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Improvement of FDM by extrapolation on multiple grids

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Abstract

The extrapolation of results obtained on a series of 3 succeeding grids with halved mesh size is tested as a variant of the multi-grid approach for solving the Laplace and Poisson equations in 2D. Based on corresponding experience with BEM for electric [1] and magnetic [2] field problems a pure power law is applied instead of the famous Richardson extrapolation [3]. On those grid points, which are common to all 3 grids, the potential values are extrapolated to an arbitrary fine discretization. On the points of the finest grid in between those of the coarser ones the potentials then are obtained by only few iterations to perform the interpolation. Both, the common 5-point discretization and the famous 9-point discretization by E. Kasper [5] are investigated and compared with respect to the possible win of accuracy by extrapolation. As an interesting result of this kind of extrapolation, the accumulated local discretization errors of the 5-point discretization are partially cured and the high accuracy by the 9-point formula of Kasper makes extrapolation inefficient. Like for classical MG (multi grid) [6] the acceleration of potential calculations on grids of large size is substantial. © 2008 Elsevier B.V. All rights reserved.

1. Introduction

In order to perform numerical calculations for continuous operators like integration, differentiation, or solving ordinary or partial differential equations it is first necessary to discretize the function and use difference equations for differential operators. This is always associated with a loss of accuracy, which is reduced by smaller and smaller steps of discretization. Even fastest computers with largest memory are easily driven to their limits by using finer and finer steps. As a way out, extrapolation of calculations obtained with different size of discretization to an arbitrary fine discretization will recover some of the accuracy lost. This has been first introduced by Richardson [3] in 1910 and only in the last decade been used for the improvement of solutions with the boundary element method (BEM) [1,2,7,8,9]. In this paper the extrapolation method will be applied to the classical finite difference method (FDM) of solving partial differential equations. The spherical condensor is used as a test bed, because the solution is known analytically. The axisymmetric 2D discretization produces first derivatives, which reduce the accuracy [4] allowing to judge the merits of the procedure. Other work on the study of the accuracy of numerically calculated fields [1, 10] or ray tracing [11, 12] also make use of this test bed.

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2. Numerical procedures

   In order to perform extrapolation, fully converged solutions are needed. Grid sizes of 32×32, 64×64, and 128×128 are chosen. Here fully converged solutions are obtained by iterating on a potential map first filled with the exact solution. For problems with an unknown exact solution, a full converged solution may be obtained by iterating sufficiently often or by using the multigrid algorithm. In those grid points, which belong to all 3 grids (these are the points on the 32×32 grid lines) extrapolation is then performed. The 3 hierarchical grids automatically obey the relation of a constant refinement factor, which allows for a very simple extrapolation formula, using the power law [1,2]. Finally the extrapolated potentials on the coarse 32×32 grid are interpolated to the finest 128×128 grid and compare with the exact solution. The interpolation is done by 16 iterations on the 128×128 grid , while the potentials on the 32×32 grid lines are kept constant.

3. Extrapolation of potentials obtained by a 5-point discretization

   The 5-point discretization for axisymmetric problems is well explained in textbooks and in ref. [4]. It uses the 2 neighbouring grid points vertically and the 2 ones horizontally to provide a difference equation as an approximation for the Laplace solution for the central grid potential. This is the most common procedure in FDM programs and used e.g. in EGUN and IGUN [13]. The result is shown in Fig. 1. From 32×32 to 64×64 to 128×128 grid the errors are falling by a factor 4, as predicted by theory. Extrapolation then improves the maximum error by a factor of 12.7, which otherwise could only be obtained on a 407×407 grid, needing about 2063 times more computation time.
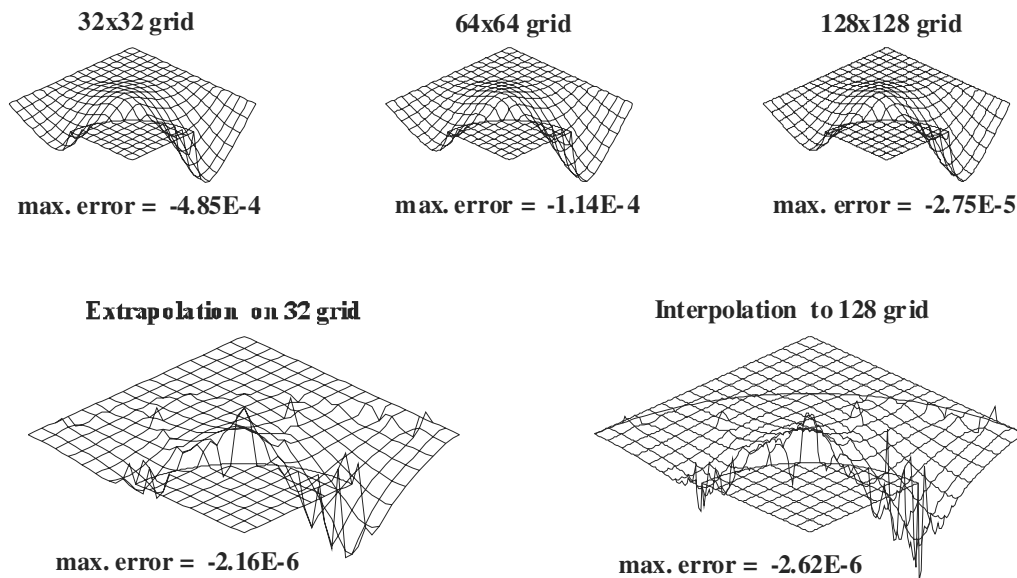
32x32 grid	64x64 grid	128x128 grid
max. error = -4.85E-4	max. error = -1.14E-4	max. error = -2.75E-5

Extrapolation on 32 grid	Interpolation to 128 grid
max. error = -2.16E-6	max. error = -2.62E-6

Fig. 1: Potential errors for half of a spherical condensor by 5-point discretization.

Upper row : potential errors on 32×32, 64×64, and 128×128 grid.

Lower row – left: potential errors by extrapolating the solutions on 32×32 grid.
                   – b right: interpolation of extrapolated potentials on 128×128 grid.

4. Extrapolation of potentials obtained by a 9-point discretization

The 9-point discretization for axisymmetric problems has been developed by E. Kasper [5]. It uses all surrounding 8 neighbouring grid points to provide a difference equation as an approximation for the Laplace solution for the central grid potential. This procedure is difficult to program for grid points, where one or more of them are outside of the computational domain. This is one reason that it has not become more popular. The result for the potential errors and the extrapolation as well as the interpolation is shown in Fig. 2. The errors on the 128×128 grid are below 1.2×10^{-10}, which firstly is amazing, but secondly shows a very odd behaviour on extrapolation (Fig. 2 lower row, left). Obviously the efficient improvement of accuracy on the hierarchical grids by Kasper's formula do not give a smooth enough local variation of the potentials that extrapolation can be beneficial. Also remarkable is the fact that the 9-point discretization is showing an error reduction by a factor of 20 from the coarser grid to the next finer one, due to the use of higher order corrections.
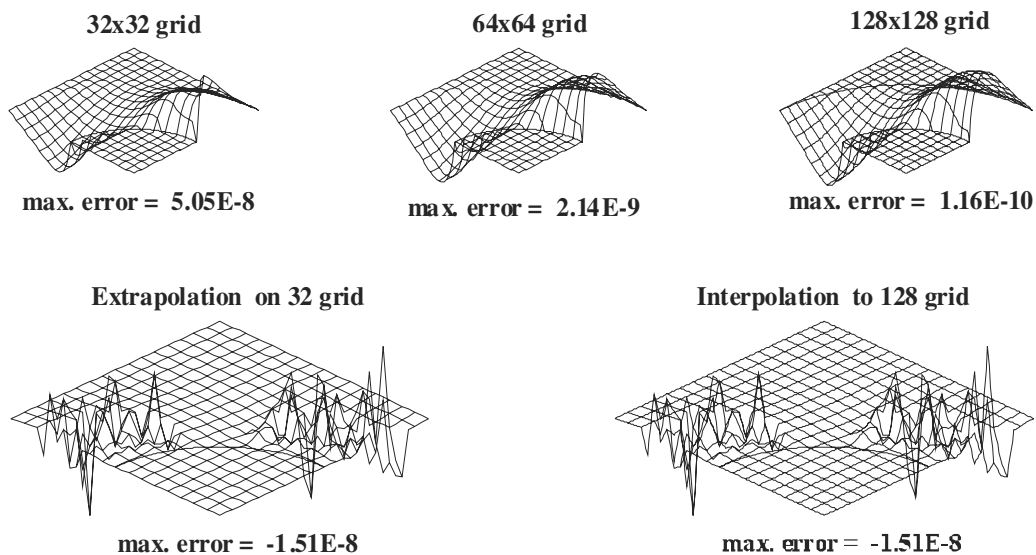
32x32 grid   **64x64 grid**   **128x128 grid**

max. error = 5.05E-8   max. error = 2.14E-9   max. error = 1.16E-10

Extrapolation on 32 grid   **Interpolation to 128 grid**

max. error = -1.51E-8   max. error = -1.51E-8

Fig. 2: potential errors for half of a spherical condensor by 9-point discretization.
upper row : potential errors on 32×32, 64×64, and 128×128 grid.
Lower row – left: potential errors by extrapolating the solutions on 32×32 grid
- right: interpolation of extrapolated potentials on 128×128 grid

5. Conclusions

Extrapolation to arbitrary fine discretization is useful for 2D FDM methods only for 5-point discretization. Here the win in computing time at comparable accuracy for a spherical condensor amounts to more than 2000. For 9-point discretization extrapolation does not make sense.

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