with discretization sizes of h, rh, and r^2h where r is the refinement ratio, the order of accuracy of the numerical result can be obtained via

$$p = \frac{\ln\left(\frac{N(r^2h)-N(rh)}{N(rh)-N(h)}\right)}{\ln r} \tag{2}$$

Once the order of accuracy is determined via analysis of the numerical scheme and verified via analysis of the numerical results, the numerical approximations can be combined to remove the leading order error term. Consider the results for two discretization sizes h and rh, or

$$\begin{aligned} N(h) &= N(0) + Ah^p + O(h^{p+1}) \\ N(rh) &= N(0) + Ar^p h^p + O(h^{p+1}) \end{aligned} \tag{3}$$

By multiplying $N(h)$ by r^p and subtracting off $N(rh)$, the leading order error term is removed. Thus, given a refinement ratio r, an order of accuracy p, and numerical approximations $N(h)$ and $N(rh)$, the Richardson extrapolation formula for improving the order of accuracy to at least $p + 1$ is

$$\frac{r^p N(h) - N(rh)}{r^p - 1} = N(0) + O(h^{p+1}) \tag{4}$$

If the original numerical scheme does not have an error term of the form h^{p+1}, then the order of accuracy of the new numerical scheme is based on the next lowest error term.

This simple formula does not depend on the quantity being approximated, whether an integral, a derivative, a solution to an ordinary differential equation or a solution to a partial differential equation. It does not require knowledge of the underlying methodology, except that the order of accuracy must be known. For many modern computational tools, which are often viewed as "black boxes," Richardson extrapolation can be viewed as a manipulation tool for the input or output of these black boxes without interfering with the details of the implementation within the black box. Thus, its application to proven numerical methodologies and well-established computational tools should require minimal effort to increase the accuracy of these methodologies and tools.

In regards to the solution of partial differential equations, Richardson extrapolation can be applied under certain circumstances to the solutions on two related meshes of step size $2h$ and h. Roache [2,3] used Richardson extrapolation to increase the accuracy of the solution to a Poisson equation on a two-dimensional structured mesh, by applying Richardson extrapolation to the mesh locations common to the coarse and refined meshes. In [4], Roache and Knupp developed the "completed Richardson extrapolation" technique by interpolating the coarse solution to the fine mesh and applying Richardson extrapolation on the fine mesh locations. They applied this

approach to an elliptic equation and to a steady advection–diffusion equation in one-dimension, achieving the expected increase in accuracy. The application of Richardson extrapolation to the numerical solution of partial differential equations can be successful when the solution does not experience significant dispersion, assuming that the algorithm has been properly implemented and that the sequence of meshes are built correctly.

Richards [5] applied the "completed Richardson extrapolation" technique to a time-dependent problem where Richardson extrapolation was applied to the solution at each time level as the time-dependent equations were solved. Sun and Zhang [6] applied a similar technique to a steady advection–diffusion equation with spatially varying coefficients in one-dimension and for a uniform two-dimensional structured mesh. In their approach, Richardson extrapolation was applied to the solution on the coarse and fine mesh after each iteration in the solution procedure. Furthermore, they used the governing partial differential equation to determine the interpolated values from the coarse mesh solution onto the missing mesh locations in the fine mesh. Rather than considering the solution, the methodology presented herein applies Richardson extrapolation to the residuals on the coarse and fine meshes at each time level, creating a higher-order accurate discretization; whereas the approach of Sun and Zhang directly applied Richardson extrapolation to the solution at each time level. These two approaches are quite similar.

The defect correction method [7] is similar to the proposed methodology in that residual calculations from various sources are combined to increase the accuracy of the numerical scheme. When the defect correction method is applied to the numerical solution of partial differential equations, a stable lower order accurate method is combined with a less stable higher order accurate method, resulting in a combined method that retains the stability of the lower order accurate method while converging iteratively to the order of accuracy of the higher order accurate method [8, 9]. The defect correction method requires implementation of two separate discretization methods, unlike the proposed methodology; furthermore, the order of accuracy is limited to the accuracy of the better discretization, unlike the proposed methodology which increases the order of accuracy. In the local defect correction method [10, 11], the residual calculated on the coarse mesh is adjusted by using the solution on the refined mesh as if it were the exact solution to approximate the truncation error within the coarse mesh solution. This defect correction is then added to the residual calculation to reduce truncation error within the coarse mesh simulation, without reference to the order of accuracy of the numerical discretization. The methodology proposed within this paper takes advantage of the fact that the residual has a certain order of accuracy and a known form based on this order of accuracy. By using this information within Richardson extrapolation, the leading order error term is methodically removed, increasing the order of accuracy. Thus, this proposed methodology may be considered a special case of the local defect correction method.

One of the main long-term goals for this research involves the application of this methodology to the numerical solution of partial differential equations on unstructured meshes. Currently, most methods for solving hyperbolic systems of equations, such as the compressible Navier-Stokes equations, on unstructured meshes, are limited to second-order spatial accuracy [12, 13] because of the use of linear reconstruction of the flow variables within each control volume. Several different methodologies have been proposed to increase the accuracy of unstructured flow solvers, including the spectral difference method [14, 15], spectral volume method [16], discontinuous Galerkin method, residual splitting methods [17], and higher-order WENO schemes [18]. In each of these methods, the original second-order algorithm, the related data structures, and parallel message passing schemes must be completely redesigned because the solution in each control volume is represented in a different manner involving several additional pieces of information and several additional flux evaluations. In comparison to the large modifications that are necessary to convert these legacy codes to higher order codes, especially when considering the confidence

level that users have with the legacy codes, the proposed methodology of applying Richardson extrapolation to increase the accuracy of these codes by using their components as "black box" subroutines may be quite attractive.

In this article, a methodology for applying Richardson extrapolation to numerical solvers of partial differential equations is presented. In the first section, the difficulties associated with the application of Richardson extrapolation to the numerical solution of partial differential equations are reviewed. After that section, the proposed methodology extending Richardson extrapolation to general partial differential equations is presented, followed by several examples. The first example deals with the parabolic heat equation in one-dimension which does not experience dispersion, allowing successful application of Richardson extrapolation to both the solution and the residual. This approach is applied on both uniform and nonuniform meshes. The second example is the one-dimensional wave equation which experiences significant dispersion. Because of the dispersion, application of Richardson extrapolation to the solution at the end of the simulation fails; whereas, its application to the residual improves the accuracy. A detailed numerical analysis of the finite difference stencil being solved via Richardson extrapolation is also presented for the wave equation. The final example is the nonlinear St. Venant equations which is a one-dimensional simplification of the incompressible Navier-Stokes equations and models the flow of water in open-channels. These examples demonstrate that the order of accuracy of numerical solutions to partial differential equations can be improved by using Richardson extrapolation without reference to the discretization method or the partial differential equations being studied.

II. DIFFICULTIES WITH RICHARDSON EXTRAPOLATION

Richardson extrapolation has been used to increase the order of accuracy of numerical approximations to the derivative of a function and to the integral of a function over a bounded domain. It can also be applied to the numerical solution of ordinary differential equations, although the computational cost is greater than certain specialized methods such as the Runge-Kutta approach. Unfortunately, Richardson extrapolation applied to the numerical solution of a partial differential equation is vulnerable to certain anomalies which cause the methodology to fail to achieve higher order accurate results. These anomalies include the effects of dispersion which can cause the location of physical features to vary based on the mesh resolution, improper implementation of the algorithm so that the theoretical order of accuracy is not achieved, and poor mesh refinement resulting in a sequence of meshes that are not geometrically similar to one another so that the errors associated with each mesh are not related. When the application of Richardson extrapolation to the numerical solution of a partial differential equation fails, the cause of this failure often is erroneously identified as the mesh "not being in the asymptotic range", when the failure may be often due to other causes.

The first vulnerability deals with the influence of dispersion on the features within a solution. For a numerical scheme that permits dispersion, waves of different frequencies travel at different speeds. If the numerical wave speeds differ from the wave speed within the exact solution, then nonphysical oscillations may develop within a simulation. The oscillations associated with numerical simulations of solutions with discontinuities are directly related to dispersion. Because of these oscillations and because the wave speeds are affected by the mesh resolution, the location of these oscillations and of physical features within a simulation, such as a standing wave or a shock, may change due solely to the mesh resolution. In Fig. 1, the solution from a first-order in time and second-order in space numerical scheme for the wave equation is shown. In this figure, oscillations are clearly visible to the left of the main solution, and the crest of the main solution

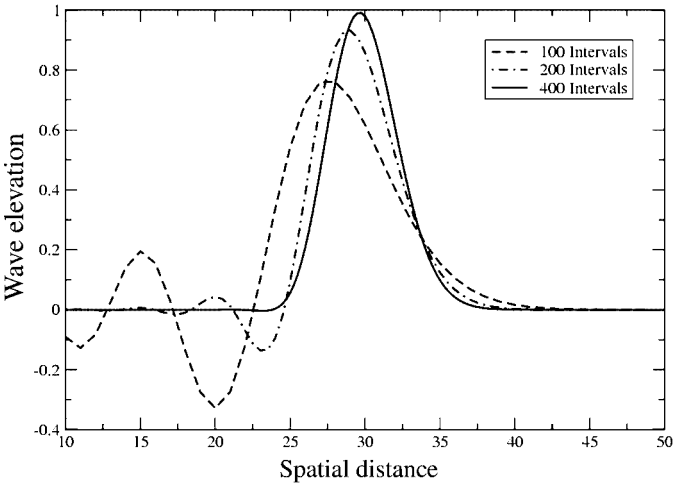

FIG. 1. Dispersion in a numerical solution to the wave equation.

changes location as the mesh is refined. Thus, Richardson extrapolation directly applied to these solutions would not improve the overall accuracy of the solution. Another example is shown in Fig. 2 which shows the steady-state water waves generated by a submerged hydrofoil [19], using a second-order unstructured mesh in two-dimensions. The frequency of these waves is clearly influenced by the mesh resolution, so that the waves are increasingly out of phase after the first wave crest, as the mesh is coarsened. Clearly, Richardson extrapolation applied to the solution will achieve a meaningful result only for the first couple of wave crests.

However, for numerical schemes and partial differential equations that do not suffer from dispersion, such as a central-difference approximation to the heat equation, Richardson extrapolation can be successfully applied to the numerical solution, assuming that the mesh is properly refined

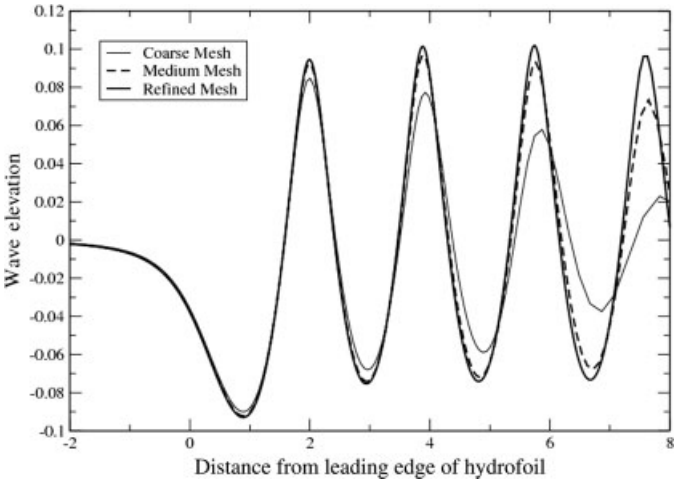

FIG. 2. Dispersion in water waves past a submerged hydrofoil.

and that the numerical scheme is properly implemented. This observation is demonstrated in the first example presented in this article and is the basis for its successful implementation by Roache on a Poisson problem [3]. Also, if the mesh is sufficiently resolved so that dispersion is unobserved, direct application of Richardson extrapolation to the solution may be feasible. In [4], the advection–diffusion problem only exhibited dispersion for the coarse meshes, resulting in a reduction in the observed order of accuracy, but for the refined meshes, the expected order of accuracy was achieved.

The second vulnerability deals with the implementation of the numerical scheme within the computational code. For Richardson extrapolation to be successful in increasing the order of accuracy, the order of accuracy of the numerical scheme must be known, in order to choose the appropriate Richardson extrapolation formula, and the numerical results must demonstrate this order of accuracy. Thus, code verification is critical for the successful application of Richardson extrapolation. However, rigorous code verification is time-consuming, and many advanced computational codes have not undergone this demanding procedure. Nonetheless, Richardson extrapolation is often attempted for these codes, with potentially dubious results.

The third vulnerability involves the manner in which the sequence of meshes was created. For a sequence of meshes to be appropriate for mesh refinement studies, the discretization in each region of the computational domain must be similar. Hence, for unstructured meshes, regions where quadrilaterals are used in a coarse mesh must be gridded using similar quadrilaterals in each successively refined mesh. Similarly, regions where triangles are used must continue to use similar triangles. Eca [20] proposed the concept of geometric similarity as a requirement for mesh refinement. In two-dimensions, h-refinement of a mesh will generally satisfy this requirement, but in three-dimensions, h-refinement only partially satisfies this requirement, so geometric similarility for three-dimensional tetrahedral meshes is still an open research issue. Furthermore, if the coarse mesh is sufficiently resolved to capture the geometric features adequately, the refined meshes may be too large because h-refinement in three-dimensions increases the number of elements by a factor of 8. For structured meshes, refinement ratios other than a factor of 2 can be used, although care still must be taken to ensure geometric similarity. For unstructured meshes, especially involving mixed elements, typical mesh generation strategies do not satisfy the requirement of geometric similarity since the prismatic boundary layer region for a refined mesh will typically not extend as far from the viscous surface, when the surface mesh is refined. An example of an unstructured mesh generation method that yielded acceptable results, without h-refinement, is given in [21].

The results cited by Baker [22] concerning the AIAA Second Drag Prediction Workshop identified significant concerns about the accuracy of the submissions when analyzed via Richardson extrapolation of the predicted lift and drag coefficients. Each submission contained a set of results from a coarse, medium, and fine mesh, from which an order of accuracy could be determined and an improved prediction could be made using this order of accuracy and Richardson extrapolation. The predicted orders of accuracy varied from 0.09 to 23.7 along with some negative and some complex orders of accuracy. Because the observed order of accuracy for the vast majority of these submission were unreasonable, the results from the application of Richardson extrapolation to improve the numerical prediction were similarly dubious. The primary reasons for these problems were the use of unverified computational tools, poor mesh refinement strategies for both structured and unstructured simulations and the choice of a test case which contained certain physical features that could be affected by numerical dispersion.

In conclusion, Richardson extrapolation when directly applied to the numerical solution of a partial differential equation often fails to increase the order of accuracy of the numerical result. This failure does not imply that Richardson extrapolation cannot or should not be applied to this

class of numerical simulations, but rather it indicates that care must be used in order to achieve the desired result. Richardson extrapolation has been successfully applied to the solution of parabolic partial differential equations on uniformly refined structured meshes in two-dimensions, showing that it can be useful under certain circumstances; however, it cannot be applied to solutions which display dispersion. Furthermore, the computational code must be verified to achieve the theoretical order of accuracy, and care must be taken in building the sequence of successively refined meshes.

III. NUMERICAL APPROACH

Finite difference, finite volume, and finite element discretization methods for partial differential equations, in implicit form, result in a nonlinear system of coupled algebraic equations that must be solved either at each time step or for the steady-state solution. This system can be written as

$$\vec{R}(\vec{Q}^{n+1}, \vec{Q}^{n}, \ldots, \vec{Q}^{n-k}, \chi, \vec{t}) = 0 \tag{5}$$

where $\vec{R}$ is referred to as the residual vector, $\vec{Q}^{n+1}$ is the solution to be determined for time level t_{n+1}, $\vec{Q}^{n}, \ldots, \vec{Q}^{n-k}$ are the known solutions at previous time levels, χ are the spatial locations, and $\vec{t}$ are the temporal locations. This formulation assumes a fixed mesh. Newton's method is often employed to seek the solution of this system of equations via the following

$$\frac{\partial \vec{R}}{\partial \vec{Q}^{n+1}} \Delta Q^{n+1,m+1} = -\vec{R}(\vec{Q}^{n+1,m}, \vec{Q}^{n}, \ldots, \vec{Q}^{n-k}, \chi, \vec{t}) \tag{6}$$

with $\vec{Q}^{n+1,m+1} = \vec{Q}^{n+1,m} + \Delta Q^{n+1,m+1}$ and $\vec{Q}^{n+1,0} = \vec{Q}^{n}$.

Assuming the solution is sufficiently smooth and that the differential operator does not introduce any discontinuities into a smooth solution, then the residual vector varies smoothly throughout the computational domain. In the presence of discontinuities, these discontinuities divide the computational domain into regions where the residual vector varies smoothly bounded by regions containing these discontinuities. Typically, when discontinuities are present in the solution, a limiter or some other artificial means is used to maintain stability at the expense of accuracy. In regions where the solution and the residual vector vary smoothly, Taylor's series expansions, Richardson extrapolation and interpolations between solutions and residuals work well, and higher order accurate numerical approximations are possible.

The residual vector generated via the finite volume or the finite element approach will typically include an extra factor of the area of the control volume or control element because of the integration involved in generating the discretized equation. The finite difference scheme typically involves a division using the appropriate length scales. Hence, the residual for finite difference approximation directly corresponds to the differential operator, while the residual for the finite volume and finite element approximations include an extra factor of the area over which the integration occurs. On a uniform mesh, the resulting residuals vary smoothly for any of these approaches because the area is constant; however, on a nonuniform mesh, the residuals for the finite difference approximation will vary smoothly, while the residuals divided by the area for the finite volume or finite element approximations will vary smoothly. This observation is critical for the effectiveness of the Richardson extrapolation algorithm for nonuniform meshes.

Richardson extrapolation involves the communication of certain information between a coarse mesh and a refined mesh. As noted in the previous section, direct application of Richardson

extrapolation to the solution fails due to the possibility of dispersion shifting the solutions significantly from one mesh level to the next. Nevertheless, this approach is a good way to introduce the relevant concepts.

Consider two meshes, χ_1 and χ_2 where χ_2 is a refined mesh consisting of all of the points in χ_1 with spacing that is half of the spacing in χ_1. In one-dimension, χ_2 has twice as many intervals as χ_1. A solution to the partial differential equation at the required time level is determined on both meshes, with Q_1 and Q_2 representing the solutions on χ_1 and χ_2, respectively. One approach for improving the accuracy is to apply Richardson extrapolation to the solution points in the coarse mesh, since these points are in common, or

$$Q_1^{\mathrm{RE}} = \mathrm{RE}(Q_1, Q_2, \chi_1, p) \tag{7}$$

where $\mathrm{RE}(Q_1, Q_2, \chi_1, p)$ represented a pth order accurate Richardson extrapolation of solutions Q_1 and Q_2 using the mesh locations in χ_1. Using this approach, the accuracy of the solution on the coarse mesh is improved, as described in [2,3]. Later work of Roache and Knupp [4] demonstrated that Richardson extrapolation using these same solutions can increase the order of accuracy on the refined mesh, if the solution on the coarse mesh is interpolated to the new locations on the fine mesh, or

$$Q_1^{(i,\chi_2)} = I^{\alpha}(Q_1, \chi_1, \chi_2) \tag{8}$$

The operator $I^{\alpha}(Q_1, \chi_1, \chi_2)$ is an interpolation operator of order α of the solution Q_1 on mesh χ_1 to mesh χ_2, and $Q_1^{(i,\chi_2)}$ is the interpolated solution. Using this interpolated solution, a solution of improved accuracy on the refined mesh can be determined via

$$Q_2^{\mathrm{RE}} = \mathrm{RE}\big(Q_1^{(i,\chi_2)}, Q_2, \chi_2, p\big) \tag{9}$$

For this approach to work, the interpolation must be sufficiently accurate so that the error introduced into the interpolated solution is of higher degree than the order of accuracy of the numerical method.

In Fig. 3, the solution on the coarse mesh and the refined mesh are shown. The solution on the coarse mesh is interpolated to the refined mesh locations. Then, Richardson extrapolation is used to combine the interpolated coarse mesh solution and the refined mesh solution. Applying this procedure to the solution after each iteration should remove the susceptibility to dispersion error, which is demonstrated in [6].

In the proposed method, Richardson extrapolation is applied to the numerical calculation of the discretized equations on a sequence of meshes rather than to the solution on these meshes. As stated in Eq. (5), the discretized governing equations can be written as

$$\vec{R}(\vec{Q}^{n+1}, \vec{Q}^{n}, \ldots, \vec{Q}^{n-k}, \chi, \vec{t}) = 0 \tag{10}$$

The equation generated from an explicit or implicit finite difference, finite volume or finite element method can be written in the form given earlier. This equation must be solved for the value at the new time level. For an explicit scheme, the unknown Q^{n+1} can be determined explicitly, or $Q^{n+1} = F(\vec{Q}^{n}, \ldots, \vec{Q}^{n-k}, \chi, \vec{t})$; whereas for an implicit scheme, the unknown Q^{n+1} can be determined via Newton's method as stated in Eq. (6).

Being that this equation is derived from a discretization method of a certain order of accuracy, the order of accuracy information is contained within this equation. By plugging a continuous solution into the residual equation, the original partial differential equation along with the order

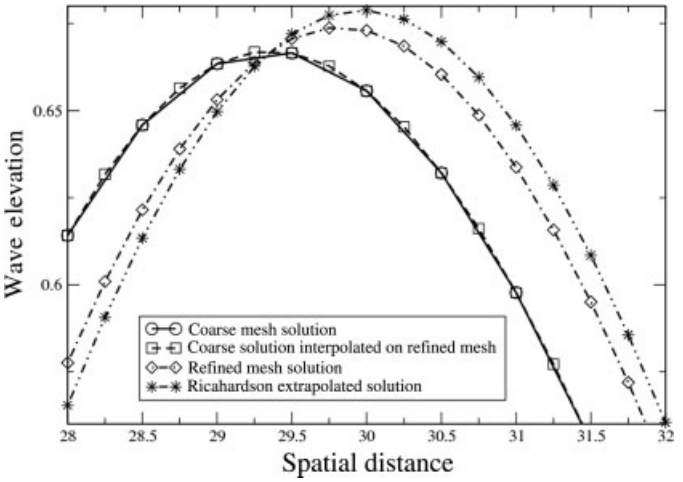

FIG. 3. Interpolation and Richardson extrapolation.

of accuracy is obtained for a finite difference stencil. For instance, consider the finite difference approximation to the elliptic equation $u_{xx} = 0$ or

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = 0 \tag{11}$$

By plugging in the continuous solution and applying Taylor's series expansion about x_j and assuming that $x_{j+1} = x_j + h$ and $x_{j-1} = x_j - h$, this equation becomes

$$R(h) = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = u_{xx} + u_{xxxx}\frac{h^2}{12} + u_{xxxxxx}\frac{h^4}{360} + O(h^6) = 0 \tag{12}$$

Thus, this stencil is second-order accurate in space. To increase the accuracy of this stencil using Richardson extrapolation, via Eq. (4), h is replaced by $2h$, and the residuals based on h and $2h$ are combined to eliminate the leading order error term, or

$$\frac{4R(h) - R(2h)}{3} = \frac{-u_{j+2} + 16u_{j+1} - 30u_j + 16u_{j-1} - u_{j-2}}{12h^2} = u_{xx} - u_{xxxxxx}\frac{h^4}{90} + O(h^6) = 0 \tag{13}$$

Similarly, the methodology proposed herein performs the same calculation, except that this calculation is performed on the numerical results generated by a computer code rather than on the stencil itself. Hence, on the coarse mesh, the residual equation is

$$R_{\text{coarse}}\left(\vec{Q}^{n+1}_{\text{coarse}}, \vec{Q}^{n}_{\text{coarse}}, \ldots, \vec{Q}^{n-k}_{\text{coarse}}, \chi_{\text{coarse}}, \vec{t}\right) = 0 \tag{14}$$

while on the refined mesh, generated by subdivision or h-refinement, the residual equation is

$$R_{\text{refined}}\left(\vec{Q}^{n+1}_{\text{refined}}, \vec{Q}^{n}_{\text{refined}}, \ldots, \vec{Q}^{n-k}_{\text{refined}}, \chi_{\text{refined}}, \vec{t}\right) = 0 \tag{15}$$

Once the coarse and refined residuals are calculated, they are combined via Richardson extrapolation or

$$R_{\text{RE}} = \frac{2^p R_{\text{refined}} - R_{\text{coarse}}}{2^p - 1} \tag{16}$$

By combining the residuals on the coarse and refined meshes and then driving them to zero, the higher order accurate residual is calculated. For an explicit scheme, this higher order result is used to update the solution at the next time level. For an implicit scheme, this higher order residual forms the right hand side to Eq. (6), from which the update to the new solution is determined. This higher order residual either exists on the coarse mesh or on the refined mesh. If the coarse mesh is used, then the refined mesh residuals must be restricted to the common nodes, and the new solution is interpolated onto the non-common nodes for the refined mesh. If the refined mesh is used, then the residuals on the coarse mesh are interpolated onto the new nodal locations on the refined mesh, using an interpolation stencil that is more accurate than the numerical discretization scheme. All examples demonstrated herein interpolate the coarse residual onto the refined mesh, so that the operations are performed on the refined mesh.

Because it views the numerical solver as a black box, this Richardson extrapolation-based algorithm reuses the residual calculations within a computational tool, the existing data structures are unchanged, and the same matrix solution algorithm is used. Hence, to achieve the higher order result, only the residual on the coarse and refined meshes must be calculated.

The Richardson extrapolation based algorithm applied to the residual involves the following components:

1. Calculate the residual for the coarse mesh $R_{\text{coarse}}(\vec{Q}^{n+1}_{\text{coarse}}, \vec{Q}^{n}_{\text{coarse}}, \ldots, \vec{Q}^{n-k}_{\text{coarse}}, \chi_{\text{coarse}}, \vec{t})$
2. Calculate the residual for the refined mesh $R_{\text{refined}}(\vec{Q}^{n+1}_{\text{refined}}, \vec{Q}^{n}_{\text{refined}}, \ldots, \vec{Q}^{n-k}_{\text{refined}}, \chi_{\text{refined}}, \vec{t})$
3. Divide through by length scale factor, if necessary.
4. Interpolate the coarse residual onto the refined mesh.
5. Apply Richardson extrapolation on the interpolated coarse mesh residual and the refined mesh residual.
6. Multiply through by length scale factor, if necessary.
7. Solve for the update to the solution, either explicitly or implicitly.
8. Add these updates to the current approximation to the new solution on the refined mesh.
9. Restrict the new refined mesh solution to the coarse mesh via direct copying for the common nodes.

IV. ADDITIONAL COMMENTS

A. Discontinuities and Interpolation

The methodology for applying Richardson extrapolation to the residual resulting from a discretization of a governing partial differential equation requires that the solution be smooth. Both, Richardson extrapolation and the interpolation assume that the solution is sufficiently smooth so that the Taylor's series expansion exists near each mesh point and that the interpolation is meaningful. However, hyperbolic systems of partial differential equations admit and even generate discontinuities, invalidating the underlying assumptions of this methodology.

To overcome these difficulties, limiters, artificial viscosity, or other methods may be used to reduce the oscillations that are generated by numerical methods near these points of discontinuity. Limiters, in essence, reduce the order of accuracy of the numerical scheme, thereby, introducing additional dissipation into the scheme. Artificial viscosity controls the oscillations generated by the numerical scheme by adding extra viscosity or dissipation to damp out these oscillations and smooth the solution adequately. Both approaches need to be investigated for use within this proposed Richardson extrapolation-based methodology.

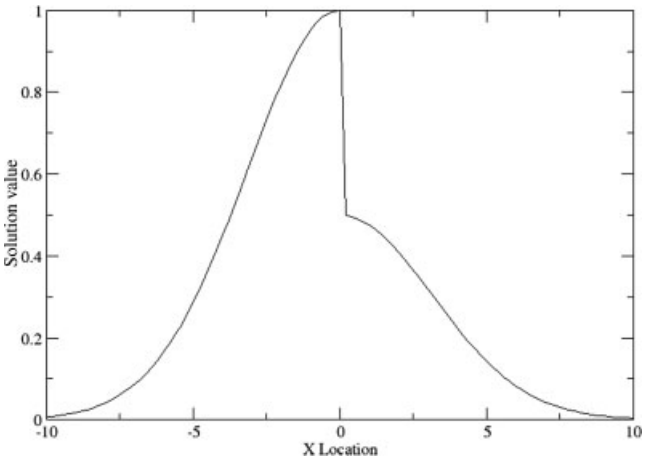

FIG. 4. An example of a discontinuous solution.

Another approach to handing simulations that contain discontinuous solutions or solutions that vary rapidly is to recognize that these discontinuities subdivide the computational domain into distinct regions where the solution is smooth. Within each such region, the assumptions underpinning Richardson extrapolation and the interpolations remain valid. Thus, the interpolation stencils should be biased so that only the information within each region is used.

For instance, a discontinuous initial solution for the wave equation is shown in Fig. 4. The values to the right of $x = 0$ are half of the values to the left. The residual from a second-order accurate finite volume stencil is shown in Fig. 5, where the effects of the discontinuity are clearly seen in a spike in the residual, near $x = 0$. However, on either side of this spike, the residual varies smoothly. By ignoring the nodes that experience the effects of the discontinuity, directionally biased interpolations can be constructed so that Richardson extrapolation can be used to increase the order of accuracy of the discretized equations away from this discontinuity. Obviously,

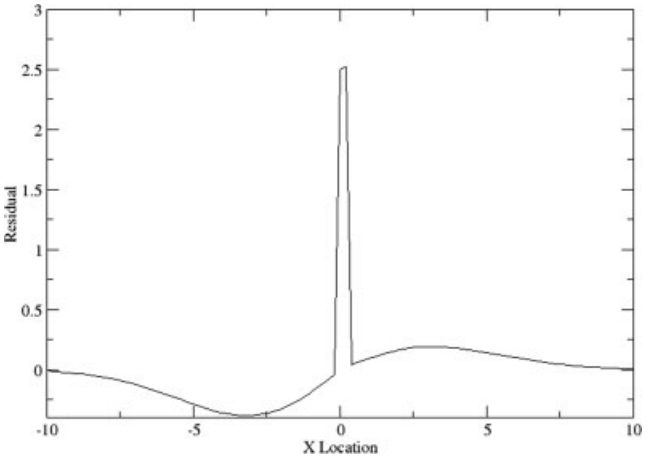

FIG. 5. Residual for a discontinous solution.

Richardson extrapolation should probably not be used for the nodes that directly experience the effects of the discontinuity as the order of accuracy at these nodes is not well defined. Detection of these discontinuities and development of methods to keep information from being erroneously used across discontinuities remains an open research topic.

These biased interpolation stencils are also needed for certain types of boundary conditions. Some boundary conditions can be applied in a manner that is consistent with the interior discretization. These cases may include farfield, inflow, and outflow boundary conditions. However, Dirichlet boundary conditions, such as no-slip or slip-wall boundary conditions that either enforce zero velocity or tangential velocity, are often implemented in a manner that is not consistent with the interior discretization. Hence, care must be taken at the boundaries when interpolating between different types of boundary conditions and interior discretizations, in order to achieve consistent interpolations.

B. Two-Dimensional Unstructured Meshes

As is shown in the examples section, this methodology achieves excellent results for one-dimensional simulations. Thus, the natural question is whether it can be extended to two- and three-dimensions. Roache and Knupp [2,3] demonstrated this methodology on a two-dimensional rectilinear mesh. As there are several approaches for increasing the order of accuracy on structured meshes, the long-term goal of this research is to apply this methodology to unstructured meshes, where second-order spatial accuracy is the standard. Current efforts to increase the spatial order of accuracy of unstructured meshes include the spectral difference method, spectral volume method and the discontinuous Galerkin method, both of which involve significant rewrites of existing codes. The primary advantage of the Richardson extrapolation approach is that no rewriting of the proven and trusted numerical algorithms within existing codes is necessary.

On unstructured triangular meshes, a sequence of meshes can be obtained via h-refinement. Appropriate information must be maintained in order to communicate information between meshes. Richardson extrapolation can be applied immediately to the nodes that are common to all meshes in the sequence. As is the case for the one-dimensional case, interpolation must be performed to pass the information to the new points in the refined meshes.

For node-based finite volume methods and most finite element methods, the variables Q and their gradients ∇Q are stored at the nodes. H-refinement subdivides each edge by placing a point at the midpoint of each edge $\vec{x}_{ij}$. Using the variable and the gradient at the two endpoints at x_i and x_j, a fourth-order accurate interpolation can be achieved at the midpoint via

$$Q_{ij} = \frac{1}{2}(Q_i + Q_j) + \frac{1}{8}(\nabla Q_i \cdot \vec{r}_{ij} - \nabla Q_j \cdot \vec{r}_{ij})$$

where $\vec{r}_{ij} = x_j - x_i$.

The gradient can be estimated to second-order accuracy via Green's Theorem or via a least-squares approach. More complicated stencils have been used to estimate higher derivatives for use within higher order methods, but these stencils are not as compact as those typically used to estimate the gradient.

The interpolation stencil given above restricts the order of accuracy of this Richardson extrapolation approach to fourth-order accuracy, because the interpolation stencil must be more accurate than the leading order error term being removed via Richardson extrapolation.

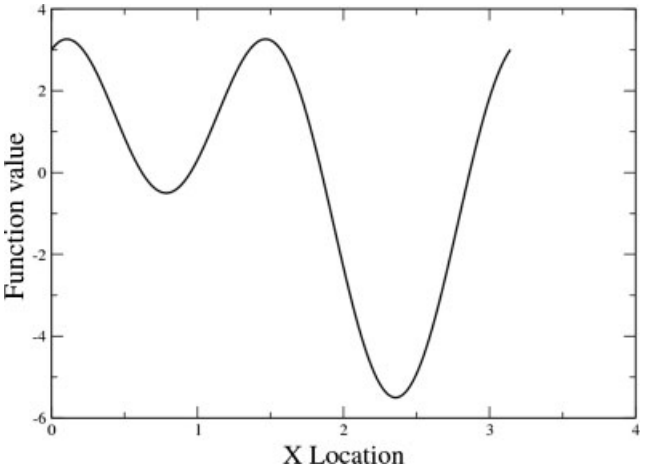

FIG. 6. Exact solution for heat equation with source term.

V. NUMERICAL EXAMPLES

A. Heat Equation in 1D

The first test case deals with the one-dimensional heat equation with a source term so that the solution is non-trivial. In particular, the governing equation is

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + AB^2 \sin(Bx) + CD^2 \cos(Dx) \tag{17}$$

which has the steady-state solution of $u(x) = A\sin(Bx) + C\cos(Dx)$. This equation is solved on the interval $[0, \pi]$ with the choice of coefficients as $d = 1$, $A = 2.5$, $B = 2$, $C = 3$, and $D = 4$. One complete period is obtained from $[0, \pi]$. A picture of the solution is shown in Fig. 6.

This equation is solved on a uniform mesh consisting of N intervals, labeled x_0, through x_N, so that $h = x_{i+1} - x_i$. This equation is solved implicitly using a finite volume approach, by integrating from $x_{i-1/2}$ to $x_{i+1/2}$, using a linear interpolation of values to the face, resulting in the following stencil

$$\begin{aligned}\frac{u_i^{n+1} - u_i^n}{k} = d\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} - AB\sin(Bx_{i+1/2}) \\ + CD\cos(Dx_{i+1/2}) + AB\sin(Bx_{i-1/2}) - CD\cos(Dx_{i-1/2})\end{aligned} \tag{18}$$

This spatial discretization is second-order accurate.

The error in the solution on a sequence of meshes is calculated via the L1-error to measure the difference between the computed solution and the exact solution or

$$\text{Error}(u^{\text{computed}}, u^{\text{exact}}) = \frac{\sum_{i=1}^{N} \left| u_i^{\text{computed}} - u_i^{\text{exact}} \right|}{N} \tag{19}$$

All errors reported herein are based on this error metric. The error results for the heat equation at steady-state are shown in Table I, along with the observed order of accuracy p. For this problem, the observed order of accuracy is second order which agrees with the theoretical order.

TABLE I. Observed errors for solution of heat equation on uniform mesh.

Intervals N	Error	Order of accuracy
50	0.024850796485	NA
100	0.006205401065	2.0016957
200	0.001550853829	2.0004617
400	0.000387682045	2.0001169
800	0.000096918540	2.0000297
1600	0.000024229515	2.0000067

At steady-state, this numerical scheme does not exhibit any dispersion, so Richardson extrapolation can be directly applied to the solution. Using a fourth-order accurate interpolation of the coarse solution onto the refined mesh and applying Richardson extrapolation to the refined mesh solutions, the leading order error term can be eliminated. Because the numerical scheme uses a central difference formulation, there are no odd order error terms, so the observed order of accuracy jumps from second to fourth order, as is shown in Table II.

Richardson extrapolation has also been applied to the residuals at each iteration. These results are presented in Table III, demonstrating a similar jump in the observed order of accuracy from second to fourth order. For this parabolic equation since it is sufficiently smooth and the numerical scheme does not exhibit any dispersion, Richardson extrapolation can be applied multiple times on a sequence of meshes. It can be applied to be a triplet of successively refined meshes to increase the order of accuracy from second order to fourth order to sixth order, assuming that the interpolation performed at each step is at least sixth order accurate.

Additional simulations were performed to demonstrate that Richardson extrapolation can be applied on nonuniform meshes. The basic numerical scheme is altered to handle non-uniform meshes resulting in the following stencil

$$\begin{aligned}\frac{u_i^{n+1}-u_i^n}{k}(x_{i+1/2}-x_{i+1/2}) = d\left(\frac{u_{i+1}^{n+1}-u_i^{n+1}}{x_{i+1}-x_i}-\frac{u_i^{n+1}-u_{i-1}^{n+1}}{x_i-x_{i-1}}\right)\\ -AB\sin(Bx_{i+1/2})+CD\cos(Dx_{i+1/2})\\ +AB\sin(Bx_{i-1/2})-CD\cos(Dx_{i-1/2})\end{aligned} \tag{20}$$

One method for generating geometrically similar nonuniform meshes is for the point spacing in the mesh to grow geometrically. In this mesh, the point spacing grows geometrically from the smallest spacing on the left to the largest on the right, with the initial spacing of $h_c = \frac{\pi(1-r)}{1-r^n}$ where r is the growth rate and n is the number of intervals. Given a coarse mesh initial space of h_c and a growth rate of r, the refined mesh initial spacing is $h_r = \frac{h_c}{1+\sqrt{r}}$ and the growth rate is $s = \sqrt{r}$.

TABLE II. Richardson extrapolation applied to the solution of the heat equation.

Mesh pair N	Error	Order of accuracy
50/100	0.000103128885	NA
100/200	0.000006221737	4.0509872
200/400	0.000000393527	3.9827829
400/800	0.000000024775	3.9894924
800/1600	0.000000001554	3.9948569

TABLE III. Richardson extrapolation applied to the residual of the heat equation.

Mesh pair N	Error	Order of accuracy
50/100	0.000021530633	NA
100/200	0.000001331410	4.0153637
200/400	0.000000082966	4.0042934
400/800	0.000000005181	4.0011159
800/1600	0.000000000324	4.0001766

Using this type of non-uniform mesh, the solution without applying Richardson extrapolation is second order accurate. The results from applying Richardson extrapolation once to the residual is shown in Table IV which indicates an improvement in the order of accuracy to fourth order. Hence, for this type of non-uniform mesh, Richardson extrapolation is able to increase the accuracy appropriately.

B. Wave Equation in 1D

The constant coefficient, first-order wave equation in one-dimension is a hyperbolic partial differential equation that can be written as

$$u_t + a u_x = 0$$

It has the exact solution of $u(x,t) = u_0(x - at)$ where $u_0(x)$ is the initial condition. Various boundary conditions can be applied, including periodic and boundary conditions consistent with the propagation of wave information either into the domain or out of the domain.

This test case uses a Gaussian initial condition centered at $x = -30$, or $u_0 = e^{-(x+30)^2/10}$ on a uniform mesh ranging from $x = -50$ to $x = 50$. The wave speed is set to $a = 2$, so the solution moves from left to right. The left boundary is set to be consistent with the Gaussian function, or $u(x = -50, t) = e^{-(-at-20)^2/10}$, while the right boundary is a flow-through boundary, determined via $u_{xx}(x = 50, t) = 0$.

The governing partial differential equation is discretized using an implicit backward-time/central-space discretization, or

$$\frac{3u_i^{n+1} - 4u_i^n + u_i^{n-1}}{2k} + \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2h} = 0 \tag{21}$$

This discretization is second-order in time and second order in space. Because the goal of this study is to use Richardson extrapolation to increase the spatial order of accuracy, the second-order temporal discretization is used with a small time step, so that the influence of the error associated with the temporal derivative is minimized.

TABLE IV. Richardson extrapolation applied to the residual of the heat equation on a non-uniform mesh.

Mesh pair N	Error	Order of accuracy
50/100	0.011989652994	NA
100/200	0.000397969733	4.9163374
200/400	0.000022704957	4.1315795
400/800	0.000001398388	4.0211708
800/1600	0.000000087080	4.0052729

TABLE V. Error in original wave equation solver.

Intervals	Error	Order of accuracy
50	0.1290134	NA
100	0.0607230	1.08721
200	0.0187772	1.69326
400	0.0047005	1.99810
800	0.0011822	1.99134
1600	0.0002957	1.99927

The dispersive nature of these methods is clearly observed in Fig. 1, where the number of intervals ranges from 100 to 400 by factors of 2. The solution is progressed 30 units of time, so that the center of exact solution is at $x = 30$. The error between the computed solution and the exact solution as an L1-norm is presented in Table V, showing second-order accuracy in space, although for the coarser meshes, a lower order of accuracy is observed.

Ahead of the wave, the solution is smooth, but behind the wave, the solution exhibits extreme oscillations due to the dispersion in the central difference stencil. The troughs and peaks of these oscillations change location based on the spatial resolution. Furthermore, the location of the highest point which should be at $x = 30$ drifts backwards due to the slowing of the wave speeds. Because the locations of the waves do not agree as the resolution changes, direct application Richardson extrapolation to the solution leads to poor results, as is shown in Fig. 7. The extrapolated solution shows some improvement near the main crest, but the oscillations behind the wave form are worse than for the unextrapolated solution on the refined mesh. Previous investigations into the use of Richardson extrapolation to partial differential equations have recognized this phenomenon and have erroneously concluded that Richardson extrapolation was of limited benefit for hyperbolic partial differential equations.

The error for the application of Richardson extrapolation to the solution at the final time level is shown in Table VI. The order of accuracy for the coarse meshes has been adversely affected by

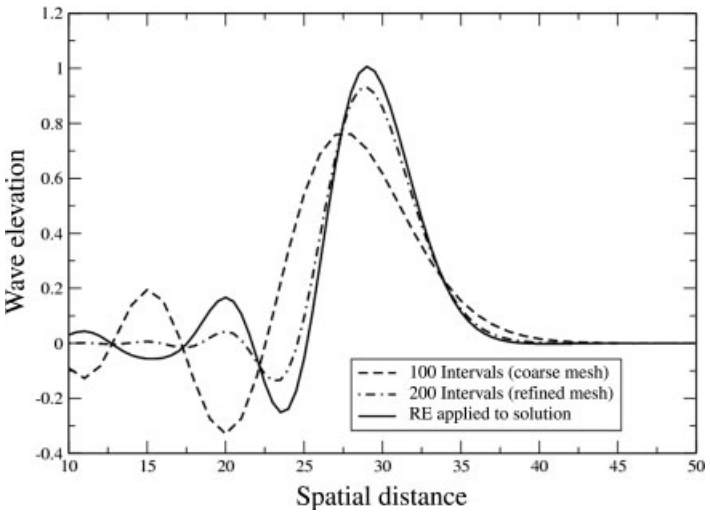

FIG. 7. Richardson extrapolation applied to solution of wave equation at final time level.

TABLE VI. Error in wave equation solver with Richardson extrapolation applied to final solution.

Intervals	Error	Order of accuracy
50/100	0.08339633	NA
100/200	0.02690852	1.63192
200/400	0.00404490	2.73389
400/800	0.00029724	3.76640
800/1600	0.00001887	3.97746

numerical dispersion, for the coarser meshes. Once the mesh resolution is sufficiently refined so that numerical dispersion is of little consequence, the predicted fourth order accuracy is observed. However, if the simulation were performed on a long enough mesh, the effects of dispersion would eventually become detrimental to the accuracy of this approach.

The Richardson extrapolation procedure applied to the residual at each time level works well, in spite of the dispersive nature of the underlying scheme. In Fig. 8, the second-order in space solutions on the coarse and refined meshes are shown along with the solution generated by the application of Richardson extrapolation to the residual during the solution process and the solution calculated via a second order in time, fourth order in space central difference stencil. The extrapolated solution is clearly much more accurate than the second-order solution and agrees well with the fourth order in space stencil. The Richardson extrapolated solution exhibits slight dispersion ahead of the wave. The error in the Richardson extrapolated solution is given in Table VII, showing fourth order spatial accuracy.

C. Analysis of Stencil Used in Wave Equation

Because the fourth-order central difference stencil to the first derivative can be derived via an application of Richardson extrapolation to the second-order stencil given by $u_x = \frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2h}$, one might expect that the Richardson extrapolated solution should agree with the fourth-order

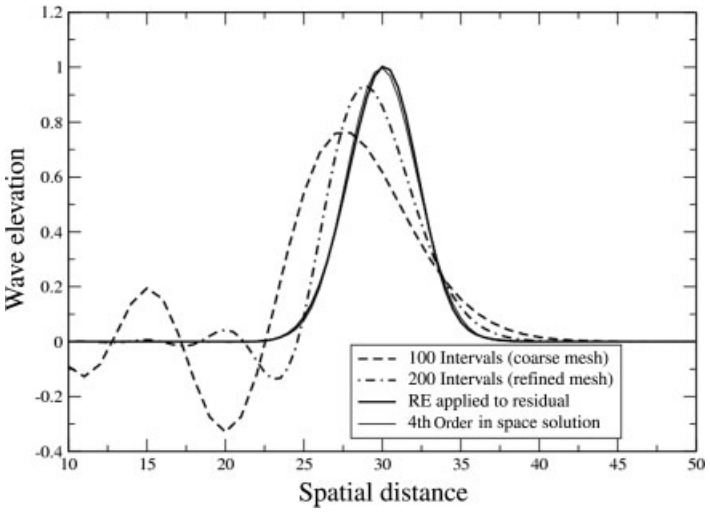

FIG. 8. Richardson extrapolation applied to residual during the simulation of wave equation.

TABLE VII. Error in wave equation solver with Richardson extrapolation applied to residual.

Intervals	Error	Order of accuracy
50/100	0.001906280	NA
100/200	0.0003634528	2.39092
200/400	0.0000384607	2.61252
400/800	0.0000027630	3.79908
800/1600	0.0000001797	3.94257

solution. As shown in Fig. 8, the solutions differ slightly. This difference is a result of the use of an interpolation to project the residual from the coarse mesh onto the refined mesh, altering the characteristics of the solutions, and is analyzed later.

The implicit second order in space and time stencil used to discretize the wave equation is

$$\frac{3u_{2i}^{n+1} - 4u_{2i}^{n} + u_{2i}^{n-1}}{2k} + \frac{u_{2i+1}^{n+1} - u_{2i-1}^{n+1}}{2h} = 0 \tag{22}$$

To distinguish between the coarse and refined meshes, the spatial index has been adjusted to represent the locations in the refined mesh. The even nodes in the refined mesh correspond to locations in the coarse mesh. The stencil shown above is for the node $2i$ which is common to both meshes.

The application of Richardson extrapolation to the residual vector based on this stencil on the refined mesh results in two different stencils. The first stencil is formed for the nodes that are in common between the coarse and refined meshes and is the standard stencil that is generated from the application of Richardson extrapolation to the stencil using equation (4) with $r = 2$ and $p = 2$. This stencil is

$$\frac{3u_{2i}^{n+1} - 4u_{2i}^{n} + u_{2i}^{n-1}}{2k} + \frac{-u_{2i+2}^{n+1} + 8u_{2i+1}^{n+1} - 8u_{2i-1}^{n+1} + u_{2i-2}^{n+1}}{12h} = 0 \tag{23}$$

Performing a Taylor's series expansion of this stencil yields

$$u_t + u_x - \frac{u_{ttt}}{3}k^2 - \frac{u_{5x}}{30}h^4 + O(k^3, h^6) = 0 \tag{24}$$

which is clearly second order in time and fourth order in space. The modified equation [23] for this finite difference stencil is

$$u_t + u_x + \frac{k^2}{3}u_{xxx} + \frac{k^3}{4}u_{4x} + \left(-\frac{h^4}{30} + \frac{9k^4}{20}\right)u_{5x} + \frac{5k^5}{8}u_{6x} + \left(-\frac{h^6}{252} - \frac{h^4k^2}{30} + \frac{57k^6}{56}\right)u_{7x} + \left(-\frac{h^4k^3}{30} + \frac{105k^7}{64}\right)u_{8x} + \left(-\frac{h^8}{4320} - \frac{h^6k^2}{252} - \frac{3h^4k^4}{40} + \frac{1585k^8}{576}\right)u_{9x} \cdots = 0 \tag{25}$$

The second stencil is generated for the node on the refined mesh that is not included in the coarse mesh, so an extra interpolation step is required. On the refined mesh, the stencil at x_{2i-1} is

$$N(h) = \frac{3u_{2i-1}^{n+1} - 4u_{2i-1}^{n} + u_{2i-1}^{n-1}}{2k} + \frac{u_{2i}^{n+1} - u_{2i-2}^{n+1}}{2h} \tag{26}$$

The interpolation stencil must be greater than second order accurate so that the error introduced by interpolation is of a higher order accuracy that the leading order error term in the stencil. The lowest centered fourth order interpolation involving locations on the coarse mesh is

$$R_{2i-1} = \frac{-R_{2i+2} + 9R_{2i} + 9R_{2i-2} - R_{2i-4}}{16} \tag{27}$$

where the coarse mesh residual is

$$R_{2i} = \frac{3u_{2i}^{n+1} - 4u_{2i}^{n} + u_{2i}^{n-1}}{2k} + \frac{u_{2i+2}^{n+1} - u_{2i-2}^{n+1}}{4h} = 0$$

Using this interpolation formula, the second order residual interpolated from the coarse mesh onto the refined mesh is

$$\begin{aligned} N(2h) = \frac{1}{16}\Bigg(& -\frac{3u_{2i+2}^{n+1} - 4u_{2i+2}^{n} + u_{2i+2}^{n-1}}{2k} + 9\frac{3u_{2i}^{n+1} - 4u_{2i}^{n} + u_{2i}^{n-1}}{2k} \\ & + 9\frac{3u_{2i-2}^{n+1} - 4u_{2i-2}^{n} + u_{2i-2}^{n-1}}{2k} - \frac{3u_{2i-4}^{n+1} - 4u_{2i-4}^{n} + u_{2i-4}^{n-1}}{2k}\Bigg) \\ & + \frac{-u_{2i+4}^{n+1} + 9u_{2i+2}^{n+1} + 10u_{2i}^{n+1} - 10u_{2i-2}^{n+1} - 9u_{2i-4}^{n+1} + u_{2i-6}^{n+1}}{64h} \end{aligned} \tag{28}$$

After applying Richardson extrapolation via Eq. (4), the fourth-order accurate in space stencil is

$$\begin{aligned} & \frac{1}{48}\Bigg(\frac{3u_{2i+2}^{n+1} - 4u_{2i+2}^{n} + u_{2i+2}^{n-1}}{2k} - 9\frac{3u_{2i}^{n+1} - 4u_{2i}^{n} + u_{2i}^{n-1}}{2k} \\ & \quad + 64\frac{3u_{2i-1}^{n+1} - 4u_{2i-1}^{n} + u_{2i-1}^{n-1}}{2k} - 9\frac{3u_{2i-2}^{n+1} - 4u_{2i-2}^{n} + u_{2i-2}^{n-1}}{2k} + \frac{3u_{2i-4}^{n+1} - 4u_{2i-4}^{n} + u_{2i-4}^{n-1}}{2k}\Bigg) \\ & \quad + \frac{u_{2i+4}^{n+1} - 9u_{2i+2}^{n+1} + 118u_{2i}^{n+1} - 118u_{2i-2}^{n+1} + 9u_{2i-4}^{n+1} - u_{2i-6}^{n+1}}{192h} \\ & = u_t + u_x - \frac{4u_{ttt}}{3}k^2 + \left(\frac{11u_{5x}}{120} + \frac{u_{txxxx}}{8}\right)h^4 + O(k^3 + h^6 + k^2h^4) \end{aligned} \tag{29}$$

Thus, this stencil is second order accurate in time and fourth order accurate in space. Unlike the other stencil, it contains mixed partial derivatives, and the coefficient of u_{5x} is almost three times larger for this stencil than for the other one. The modified equation for this finite difference stencil is

$$\begin{aligned} & u_t + u_x + \frac{k^2}{3}u_{xxx} + \frac{k^3}{4}u_{4x} + \left(-\frac{h^4}{30} + \frac{9k^4}{20}\right)u_{5x} + \frac{5k^5}{8}u_{6x} + \left(\frac{5h^6}{63} - \frac{h^4k^2}{30} + \frac{57k^6}{56}\right)u_{7x} \\ & \quad + \left(-\frac{h^4k^3}{30} + \frac{105k^7}{64}\right)u_{8x} + \left(\frac{209h^8}{4320} + \frac{5h^6k^2}{63} - \frac{3h^4k^4}{40} + \frac{1585k^8}{576}\right)u_{9x} \cdots = 0 \end{aligned} \tag{30}$$

The modified equation is the actual partial differential equation being solved by the numerical approach. The modified equation for the nodes that are in the refined mesh but not in the coarse mesh differs only slightly from the modified equation for the common nodes, differing slightly in seventh and higher odd order spatial derivatives. The odd order spatial derivatives affect the dispersion, which is consistent with the numerical observations, where the Richardson extrapolation result is slightly ahead of the traditional fourth-order accurate simulation.

D. St. Venant Equations

The St. Venant equations are a one-dimensional simplification of the incompressible Navier-Stokes applied to water flow in open channels [24]. They consist of an equation for the conservation of mass and an equation for the conservation of momentum. The depth of flow h represents the mass, and the discharge $p = hu$ represents the momentum. Additionally, the bed slope S_o transfers potential energy into momentum, and the friction component S_f removes momentum from the system via contact with the channel walls and channel bottom. The governing equations are expressed as

$$\frac{\partial h}{\partial t} + \frac{\partial p}{\partial x} = 0$$
$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left(\frac{p^2}{h} + \frac{1}{2}gh^2\right) = gh(S_o - S_f) \tag{31}$$

The friction models used herein is $S_f = \frac{n^2 p^2}{h^2 R_w^{4/3}}$, where n was Manning's friction coefficient, the wetted perimeter was $R_w = \frac{W*h}{W+2h}$ where W is the width of the channel. For large widths, the wetted perimeter is approximated as h, in order to simplify the code verification process [25, 26]. A standard finite volume method is used to discretize the governing equations, using the flux differencing method with Roe-averaged variables, similar to the method used by Whitfield [27] for the two-dimensional shallow water equations.

As exact solutions to these equations are difficult to derive, a source term is added so as to drive the solution to a predetermined solution [25, 26]. In particular, these solutions are chosen as $h(x) = e^{-x/1500}$ and $p(x) = e^{x/1500}$, where 1500 is the length of the channel. Thus, code verification can be performed by comparing the converged numerical solution with the exact solution. Because the solution consists of two components, the L1-norm for each component was reported. For these results, $g = 0.5$, the slope $S_0 = 0.004$ and Manning's friction coefficient $n = 0.012$.

The results for the original St. Venant code are shown in Table VIII. These results show that the original implementation including the source terms is second-order accurate, as expected. In Table IX, the errors in the converged solution for the application of Richardson extrapolation to the residual are presented. Since an upwinded scheme is employed, only one order of accuracy

TABLE VIII. Observed errors ($\times 10^7$) for solution of St. Venant equations on uniform mesh.

Intervals N	Error in h	Order of accuracy	Error in p	Order of accuracy
50	22.505386673	NA	320.23331353	NA
100	4.750499641	2.24412	84.79340683	1.91710
200	1.072750668	2.14676	21.81436158	1.95867
400	0.253468537	2.08143	5.53087333	1.97970
800	0.061505831	2.04301	1.39292705	1.99989
1600	0.015142389	2.02213	0.34947572	1.98436

TABLE IX. Observed errors ($\times 10^7$) for Richardson extrapolation applied to the residual.

Mesh pair N	Error in h	Order of accuracy	Error in p	Order of accuracy
50/100	1.7314635585	NA	9.7760937413	NA
100/200	0.2147855623	3.01102	1.2199231067	3.00247
200/400	0.0267858355	3.00335	0.1523610558	3.00122
400/800	0.0033458781	3.00101	0.0190370934	3.00061
800/1600	0.0004181255	3.00038	0.0023791339	3.00040

improvement is expected. Since the observed order of accuracy approaches 3, the application of Richardson extrapolation to the residual for this hyperbolic system of equations has increased the order of accuracy appropriately.

VI. CONCLUSIONS

Richardson extrapolation has been extended to general partial differential equations by its application to the discretized result (i.e., the residual) developed within existing numerical partial differential equations solvers. In previous work, Richardson extrapolation was performed only on the converged solution on a sequence of refined meshes. When applied only to the converged solution for parabolic equations or for simulations that do not exhibit numerical dispersion, Richardson extrapolation achieves the expected improvement in the order of accuracy; however, when it is applied only to the converged solutions that have dispersion, as often occurs for hyperbolic equations, Richardson extrapolation has a reputation of generating poor results. The reason for this failure has often been wrongly attributed to the mesh "not being in the asymptotic region" when the failure may be due to dispersion.

By applying Richardson extrapolation directly to the residual on a sequence of meshes, so that the residual is higher order, the solution maintains the similar higher order accuracy. This method has been demonstrated on the parabolic, linear heat equation along with the application of Richardson extrapolation to the solution. Both approaches achieve the expected increase in the order of accuracy. The method has also been applied to the linear wave equation and to the nonlinear St. Venant equations, achieving the improved accuracy for the application of Richardson extrapolation to the residual.

Because this methodology uses the components within an existing numerical partial differential equation solver, this approach can be implemented within a wide range of solvers, based on finite difference, finite volume or finite element discretizations. Future work will focus on applying this approach to two-dimensional and then to three-dimensional unstructured triangular and tetrahedral codes for solving nonlinear hyperbolic systems of partial differential equations, such as the Navier-Stokes equations.

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