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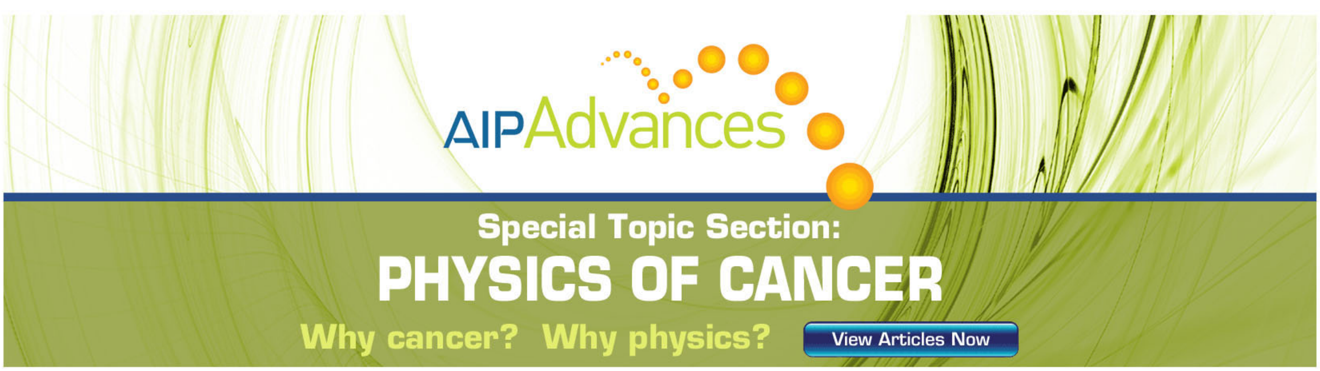

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Simple Phenomenological Theory of Turbulent Shear Flows

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A rate equation is proposed to govern the variation of the effective turbulent viscosity. The effects of generation, convection, diffusion, and decay are each represented by appropriate terms leaving only two empirical constants to be determined by experiment. This rate equation together with the equations of motion form a closed system applicable to quasiparallel turbulent shear flows. For an incompressible turbulent boundary layer with zero pressure gradient, solutions were obtained by assuming local similarity and a linear growth of the boundary-layer thickness. Another problem, the turbulent-nonturbulent interface at the outer edge of the boundary layer was treated by using the further assumption that the large scale motion of the interface has no significant contribution to the Reynolds stress. It can be shown that for a nearly homogeneous domain, Prandtl's mixing length theory is a limiting case of the present theory.

I. INTRODUCTION

The phenomenological theories associated with such names as Taylor,[1] Prandtl,[2] von Kármán,[3] as well as others, have recently received renewed attention. There always were, of course, empirical formulas used to calculate a particular kind of turbulent shear flow, such as a shape parameter equation or an eddy viscosity formula for certain kinds of turbulent boundary layers. The phenomenological theories considered here are different from the mere empirical formulas. It is hoped that a hypothesis based on the physics of turbulent motion would yield a more general theory that can be applied to different kinds of turbulent shear flow irrespective of whether it is a turbulent flow along a wall, a free turbulent flow, or a mixed case. Various earlier attempts have met with different degrees of success. Just to mention a few, there is the simple two-layer concept of Clauser.[4] More recently, by generalizing Miles' results for wave generation by wind, Phillips[5] proposed a new kind of eddy viscosity, the ratio of the gradient of the Reynolds stress, and the second derivative of the mean velocity. His eddy viscosity is proportional to the product of the energy in the cross-flow component of the velocity fluctuations and the convected integral time scale. This idea is very interesting since it postulates a mechanism for the turbulent shear stress. Emphasis is more on the physics of the turbulence than on the computation of the mean flow. Nevertheless, further studies must be made to calculate the convected integral time scales as well as the mean square of the cross-flow velocity component; a direct application is impossible without relying on actual measurements of these quantities. On the other hand, in an attempt to predict the turbulent boundary-layer development under arbitrary pressure gradient, Bradshaw *et al.*[6] transformed the turbulent energy equation into an equation for the turbulent shear stress but at the expense of introducing three further empirical functions to express the turbulent intensity, its dissipation, and diffusion in terms of the turbulent shear stress. The results depend, of course, on how

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[1] G. I. Taylor, Phil. Trans. Roy. Soc. London **A215**, 1 (1915).
[2] L. Prandtl, Z. Angew. Math. Mech. **5**, 136 (1925).
[3] Th. von Kármán, in *Proceedings of the Third International Congress on Applied Mechanics* (A. B. Sveriges Litograsiska Tryckerier, Stockholm, 1931), Pt. 1, p. 85.
[4] F. H. Clauser, in *Advances in Applied Mechanics*, H. L. Dryden, Th. von Kármán, and G. Kuerti, Eds. (Academic Press Inc., New York, 1956), Vol. 4, p. 1.
[5] O. M. Phillips, J. Fluid Mech. **27**, 131 (1967).
[6] P. Bradshaw, D. H. Ferris, and N. P. Atwell, J. Fluid Mech. **28**, 593 (1967).

successfully these functions were chosen. Another new approach was suggested by Harlow and Nakayama.[7] Based on the turbulent energy equation, a transport equation for eddy viscosity is derived by introducing additional functions, which in turn are assumed to be governed by other transport equations involving correlation functions simplified by some flux approximations. Finally, it results in a system of a large number of equations containing a large number of empirical constants and assumed functional forms which must be known or approximated in order to calculate any particular turbulent flow.

The phenomenological theory proposed here is a very modest one. It is formulated in such a way as to take into account the relevant mechanism of the turbulent motion but only with the minimum complications. The theory overcomes the objection of "localness" of older theories by bringing in the past history of the turbulent flow and also the lateral diffusion of turbulence. However, it is still quite simple since for the computation of turbulent shear flows, only one more equation is added to the usual equations of motion. This new rate equation governs the turbulent viscosity, and it has only two universal constants to be determined from the experiment. If, in addition, the "law of wall" is regarded as a universal law, then the number of universal constants for wall turbulence is reduced to one. Naturally, elaborations and refinements are always possible at the expense of simplicity.

II. THE BASIC EQUATIONS

In order to form a closed system of governing equations for the mean motion, it is necessary to determine the distribution of the Reynolds stress. One can always derive an equation for the Reynolds stress by averaging the appropriate products of the equation of motion. The various terms in the equation for the Reynolds stress then can be identified as convection, diffusion, production, and dissipation, respectively (see, e.g., Rotta,[8] Sec. 8). Any direct use of such an equation is hampered by the large number of unknown correlation functions involved and some hypothesis on their further relations is needed to accomplish "closure."

Here, our approach is to make a simple "educated guess" concerning the various terms based on the corresponding physical mechanism such as diffusion, production, and dissipation. The dependent variable chosen is the turbulent viscosity which, of course, relates directly to the Reynolds stress. This was chosen in preference to the turbulent energy simply in order to avoid the need for one further assumption relating the turbulent intensity to the Reynolds stress. The Reynolds stress can be expressed as the product of the eddy viscosity and of the mean velocity gradient. This relation serves here only as a definition for the eddy viscosity ϵ

$$\epsilon = \frac{\overline{u'v'}}{du/dy}. \qquad (1)$$

The total viscosity $\epsilon + \nu = n$ is assumed to be governed by a rate equation and thus, it is dependent on the past history through convection and on nearby values through diffusion.

The eddy viscosity may be regarded as the ability of a turbulent flow to transport momentum. This ability must be directly related to the general "level of activities," and therefore, to the total turbulent energy. The structure of the turbulent energy equation is, in fact, quite similar to the diffusion equation or more precisely to the rate equation of a chemical species. Here, one should make a small digression.

In general, any transportable scalar quantity F subject to the conservation laws is transported according to the equation

$$\frac{DF}{Dt} \equiv \frac{\partial F}{\partial t} + (\mathbf{u}\cdot\nabla)F = \nabla\cdot\varphi_F + \text{production} - \text{decay}, \qquad (2)$$

where φ_F is the flux of the quantity F due to diffusion.

The usual form for the flux is

$$\varphi_F = D_F \nabla F, \qquad (3)$$

where D_F is the coefficient of diffusion for the particular quantity F. In the present case the transportable quantity is the total turbulent viscosity $n = \epsilon + \nu$, where ν is the molecular viscosity. The flux of n in the diffusion term will then become

$$\varphi_n = D_n \nabla n, \qquad (4)$$

where D_n is now the diffusivity of the quantity n. Since the turbulent motion diffuses by itself, it is reasonable to assume that the diffusivity $D_n = n$. In any case, the turbulent Prandtl number and Schmidt number are both nearly unity so $D_n/n \approx 1$.

The general rate equation for the total viscosity n is proposed as

$$\frac{\partial n}{\partial t} + (\mathbf{u}\cdot\nabla)n = \nabla\cdot(n\nabla n) + G - D, \qquad (5)$$

[7] F. H. Harlow and P. I. Nakayama, Phys. Fluids **10**, 2323 (1967).
[8] J. C. Rotta, in *Progress in Aeronautical Sciences* (Pergamon Press, Inc., New York, 1962), Vol. II.

where **u** is the velocity vector, G represents the generation term, and D, the decay term. This formulation was first presented by Kovasznay and Nee.[9]

The actual forms of G and D are "guessed" using plausibility arguments; nevertheless, the success of any such theory depends on the particular form chosen. If they are chosen in a form that is too complicated, there will be little hope of calculating any concrete flow problems; on the other hand, if they are oversimplified, the results may be easy to calculate but not close enough to reality to be of any real value.

First, let us consider the generation term. From a rough analogy with the production of turbulent energy, it is clear that it must increase monotonically with the magnitude of the mean vorticity $|\partial U/\partial y|$ and also with the increasing level of turbulent agitation, therefore, with n. There are, of course, many functional forms possible.

By choosing the simplest form compatible with the above requirement, the generation term is assumed.

$$G = A(n - \nu)\left|\frac{\partial U}{\partial y}\right|, \tag{6}$$

where A is a universal constant of order unity.

For the decay term some clues may be obtained from the analogy of n with turbulent energy. The rate of decay of the energy of high-intensity homogeneous turbulence (such as grid turbulence) is, to a very rough approximation, inversely proportional to the square of the energy

$$\frac{d\overline{u^2}}{dt} = -\beta(\overline{u^2})^2. \tag{7}$$

This leads to a decay law

$$\overline{u^2} \cong 1/\beta t. \tag{8}$$

The rate equation (5) in the absence of all other terms (generation, convection, and diffusion) will reduce to

$$\frac{\partial n}{\partial t} = -D. \tag{9}$$

In an analogy with Eq. (7), we choose $D \cong \beta n^2$. In order to have a more appropriate behavior when $n \to \nu$ (or $\epsilon \to 0$) and for dimensional consideration, the actual form chosen will be

$$D = \frac{B}{L^2}\, n(n - \nu), \tag{10}$$

where B is another universal constant.

Here, the characteristic length L is introduced in order to make B a nondimensional "universal" constant. In general, L will be a function of y. In the region of the outer edge of a turbulent flow one may choose $L = \delta$, the boundary-layer thickness, but closer to the wall the choice $L = y$ appears to be reasonable in view of the fact that the characteristic length for decay must be of the same order of magnitude as the scale of the largest eddy responsible for the viscous dissipation of turbulence. The dependence of the decay term on the distance from the wall appears to be quite necessary to account for the high rate of dissipation in the proximity of solid boundaries, where the maximum dimension of the dissipating eddies in the direction perpendicular to the flow must be the same as the distance from the wall.

Now we are ready to consider an incompressible steady turbulent flow with no body forces present. The turbulent flow is assumed to have reached its "asymptotic" or "fully developed" state (meaning that it is far from transition). For quasiparallel flows, namely, in flows where the y directional momentum is so small that its effect can be neglected, the boundary-layer approximation can be assumed to be valid.

For a two-dimensional steady quasiparallel flow, the governing equations for the mean motion can be written as

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\partial}{\partial y}\left(n\frac{\partial U}{\partial y}\right), \tag{11}$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \tag{12}$$

$$U\frac{\partial n}{\partial x} + V\frac{\partial n}{\partial y} = \frac{\partial}{\partial y}\left(n\frac{\partial n}{\partial y}\right) + A(n - \nu)\left|\frac{\partial U}{\partial y}\right| - \frac{B}{L^2}\, n(n - \nu). \tag{13}$$

The variable n is the sum of ν and ϵ. Townsend[10] has pointed out that in the turbulent dispersion of heat, the molecular diffusion and the turbulent diffusion cannot be simply added for the reason that the turbulent strain field also increases the molecular transfer rates by increasing local instantaneous gradients. But by allowing ϵ to be a function of the velocity gradient history since it is governed by the rate equation, the interaction between ϵ and ν is considered to be a part of turbulent mechanism governed by the rate equation, therefore, it can be absorbed in n.

[9] L. S. G. Kovasznay and V. W. Nee, Bull Am. Phys. Soc. **10**, 683 (1965).

[10] A. A. Townsend, Proc. Roy. Soc. (London) **A209**, 418 (1951).

Inside a fully turbulent region, the Reynolds stress dominates so $n \gg \nu$. In addition, in that region n varies relatively slowly so the convection and diffusion terms may each become so small that their contribution can be considered as negligible compared with generation and decay terms. In this limiting case the generation term will be left to balance the decay term, and the rate equation is reduced to an algebraic equation

$$A(n - \nu) \left|\frac{dU}{dy}\right| = \frac{B}{L^2} n(n - \nu), \tag{14}$$

immediately giving the formula for the total viscosity

$$n = \frac{A}{B} L^2 \left|\frac{dU}{dy}\right|. \tag{15}$$

That is the well-known Prandtl's formula with the mixing length $l = L(A/B)^{1/2}$.

For wall turbulence, $l = Ky$ will be assumed with $K = 0.4$, (see Schlichting[11]). Accordingly, A and B must satisfy the relation

$$A = K^2 B \tag{16}$$

and there will be only one disposable constant (either A or B) to be determined from comparison with experiments.

III. TURBULENT BOUNDARY LAYER WITH ZERO PRESSURE GRADIENT

In order to test the validity of the proposed method, some simple cases should be calculated. Direct forward integration of Eqs. (11), (12), and (13) for a two-dimensional turbulent boundary layer with zero pressure gradient has not been attempted; first, because it would have required a considerable amount of numerical calculation. As a more modest attempt the system of partial differential equations was first reduced to a system of ordinary differential equations by using the assumption of a local similarity of the turbulent boundary-layer flow.

In contrast with the laminar case, the turbulent layer even with zero pressure gradient does not have complete similarity for all Reynolds numbers. For a large Reynolds number flow, the turbulent energy transfer requires more steps in the energy cascade to reach the smallest eddy sizes where most of the viscous dissipation takes place, than for lower Reynolds numbers; consequently, two turbulent flows with different Reynolds numbers cannot be completely similar. Or, in another way, this is shown by the differences in the thickness of the viscous sublayers relative to the total boundary-layer thickness.

Nevertheless, local similarity in the turbulent boundary layer can still be assumed with a good degree of approximation, if the thickness of the sublayer is not included. The experiments conducted by Clauser[4] and also by Klebanoff and Diehl[12] have shown that a zero pressure gradient turbulent boundary layer exhibits an "x-wise stability" in the sense that when the flow is disturbed, it tends to return to its "natural state." Consequently, the mean velocity distribution at any cross section ($x = \text{const}$) along the plate depends primarily on the local parameters and only mildly on past history.

The zero pressure gradient turbulent boundary layer grows monotonically. The growth rate has been found experimentally to be approximately proportional to the $\frac{4}{5}$th power of x within a wide range of Reynolds numbers. When using the assumption of local similarity, only the local growth rate will enter into the analysis so the local boundary-layer thickness can be expressed as a linear function of x, and the growth rate coefficient is determined by the local parameters.

Here, it is convenient to make a further assumption. The inertia term in the momentum equation is approximated by the introduction of an effective crossing velocity or entrainment velocity V_e

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -V_e \frac{\partial U}{\partial y}. \tag{17}$$

For the case of self-similar boundary layers, the introduction of the effective cross velocity V_e offers a real advantage by eliminating the derivatives with respect to x. Introducing the stream function ψ (the usual definition)

$$\psi(x, y) = U_\infty[y - \alpha F(\eta)]. \tag{18}$$

The nondimensional distance from the wall will be defined as

$$\eta = y/\delta(x), \tag{19}$$

where $\delta(x)$ is the boundary-layer thickness; finally, the growth rate constant is defined as

$$\alpha = \frac{d\delta(x)}{dx}. \tag{20}$$

The mean velocity components then become

$$U = U_\infty[1 - F'(\eta)], \tag{21}$$

$$V = U_\infty \alpha[F(\eta) - \eta F'(\eta)], \tag{22}$$

[11] H. Schlichting, *Boundary Layer Theory* (McGraw-Hill Book Company, New York, 1960), p. 556.

[12] P. S. Klebanoff and Z. W. Diehl, NACA Report 1110 (1952).

and the inertia terms become

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \frac{U_\infty^2 \alpha F''}{\delta}(\eta - F)$$

$$= -\alpha U_\infty(\eta - F)\frac{\partial U}{\partial y}. \qquad (23)$$

The "effective crossing velocity" can now be defined as

$$V_e(\eta) = \alpha U_\infty(\eta - F). \qquad (24)$$

In a similar manner, the convection terms in the rate equation (13) for the turbulent viscosity can also be expressed with the same $V_e(\eta)$.

In order to estimate V_e let us consider a simple "power law" for the mean velocity profile

$$\frac{U}{U_\infty} = \begin{cases} \eta^{1/m} & \text{for} \quad \eta < 1 \\ 1 & \text{for} \quad \eta \geq 1. \end{cases} \qquad (25)$$

The stream function then becomes

$$F(\eta) = \begin{cases} \eta - \dfrac{m}{m+1}\eta^{(m+1)/m} & \text{for} \quad \eta < 1 \\ \dfrac{1}{m+1} & \text{for} \quad \eta \geq 1. \end{cases} \qquad (26)$$

The effective crossing velocity

$$V_e(\eta) = \begin{cases} \alpha\, U_\infty \dfrac{m}{1+m}\eta^{(m+1)/m} & \text{for } \eta < 1 \\ \alpha\, U_\infty\left(\dfrac{m}{1+m} + \eta - 1\right) & \text{for } \eta \geq 1. \end{cases} \qquad (27)$$

As a numerical example, let us assume $m = 8$. The variation of $V_e(\eta)$ across the layer is then quite close to the linear one (Fig. 1),

$$V_e(\eta) \approx \alpha U_\infty \eta \qquad (28)$$

so in the subsequent analysis we assume that the V_e varies linearly with the distance from the wall.

The momentum equation (11) and the rate equation (13) now can be written as

$$-V_e\frac{dU}{dy} = \frac{d}{dy}\left(n\frac{dU}{dy}\right), \qquad (29)$$

$$-V_e\frac{dn}{dy} = \frac{d}{dy}\left(n\frac{dn}{dy}\right) + A(n - \nu)\left|\frac{dU}{dy}\right| - \frac{B}{y^2}n(n - \nu). \qquad (30)$$

We now introduce the nondimensional variables

$$N_1 = n/\nu,\ U_1 = U/u_*,$$
$$V_1 = V_e/u_*,\ Y_1 = u_* y/\nu.$$

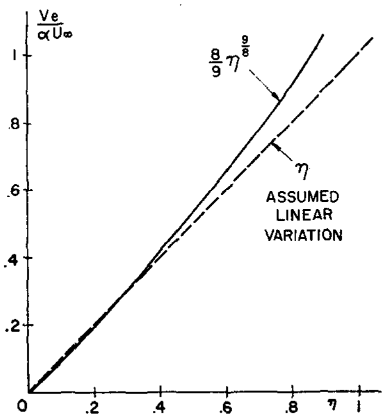

FIG. 1. The assumed linear variation of the effective crossing velocity V_e is compared with the exact variation calculated by assuming a "⅛th power law" for the mean velocity distribution.

Here, only the "wall" parameters were used for nondimensionalization. The "outer" parameters (U_∞ and δ) will enter explicitly only after integration. The nondimensional effective crossing velocity V_1 can be written as

$$V_1 = \frac{V_e}{u_*} = k\frac{u_* y}{\nu} = kY_1, \qquad (31)$$

where k, the new "growth parameter," will be determined by the outer flow parameters. The system of nondimensional equations now becomes

$$-kY_1\frac{dU_1}{dY_1} = \frac{d}{dY_1}\left(N_1\frac{dU}{dY_1}\right), \qquad (32)$$

$$-kY_1\frac{dN_1}{dY_1} = \frac{d}{dY_1}\left(N_1\frac{dN_1}{dY_1}\right) + A(N_1 - 1)\frac{dU_1}{dY_1} - B\frac{N_1(N_1 - 1)}{Y_1^2}. \qquad (33)$$

These equations are nonlinear and in order to solve them by an analog computer, eight multipliers are required to perform the necessary number of multiplications and divisions. Unfortunately, at the time the calculations were performed that number of multiplications was not available. Consequently, the system of equations was transformed in order to make them tractable by using fewer multiplers. Let us define the transformation (suggested by F. H. Clauser)

$$dY_1 = N_1\, dt \qquad (34)$$

and denote the total shear stress as

$$\tau = N_1\frac{dU_1}{dY_1}. \qquad (35)$$

Equations (32) and (33) become

$$-kY_1\tau = \tau', \tag{36}$$

$$-kY_1N_1' = N_1'' + A(N_1 - 1)\tau - BN_1^2(N_1 - 1)/Y_1^2. \tag{37}$$

The prime indicates differentiation with respect to the new variable t. Equations (36) and (37) require only five multipliers instead of the original eight for Eqs. (32) and (33).

The actual integration cannot start from the wall as the approximation is poor in, and near, the sublayer region. Near the wall, the "law of the wall" is assumed to be valid, so Eqs. (36) and (37) are integrated only from some point near but outside the sublayer within the logarithmic region.

The "law of the wall" is usually stated as

$$\frac{U}{u_*} = \frac{1}{c_1}\ln\frac{u_* y}{\nu} + c_2. \tag{38}$$

Consequently, the boundary conditions at $Y_1 = Y_0$ will be chosen as

$$U_0 = (1/c_1)\ln Y_0 + c_2,$$
$$\tau_0 = \tau_w/u_*^2 = 1,$$
$$N_0 = c_1Y_0 - \lambda Y_0^2,$$
$$N_0' = (c_1Y_0 - \lambda Y_0^2)(c_1 - 2\lambda Y_0),$$
$$\frac{dN_1}{dY_1} = c_1 - 2\lambda Y_0.$$

Y_0 is the value of Y_1 where the integration starts. The constants C_1 and C_2 have been obtained from experiments and are usually assumed to be 0.4 and 5.2, respectively. Other quantities with the subscript 0 are values of the variables at the position $Y_1 = Y_0$. As Y_0 must be chosen quite near the wall, the total shear stress τ can be approximated by the value of wall shear τ_w. The value of C_1Y_0 is obtained by dividing the wall shear by the local value of the derivative of the mean velocity. This is essentially a two-point boundary-value problem but is handled as an initial-value problem with variable initial conditions. Therefore, the parameter λ has been introduced as a "dummy parameter" to slightly vary the initial conditions in order to meet the other boundary conditions at the free stream edge.

We must, of course, first assume numerical values for the universal constants A and B appearing in the rate equation.

A set of values of A and B subject to the constraint of Eq. (16) were determined by comparing the calculated results for $n_{\max}$ with the available experimental results.

Still another nondimensional parameter "k" representing the growth rate of the layer given by Eq. (31) must be found. From general dimensional reasoning, it is expected that the constant "k" will depend on the Reynolds number of the layer

$$R_\delta = \frac{U_\infty \delta}{\nu}.$$

In the actual analog computation procedure we have chosen values for A, B, k, Y_0 then integrated and the results gave the corresponding value of R_δ.

We also know that the solution for n must satisfy the outer boundary condition $n = \nu$ outside the interface. Since in analog computation we can only adjust the initial conditions, we will do so by changing the value of the parameter λ. From the solution for n that satisfies the outer boundary conditions, the position of the interface y_i gives the boundary-layer thickness

$$\left(\frac{u_* y}{\nu}\right)_{\text{interface}} = \frac{u_* y_i}{\nu}. \tag{39}$$

The average position of the interface is $\bar{y}_i = 0.78\ \delta$.[13]

From the integrated solution of the velocity, the value of U_∞/u_* is simply the value of the solution at the interface where $U = U_\infty$

$$\left(\frac{U}{u_*}\right)_{\text{interface}} = \frac{U_\infty}{u_*}. \tag{40}$$

The value of the corresponding Reynolds number R_δ can be obtained as

$$R_\delta = \frac{U_\infty\delta}{\nu} = \left(\frac{u_*\delta}{\nu}\right)\left(\frac{U_\infty}{u_*}\right). \tag{41}$$

By definition, the skin friction coefficient is

$$c_f = 2\left(\frac{u_*}{U_\infty}\right)^2. \tag{42}$$

Now, the results can be discussed as follows:

The distribution of the turbulent viscosity obtained here depends on two parameters

$$R_\delta = \frac{U_\infty\delta}{\nu} \quad\text{and}\quad \frac{u_*}{U_\infty} = \left(\frac{c_f}{2}\right)^{1/2}.$$

It is known from the experimental results of Klebanoff[13] and Townsend[14] that the normalized turbulent viscosity $n/u_*\delta$ is nearly constant, and does not

[13] P. S. Klebanoff, NACA Technical Note 3178 (1954).
[14] A. A. Townsend, Proc. Cambridge Phil. Soc. **47**, 375 (1951).

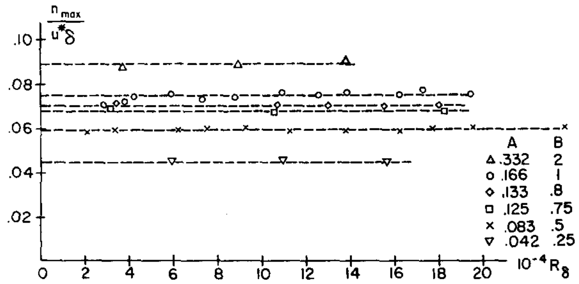

FIG. 2. Calculated maximum total turbulent viscosity as a function of the Reynolds number.

depend on R_δ (see, e.g., Hinze).[15]

From the present calculation, the maximum value of the turbulent viscosity $n_{max}/u_* \delta$ is plotted against R_δ in Fig. 2. This quantity is indeed approximately independent of the Reynolds number.

Computations were made with different sets of values for the two universal constants A and B. For each pair of values of A and B the value of $n_{max}/u_* \delta$ was calculated. For the set of values $A = 0.133$ and $B = 0.8$, $n_{max}/u_* \delta = 0.070$, the value closest to the ones obtained both by Klebanoff and Townsend.

The calculated distribution of the mean velocity, shear and turbulent viscosity distribution are all given as a function of the position without taking into account the instantaneous position of the interface due to intermittency. The experimental results all give the long time averaged velocity $\bar{U}(y)$ as a function of the distance from the wall. The present calculation can be compared directly with experiments only if the statistical distribution of the position of the superlayer is taken into account (see Corrsin and Kistler,[16] Klebanoff,[17] and recently, Fiedler and Head.[18] Thus, the average position $\bar{y}_i$ and the standard deviation σ of the superlayer are both well known from experiments. Furthermore, the probability density was found to be nearly Gaussian by all the experimenters.

Let us assume that the instantaneous interface position $y_i(t)$ has a probability density distribution $p(y_i)$; furthermore, the entire layer expands and contracts proportionally to y_i so any value of n or τ found at a position $\bar{y}$ will fluctuate slowly according to this "breathing" of the layer,

$$n(\bar{y}, t) = n[\bar{y} + y'(t)], \qquad (43)$$

[15] J. O. Hinze, *Turbulence* (McGraw-Hill Book Company, New York, 1959), p. 493, Fig. 7.17.

[16] S. Corrsin and A. L. Kistler, NACA Technical Report 1244 (1955).

[17] P. S. Klebanoff, NACA Technical Report 1247 (1955).

[18] H. Fiedler and H. R. Head, J. Fluid Mech. 25, 719 (1966).

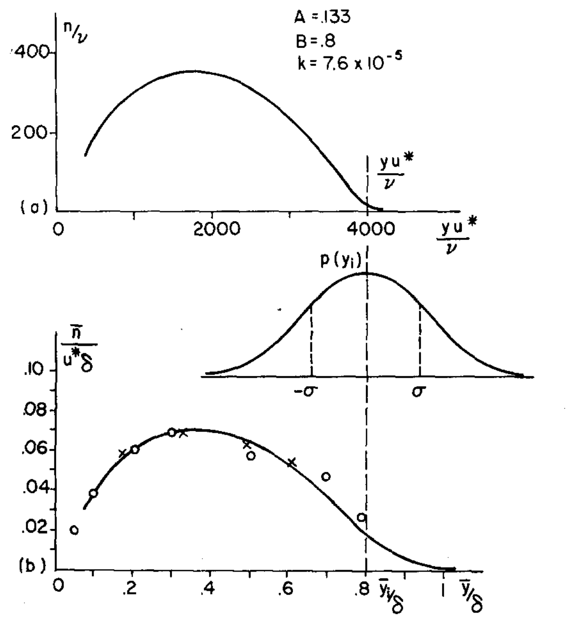

FIG. 3. Distribution of the total turbulent viscosity across the boundary layer. Above: Original calculation for fixed position of interface. Below: Corrected for the fluctuation of interface position (Gaussian distribution) and compared with experiments of Klebanoff[13] (open circles) and Townsend[14] (×).

where

$$y'(t) = \begin{cases} y_i(t) - \bar{y}_i & \text{for} \quad \bar{y} \geq \bar{y}_i \\ \dfrac{\bar{y}}{\bar{y}_i}[y_i(t) - \bar{y}_i] & \text{for} \quad \bar{y} < \bar{y}_i. \end{cases} \qquad (44)$$

When the intermittency is "averaged out," the observed wind tunnel data will give average values of

$$\bar{n}(\bar{y}) = \int_0^\infty n(\bar{y} + y')p'(y')\, dy'. \qquad (45)$$

The probability density distribution $p'(y')$ is similar to $p(y_i)$ except its standard deviation $\sigma' = (\bar{y}/\bar{y}_i)\,\sigma$ is assumed to be linearly decreasing toward the wall [Eq. (44)].

The calculated solution of the turbulent viscosity $n(y)$ obtained from present theory and the "smeared" $\bar{n}(\bar{y})$ have been compared with both Klebanoff's[13] and Townsend's[14] experimental results in Fig. 3. The curves begin in the wall region just outside the sublayer, then reach a maximum value near $y/\delta = 0.3$, and then drop rapidly near the interface. Outside the interface $n \to \nu$ as expected. The "smeared" values, of course, agree much better with experiments.

The distribution of the total shear stress has been similarly compared with the experimental results in Fig. 4. Near the wall, it was adjusted to the wall

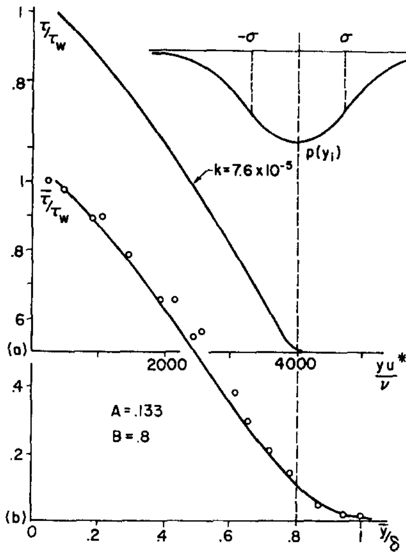

FIG. 4. Distribution of shear stress across the boundary layer. Above: Original calculation for fixed position of interface. Below: Corrected for the fluctuation of interface position and compared with experiment of Klebanoff.[13]

shear τ_w. It decreases monotonically and drops to zero just after passing the interface. The experimental points follow our computed curve very closely. The mean velocity profile shown in Fig. 5 also agrees well with experimental results. The detail of the flow near the interface is treated in more detail in the next section. The skin friction resistance law is plotted from a number of pairs of corresponding values of c_f and R_δ, each pair obtained from a separate integration. For the smooth plate, the results of the present calculation were compared with the ex-

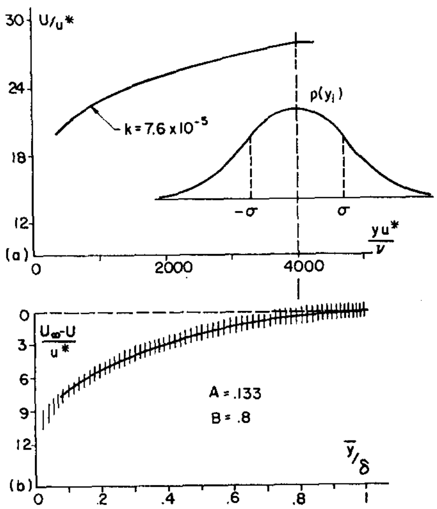

FIG. 5. Distribution of the mean velocity across the boundary layer. Above: Original calculation for fixed interface. Below: Corrected for Gaussian distribution of interface position and compared with experimental data[23] (shaded area).

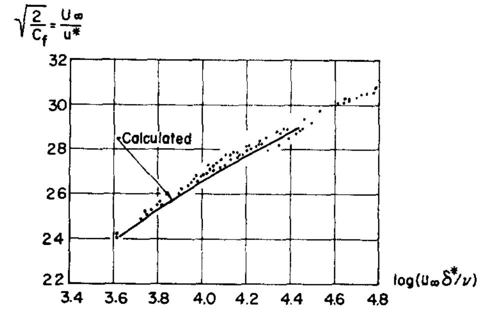

FIG. 6. Calculated skin friction coefficient for smooth plate (solid line) compared with experiments of Schultz-Grunöw[19] and Smith and Walker.[20]

periments of Schultz–Grunöw[19] and Smith and Walker[20] in Fig. 6; agreement appears satisfactory.

For rough plates, the equation for the mean velocity in the logarithmic region can be written in the form (see, e.g., Clauser[4])

$$\frac{U}{u_*} = 2.5 \ln \frac{u_* y}{\nu} + 5.2 - \frac{\Delta U}{u_*}, \qquad (46)$$

where $\Delta U/u_*$ represents the shift caused by roughness.

In order to obtain solutions for the various types of roughness, one can simply change the initial conditions for the velocity by assigning different values of $\Delta U/u_*$ at the starting point (Fig. 7). The result for each integration will then give the corresponding values of c_f and R_δ for the assigned roughness parameter $\Delta U/u_*$.

In order to demonstate the plausibility of the proposed rate equation, the four terms in the rate equation were plotted separately in Fig. 8.

IV. ASYMPTOTIC SOLUTION OF THE OUTER EDGE

As the turbulent boundary layer propagates out into nonturbulent flow, its thickness grows, and it entrains the outer nonturbulent field. This action of the turbulent front (or the superlayer) must impart vorticity to the irrotational fluid through viscous action, and in the meantime the interface itself is being continuously distorted by the motion of the large eddies inside the turbulent regime.

Here, the important assumption was made that the action of the large-scale motion merely displaces the front but is not responsible for any significant amount of "ingesting" of large "chunks" of the nonturbulent fluid.

[19] F. Schultz-Grunöw, NACA Technical Memorandum 986 (1940).

[20] D. W. Smith and J. H. Walker, NASA Technical Report R-26 (1959).

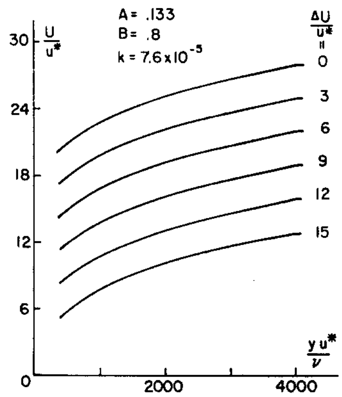

FIG. 7. Sample results of the mean velocity profiles for different roughness parameters.

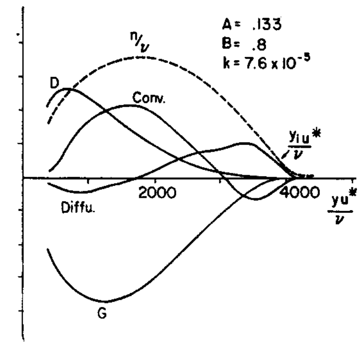

FIG. 8. Balance of the four terms of the rate equation across the boundary layer.

It is known from abundant experimental evidence[8,15] that the flow in the layer near the wall is governed primarily by local parameters while the outer part of the flow is affected more by the upstream conditions.

Let us define the mean velocity $\bar{U}(y, y_i)$ as (see Betchov and Criminale[21]) the average of the instantaneous velocity only during those times when the superlayer is located between y_i and $y_i + \Delta y_i$. The conventional mean velocity, denoted here as $\bar{U}(\bar{y})$ can be obtained by averaging $U(\bar{y}, y_i)$ over a long time just as was done for $n(y)$ [Eqs. (43), (44), and (45)],

$$\bar{U}(\bar{y}) = \int_0^\infty U(\bar{y}, y_i) p(y_i)\, dy_i. \tag{47}$$

The function $U(\bar{y}, y_i)$ is approximated as

$$U(\bar{y}, y_i) = [U(y)]_{y=\bar{y}+y'}, \tag{48}$$

while y' is defined by Eq. (44).

Near the outer edge we consider the structure of the flow relative to the interface; the interface can then be thought of as being immobilized at a constant position $\bar{y}_i$.

In the simple model here, we turn the coordinate system parallel with the interface. Note the fact that the interface is *not* a streamline, the flow enters through the interface with a mean normal velocity,

$$V_0 = U_\infty \left(\frac{d\bar{y}_i}{dx} - \frac{d\delta_*}{dx}\right), \tag{49}$$

where $\bar{y}_i(x)$ is the average position of the interface and $\delta_*(x)$ is the displacement thickness. In the region near the interface one may assume $V_0 =$ const. The origin of the new normal coordinate x is along the average interface and not along the undisturbed flow direction (or along the wall). (The angle between old and new coordinate is so small that the scales do not change significantly.)

For the purposes of the present calculations in the outer part of the layer, the flow can be assumed to be independent of x. Consequently, the mean momentum equation will reduce to

$$-V_0 \frac{dU}{dy_1} = \frac{d}{dy_1}\left(n \frac{dU}{dy_1}\right), \tag{50}$$

where the new independent variable is $y_1 = y - y_i$. As far as the rate equation is concerned, at the outer edge the dissipation term drops off at least as fast as n^2. The generation term is more significant since it is largely responsible for the creation of turbulence. The transport and diffusion terms are both high (see Fig. 8). Taking all these facts into account, it appears reasonable to neglect the decay term in the outer edge region. Thus, the rate equation reduces to

$$-V_0 \frac{dn}{dy_1} = \frac{d}{dy_1}\left(n \frac{dn}{dy_1}\right) + A(n - \nu)\frac{dU}{dy_1}. \tag{51}$$

Similar to Eq. (34), a new variable $\varphi(y_1)$ is substituted as the new independent variable

$$n\, d\varphi = \nu\, dy_1. \tag{52}$$

The momentum equation can be integrated at once, to give

$$\tau = n \frac{dU}{dy_1} = \tau_0 \exp\left[2a(\varphi_i - \varphi)\right], \tag{53}$$

where $2a = V_0/\nu$ and we define: $\tau = \tau_0$ for $\varphi = \varphi_i$, $\varphi = \varphi_i$ is now the nominal position of the interface, where $n = 2\nu$ or $\epsilon = \nu$.

After carrying out the same transformation of variables, the simplified rate equation becomes

$$\frac{d^2 n}{d\varphi^2} + 2a \frac{dn}{d\varphi} + \frac{A}{\nu^2} n(n - \nu)\left|\frac{dU}{dy_1}\right| = 0. \tag{54}$$

By using the definition of τ, the last term can be substituted from Eq. (53) and the system of two

[21] R. Betchov and W. O. Criminale, Phys. Fluids 7, 1950 (1964).

equations can be combined into a single one

$$\frac{d^2n}{d\varphi^2} + 2a\frac{dn}{d\varphi} + \frac{A\tau_0}{\nu^2}(n - \nu) \cdot \exp[2a(\varphi_i - \varphi)] = 0. \quad (55)$$

The complete solution of Eq. (55) can be obtained as

$$n = \nu + \nu \exp[a(\varphi_i - \varphi)] \cdot \left[CJ_1\left(\frac{b}{a}\exp[a(\varphi_i - \varphi)]\right) + C_1Y_1\left(\frac{b}{a}\exp[a(\varphi_i - \varphi)]\right)\right], \quad (56)$$

where $b = A\tau_0\nu^{-2}$, C and C_1 are integration constants, and J_1 and Y_1 are the first-order Bessel functions of the first and second kind, respectively.

One boundary condition can be obtained from the fact that far outside the turbulent boundary layer $\varphi \to \infty$, $n \to \nu$ (or $\epsilon \to 0$). Using this condition and defining the nominal position of the interface ($\varphi = \varphi_i$) where $n = 2\nu$, the solution becomes

$$n = \nu + \frac{\nu \exp[a(\varphi_i - \varphi)]}{J_1(b/a)} J_1\left(\frac{b}{a}\exp[a(\varphi_i - \varphi)]\right). \quad (57)$$

In the region near, but outside the interface, the value of the argument of the Bessel function is small. By means of the asymptotic expansion of the Bessel function for small values, the solution (57) can be expressed as

$$n = \nu + \nu \exp[2a(\varphi_i - \varphi)]. \quad (58)$$

In this region near the superlayer, the shearing stress becomes

$$\tau = 2\nu U_0' \exp[2a(\varphi_i - \varphi)]. \quad (59)$$

Here, the value $2\nu U_0'$ is used to replace the value τ_0 in Eq. (53). The constant U_0' denotes the mean vorticity (velocity gradient) at the nominal position of the interface (where $n = 2\nu$).

By dividing Eq. (59) by Eq. (58), the distribution of the mean velocity gradient becomes

$$U'(\varphi) = \frac{2U_0' \exp[2a(\varphi_i - \varphi)]}{1 + \exp[2a(\varphi_i - \varphi)]}. \quad (60)$$

Note that

$$\varphi \longrightarrow \infty, \qquad U' \longrightarrow 0,$$
$$\varphi = \varphi_i, \qquad U' = U_0',$$
$$\varphi \longrightarrow -\infty, \qquad U' \longrightarrow 2U_0'.$$

The asymptotic value of U' inside the fully turbulent interior of the layer is considered as a parameter to be determined.

The mean velocity distribution itself can be obtained by integrating Eq. (60) once more with respect to φ and again using the transformation $nd\varphi = \nu dy_1$,

$$\tilde{U}(\varphi) = U_\infty - U = (U_0'/a)\exp[2a(\varphi_i - \varphi)]. \quad (61)$$

Here, the boundary condition $U \to U_\infty$, since

$$\varphi \to \infty, \quad \tilde{U} \to 0 \ (U \to U_\infty)$$

has been included in the outside flow.

The eddy visocity differs from the total viscosity only by the constant kinematic viscosity ν, so we can easily obtain

$$\epsilon = \nu \exp[2a(\varphi_i - \varphi)]. \quad (62)$$

And finally, the Reynolds stress

$$\epsilon\frac{dU}{dy} = 2\nu U_0' \frac{\exp[4a(\varphi_i - \varphi)]}{1 + \exp[2a(\varphi_i - \varphi)]}. \quad (63)$$

The above relations were all given as function of the transformed independent variable φ. We must transform them back to again be expressed as functions of the physical coordinate $y_1 = y - y_i$.

The relationship between φ and y_1 can be obtained by integrating Eq. (52),

$$2ay_1 = 2a(y - y_i) = \exp[2a(\varphi - \varphi_i)] + 2a(\varphi - \varphi_i) - 1, \quad (64)$$

where y_i is defined so that $\varphi = \varphi_i$ when $y = y_i$, or $y_1 = 0$.

A simple explicit inverse relationship between y and φ cannot be given; consequently, a tabulated function $\Phi[(V_0/\nu)(y - y_i)]$ is introduced and all the asymptotic solutions are expressed by it. (See Table I.) We have, the total viscosity (Fig. 9)

$$n = \nu + \nu\Phi, \quad (65)$$

the mean velocity gradient (or mean vorticity) (Fig. 9)

$$U' = \frac{2U_0'\Phi}{1 + \Phi}, \quad (66)$$

the total shear stress

$$\tau = 2\nu U_0'\Phi, \quad (67)$$

the eddy viscosity

$$\epsilon = \nu\Phi, \quad (68)$$

the Reynolds stress

$$\epsilon\frac{dU}{dy} = 2\nu U_0' \frac{\Phi^2}{1 + \Phi}, \quad (69)$$

TABLE I. Tabulated function Φ of the asymptotic solutions of the outer edge.

$(V_0/\nu)(Y - Y_i)$	Φ
∞	0
11	
5.99	0.0067
4.92	0.018
3.95	0.049
2.86	0.135
1.63	0.3679
1.05	0.549
0.38	0.818
0	1
−1.15	1.649
−2.72	2.718
−4.98	4.48
−6.17	5.47
−8.39	7.389
−10.23	9.025
−13.68	12.18
−16.58	14.88
−22.10	20.09
−35.60	33.1
−57.60	54.6

and, finally, the mean velocity defect

$$\tilde{U} = U_\infty - U = \frac{2\nu U_0'}{V_0}\Phi. \qquad (70)$$

It is clear that the eddy viscosity ϵ, the mean velocity defect $U_\infty - U$, the shearing stress, and the Reynolds stress all increase slowly from zero in the outside flow, near the interface they grow rapidly, but inside the region they all grow almost linearly.

The mean velocity gradient or mean vorticity is also interesting. It grows from zero and asymptotically reaches the value of $2U_0'$ in a relatively short distance. The thickness of the superlayer Δ forms a Reynolds number of the order of 10 with the cross-section velocity V_0 (Fig. 9),

$$V_0\,\Delta/\nu \approx 10.$$

An estimate of the thickness of the superlayer Δ was made earlier by Corrsin and Kistler.[16] By using an argument based on diffusion, they found Δ to be of the order of the Kolmogoroff length $\Delta \approx (\nu^3/\Phi')^{1/4}$, and the propagation velocity of the turbulent front was estimated to be proportional to $(\nu w')^{1/2}$ (where Φ' is the rate of energy dissipation per unit mass, and w' is the root mean square value of the vorticity). Corrsin and Kistler's value of Δ agrees reasonably well with an estimate based on a partial analysis of Lessen.[22]

From a consideration of the balance of the convection and diffusion of vorticity, the entrainment velocity V_0 can be estimated to be of the order of

[22] M. Lessen, NACA Report 979 (1950).

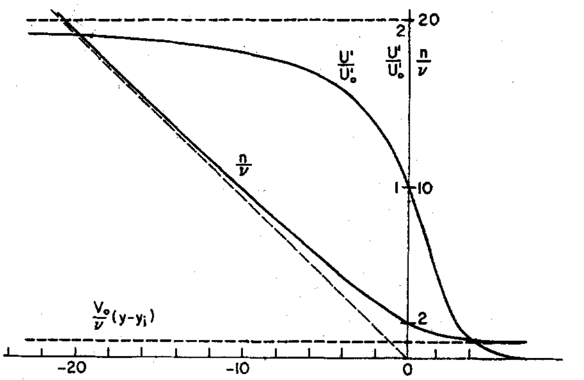

FIG. 9. The distribution of total turbulent viscosity and mean vorticity across the interface.

$(\Phi'\nu)^{1/4}$ [see Eq. (12.8) of Rotta[8]]. The Reynolds number of the superlayer based on the Kolmogoroff length obtained by Corrsin and Kistler[16] becomes

$$\frac{V_0\Delta}{\nu} = \frac{(\Phi'\nu)^{1/4}(\nu^3/\Phi')^{1/4}}{\nu} = 1,$$

which is within an order of magnitude of the value obtained above.

Experimenters have obtained the distribution of the eddy viscosity ϵ by dividing the measured value of Reynolds stress $-\rho\overline{uv}$ by the measured value of the mean velocity gradient. In this outer region, the value of the mean velocity gradient is almost zero. Therefore, the experimentally available values of ϵ are too uncertain to be compared directly with the analytic solution. However, we can match the mean velocity solutions with the experimental mean velocity defect distribution.

According to Corrsin and Kistler[16] and Klebanoff,[17] for a boundary layer with zero pressure gradient, the mean position of the interface is $\bar{y}_i \approx 0.78\,\delta$ and $y_i(t)$ fluctuates with an rms value of $\sigma \approx 0.14\,\delta$. With these values, the long time average mean velocity defect $U_\infty - \bar{U}$ can be plotted by "smearing" the solution according to the Gaussian distribution of y_i as given in Eqs. (47) and (48). Using the measured data presented in Clauser,[23] we find that for the best fit we can determine the constant $2U_0'$ inside the turbulent region. From the similarity form of the defect law, we may write

$$2U_0' = K(u_*/\delta). \qquad (71)$$

The best fit for the constant K was found to be $K = 9.4$. This constant determines the slope of the mean velocity distribution. And through the value of the friction velocity u_* in Eq. (71), the effect of the presence of the wall is brought in. The calculated asymp-

[23] F. H. Clauser, J. Aeron. Sci. 21, 2 (1954).

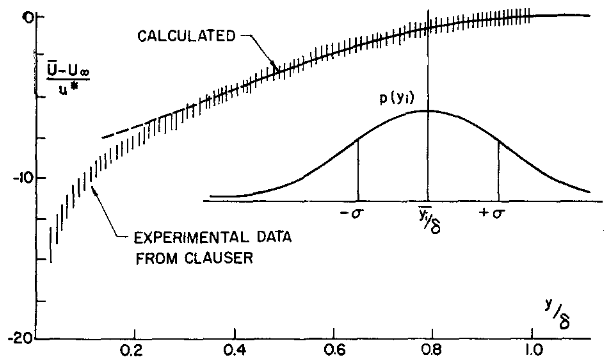

FIG. 10. Calculated asymptotic mean velocity defect profile in the outer region of the boundary layer compared with experiment of Clauser.[23]

totic mean velocity defect $U_\infty - U$ is then compared with the experimental data in Fig. 10.

Recent measurements were made in the intermittent region of the turbulent boundary layer by Kibens and Kovasznay.[24] A special electronic technique, conditional sampling, permitted the measurement of the "instantaneous mean velocity profile" through the interface and the results confirm the existence of the linear range with $K = 9.2$.

V. CONCLUSIONS

A rate equation has been proposed for the total turbulent viscosity $n = \epsilon + \nu$. It includes the effects of generation, convection, diffusion, and decay. The generation and decay terms in the rate equation are "guessed" by using dimensional reasoning and plausibility arguments. They involve two "universal constants" A and B. Prandtl's "mixing length theory" is obtained as a limiting case of present theory inside a highly turbulent region where the convection and diffusion effects are negligible. Comparison with experimental results obtained for the turbulent viscosity in turbulent boundary layers with zero pressure gradient suggests the choice of $A = 0.133$ and $B = 0.8$.

[24] V. Kibens and L. S. G. Kovasznay, Bull. Am. Phys. Soc. **13**, 827 (1968).

The rate equation, the momentum equation, and the continuity equation for the mean flow form a closed system of parabolic partial differential equations. At least in principle, a direct forward integration using a digital computer can be performed to calculate the downstream development of the mean velocity, shear stress, as well as the turbulent viscosity profile.

Explicit results for the mean velocity profile and other flow properties of the outer edge (superlayer) have been obtained in closed form by making the additional assumption that the large scale motion plays no important role in maintaining the Reynolds stress. The thickness of the superlayer was found to be of the order of $10\nu V_0^{-1}$ where V_0 is the entrainment velocity.

By accepting the "law of the wall" as valid near the boundary, the mean velocity, shear stress, and total viscosity of a locally similar turbulent boundary layer with zero pressure gradient have been calculated by analog computer. Agreement with the experimental results is satisfactory.

ACKNOWLEDGMENT

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