Spalart-Allmaras turbulence model for Compressible flows

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Report

by

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Certificate

This is to certify that this seminar report submitted by Mr. Pratikkumar Vikramark Raje (Roll No. 144010010) is approved by me for submission. Certified further that, to the best of my knowledge, the report represents work carried out by the student.

Date: Prof. Krishnendu Sinha

Declaration

I declare that this written submission represents my ideas in my own words. Wherever ideas or words or figures have been included from published works/private communications are adequately cited and has been referenced to their original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified data/idea/fact/source in my submission. I understand that any violation of the above will be the cause of disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Abstract

This work deals with the development of the standard Spalart-Allmaras (SA) one equation eddy viscosity turbulence model. The SA model saves considerable amount of computational effort compared to two-equation models like k-ϵ or k-ω while giving fairly good and more accurate results than the algebraic models. The rationale behind the formulation of its individual terms is discussed on the basis of physics of turbulence and the requirements of dimensional homogeneity, Galilean invariance and scalar form. The model gives fairly accurate results for incompressible and weakly compressible flows. However, depending on the conditions of the flow, it can be calibrated by adding new terms or modifying the already existing terms. To extend its predictive capability for compressible flows for e.g. in cases of supersonic and hypersonic flows, compressibilty corrections suggested by Catris et al. and Allmaras et al. have been discussed. Modifications for non-equilibrium flow conditions and for compressible mixing layers are also described briefly.

Keywords : turbulence, SA model, k-ϵ model, k-ω model, galilean invariance, compressibility

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Nomenclature

x, y, z	Cartesian coordinates
t	Time
R	Specific gas constant
ρ	Density
u	Velocity component in stream wise direction (x)
v	Velocity component in wall normal direction (y)
w	Velocity component in span wise direction (z)
h	Specific enthalpy
p	Pressure
T	Temperature
a	Speed of sound
M_t	Turbulent Mach number
ν	Kinematic viscosity
Re	Reynolds number

Subscripts

b	basic
v	viscous
w	wall
T, t	turbulent
i, j, k	Cartesian tensor indices

Conventions

$\bar{.}$	bar denote Reynolds averaged values
$\tilde{.}$	tilde denote Favre averaged values
$'$	prime denote Reynolds fluctuations
$''$	double prime denote Favre fluctuations

Abbreviations

NS	Navier-Stokes
$RANS$	Reynolds Averaged Navier Stokes
SA	Spalart-Allmaras
TKE	Turbulence Kinetic Energy
BL	Boundary layer
$SBLI$	Shock wave Boundary Layer Interaction
SSI	Shock Shock interaction
$FANS$	Favre-averaged Navier-Stokes
ZPG	Zero Pressure gradient
TSL	Thin shear layer

Chapter 1

Introduction

Turbulence is a flow phenomenon present everywhere in nature. Turbulent flows are characterised by a sea of unstable blobs, tubes or sheets of vortical structures called as "turbulent eddy" which are chaotically advected in the velocity field induced by themselves and by all the other vortical structures. The spatial and temporal distribution of these vortices is highly random and it exhibits a wide range of length and time scales. One thing to note here is that not only are the trajectories of eddies chaotic but the entire vorticity field is chaotic. Thus, turbulence is extremely sensitive to its initial conditions. Turbulent flows have always attracted the attention of engineers as most of the flows occuring in engineering applications are turbulent. An engineer's primary concern is to study the influence of turbulence on practical processes such as drag, mixing, heat transfer, and combustion. For eg. to control the aerodynamic drag on a plane or to design a better wing, good understanding of turbulence is required. For an engineer, not only the qualitative understanding of turbulence is of prime importance but also the ability to make qualitative predictions.

Like almost all of the physical phenomenon, turbulence can also be modeled mathematically; Navier-Stokes (NS) equations being its governing equations. It is a mathematical marvel that NS equations embody such rich and complex phenomena as turbulence. However, as turbulence consists of higly chaotic variations of flow variables, it is not possible to predict its characteristics deterministically. So, the science of turbulence is about making statistical predictions about the chaotic solutions of NS equations. In this work, we have used a simple statistical tool called as **Reynolds Decomposition** wherein we decompose all the flow quantities into time averaged values i.e. mean values and fluctuations with zero mean. We are no more interested in calculating the instantaneous flow variables (it is not feasible also) but in mean flow values. Thus, we are interested in the turbulence only to an extent that it influences the mean flow. However, on performing such decomposition of the flow variables in the NS equations and carrying out the so called Reynolds averaging, the resulting RANS equations contain an extra term compared to the NS equations in the momentum conservation equation, called as **Reynolds stress tensor**. Thus, we have more unknowns than equations. This is the closure problem in turbulence.

To close our system of equations, we need to find suitable approximations for the Reynolds stress tensor and the means of doing that are our turbulence models. A whole class of turbulence models are available there but in this work only **eddy-viscosity turbulence models** are studied. The objective of an eddy-viscosity model is to try and parameterize the influence of turbulence on the mean flow. All the eddy-viscosity models are based on the **Boussinesq hypothesis** which relates the turbulence stress with the mean strain rate tensor via a quantity called **eddy viscosity**. So, the main aim of an eddy-viscosity model is to somehow calculate this eddy viscosity to get the Reynolds stress. Now, depending upon the number of transport equations we need to solve for getting the eddy viscosity, we have zero (algebraic), one or two equation models. Each of these models have their own advantages and disadvantages.

The main goal of the present work is to study the development of an one-equation turbulence model called as the **Spalart-Allmaras (SA) turbulence model** which is used for calculating the eddy viscosity by solving a transport equation for the same. The organization of this report is as follows.

* Chapter 1 provides a brief introduction to turbulence and also the objective of this work.
* Chapter 2 deals with the Governing equations of turbulence which are the NS equations. Its main contents are the RANS equations and the problem of closure in turbulence.
* Chapter 3 deals with the turbulence modeling which is required to devise appropriate approximations for the unclosed terms in the RANS equations. A brief discussion of zero, one and two equations models is presented. The meanings of various terms in the Turbulent Kinetic energy (TKE) equation are also explained.
* Chapter 4 is central to this work. It presents the development of SA model for incompressible and weakly compressible flows in depth.
* Chapter 5 deals with the compressible forms of SA model by Catrix et al. and Allmaras et al. Modifications for non-equilibrium flows, compressible mixing layers and for other relevant flow conditions are also discussed briefly.
* Chapter 6 talks about conclusion and the possible future work.

Chapter 2

Governing equations

One of the most enthralling things about Nature is that there exists such patterns in its actions, that these patterns can be formulated into some laws using the mathematical language called as the mathematical models. We call these mathematical models the governing equations of that particular physical phenomenon under study. We can write governing equations to predict the change in position or configuration of any solid or fluid in motion or at rest. Navier-Stokes equations are the governing equations in case of fluid flows, used for predicting the fate of any fluid in motion. Solving them, for a particular set of boundary conditions (such as inlets, outlets, and walls), predicts the fluid velocity and its pressure in a given geometry.

2.1 Navier-Stokes equations

The laminar or turbulent motion of a viscous, heat conducting fluid is governed by the Navier-Stokes equations. These are vector equations obtained by applying Newton's 2^{nd}law of motion to a fluid element and they describe the conservation of linear momentum. They are supplemented by the mass conservation equation, also called continuity equation and the energy equation. Usually in the CFD literature, the term Navier-Stokes (NS) equations is used to refer to all of these 3 equations,i.e. , conservation of mass given by Equation [2.1a], momentum given by Equation [2.1b] and energy. For incompressible flows, the energy equation is not coupled with the mass and momentum equations. Hence, it can be solved independently after getting the velocity and pressure fields from the mass and momentum conservation equations. The NS equations are only valid as long as the fluid is considered to be a continuum i.e. the representative physical length scale of the system is much larger than the mean free path of the molecules that make up the fluid.

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2.1a}$$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial t_{ji}}{\partial x_j} \tag{2.1b}$$

The vectors u_i and x_i are velocity and position, t is time, p is pressure, ρ is density and t_{ji} is the viscous stress tensor defined by

$$t_{ji} = 2\mu S_{ij} \tag{2.2}$$

where μ is the molecular viscosity and S_{ij} is the strain rate tensor given by,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.3}$$

2.2 Reynolds time Averaging

As turbulence consists of higly random fluctuations of the various flow properties, a deterministic approach is not possible and one has to resort to statistical tools. Also, as turbulence consists of large number of scales, resolving each and every scale is not required for engineering applications. So, Reynolds came out with the averaging concepts, i.e. the **time average**, the **spatial average** and the **ensemble average** in 1895. Time averaging has been considered in this work for arriving to the Reynolds Averaged Navier-Stokes equations (RANS) as it is most appropriate for stationary turbulence i.e. a turbulent flow that on the average does not vary with time. Time averaging is given by,

$$F_T(\mathbf{x}) = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} f(\mathbf{x}, t)dt \tag{2.4}$$

Using these averaging concepts, Reynolds introduced a procedure called as the Reynolds decomposition in which all the flow quantities are expressed as the sum of mean and fluctuating parts. For eg.,

$$u(x, y, z, t) = \overline{u(x, y, z, t)} + u^{'}(x, y, z, t) \tag{2.5}$$

$$= U(x, y, z, t) + u^{'}(x, y, z, t) \tag{2.6}$$

where $\overline{u(x, y, z, t)} = U(x, y, z, t)$ is the time averaged mean velocity and $u^{'}(x, y, z, t)$ is the fluctuating component of the velocity. So using the definition of time averaging, we can write

$$U(\mathbf{x}) = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} u_i(\mathbf{x}, t)dt \tag{2.7}$$

Some of the important properties of Reynolds time averaging are as follows.

$$\begin{aligned} \overline{U_i(\mathbf{x})} &= U_i(\mathbf{x}) \\ \overline{u_i^{'}} &= 0 \\ \overline{\frac{\partial u_i}{\partial x_i}} &= \frac{\partial U_i}{\partial x_i} \\ \overline{u_i u_j} &= U_i U_j + \overline{u_i^{'} u_j^{'}} \end{aligned} \tag{2.8}$$

2.3 RANS

Reynold Averaged Navier-Stokes i.e. RANS equations are derived by taking the time average of the Navier-Stokes equations. To simplify the time-averaging process, convective term in the momentum equations is written in its conservative form as

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}(u_i u_j) - u_i \frac{\partial u_j}{\partial x_j} \tag{2.9}$$

Now, from mass conservation

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{2.10}$$

$$\therefore \qquad u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}(u_i u_j) \tag{2.11}$$

Substituting Equation [2.11] in the Equation [2.1b], we get Navier-Stokes equations in conservation form :

$$\rho \frac{\partial u_i}{\partial t} + \rho \frac{\partial (u_j u_i)}{x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu S_{ji}) \tag{2.12}$$

Taking time average of Mass conservation Equation [2.1a], we get

$$\overline{\frac{\partial u_i}{\partial x_i}} = 0 \tag{2.13}$$

$$\therefore \frac{\partial \overline{u_i}}{\partial x_i} = 0 \tag{2.14}$$

$$\therefore \frac{\partial \left(\overline{U_i + u_i'}\right)}{\partial x_i} = 0 \tag{2.15}$$

$$\therefore \frac{\partial U_i}{\partial x_i} = 0 \tag{2.16}$$

Similarly, by taking the time average of momentum Equation [2.12], then decomposing the instantaneous flow variables i.e. velocity and pressure into their time averaged mean and fluctuating parts and using the properties of the Reynolds time averaging given by Equation [2.8], we arrive at the momentum equation for the mean velocity given by,

$$\rho \frac{\partial U_i}{\partial t} + \rho U_j \frac{\partial (U_i)}{x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}(2\mu S_{ji} - \rho \overline{u_j' u_i'}) \tag{2.17}$$

Equations [2.16] and [2.17] are referred to as the **Reynolds-Averaged Navier-Stokes equations (RANS)**. The quantity $-\rho \overline{u_j' u_i'}$ is known as the **Reynolds-stress tensor**. and it is denoted by $\rho \tau_{ij}$. τ_{ij} is the **specific Reynolds stress tensor** given by

$$\tau_{ij} = -\overline{u_i' u_j'} \tag{2.18}$$

The mean flow and the turbulence interact via the Reynolds stresses. The Reynolds stress arises as a result of the turbulence and acts on the mean flow, shaping its evolution. It is also responsible for the maintenance of turbulent fluctuations as it channels energy out of the mean flow and into the turbulence. The Reynolds stress is not really a true stress in the conventional sense of word. It represents the flux of momentum caused by the turbulent fluctuations.

From its definition, the Reynolds stress is symmetric ,i.e $\tau_{ij} = \tau_{ji}$. Hence, we have produced 6 unknown quantities as a result of Reynolds averaging without gaining any additional equations. So, for general 3-D flows, we have 10 unknowns i.e. 4 unknown mean flow properties, viz., pressure and three velocity components along with six Reynolds-stress components. Our equations are mass conservation given by Equation [2.16] and the three components of Equation [2.17] i.e. total of 4 equations. This means we cannot solve for all the unknowns,i.e. our system is not **closed**. To close the system, we must find enough equations to solve for our unknowns.

2.4 The Closure Problem

In a quest to get additional equations, we can take moments of the NS equations, i.e. we multiply the NS equation by a fluctuating property and time average the product. In doing so, we can derive a differential equation for the Reynolds-stress tensor. Let $N(i)$ denote the Navier-Stokes operator, viz.,

$$N(u_i) = \rho\frac{\partial u_i}{\partial t} + \rho u_k\frac{\partial u_i}{x_k} + \frac{\partial p}{\partial x_i} - \mu\frac{\partial^2 u_i}{\partial x_k \partial x_k} \tag{2.19}$$

The NS equation can be written symbollically as

$$N(u_i) = 0 \tag{2.20}$$

In order to derive an equation for Reynolds stress tensor, we form the following time average.

$$\overline{u_i^{'} N(u_j) + u_j^{'} N(u_i)} = 0 \tag{2.21}$$

Now, again following the same procedure as that followed for deriving the RANS equations, i.e. splitting the instantaneous quantites into their mean and fluctuating parts and using the properties of the Reynolds time averaging given by Equation [2.8], we arrive at the Reynolds-Stress equation given by,

$$\begin{aligned}\frac{\partial \tau_{ij}}{\partial t} + U_k\frac{\partial \tau_{ij}}{\partial x_k} = &-\tau_{ik}\frac{\partial U_j}{\partial x_k} - \tau_{jk}\frac{\partial U_i}{\partial x_k} + \epsilon_{ij} + \overline{\frac{u_i^{'}}{\rho}\frac{\partial p^{'}}{\partial x_j}} + \overline{\frac{u_j^{'}}{\rho}\frac{\partial p^{'}}{\partial x_i}} \\ &+ \frac{\partial}{\partial x_k}\left[\nu\frac{\partial \tau_{ij}}{\partial x_k} + \overline{u_i^{'} u_j^{'} u_k^{'}} + \overline{p^{'} u_i^{'}}\right]\end{aligned} \tag{2.22}$$

where

$$\epsilon_{ij} = 2\nu \overline{\frac{\partial u_i^{'}}{\partial x_k} \frac{\partial u_j^{'}}{\partial x_k}} \tag{2.23}$$

Thus, we have achieved our goal of getting 6 extra equations. However, in doing so we have also produced 22 unknowns given below.

$$\overline{u_i^{'} u_j^{'} u_k^{'}} \rightarrow 10 \text{ unknowns}$$

$$2\nu \overline{\frac{\partial u_i^{'}}{\partial x_k} \frac{\partial u_j^{'}}{\partial x_k}} \rightarrow 6 \text{ unknowns}$$

$$\overline{\frac{u_i^{'}}{\rho} \frac{\partial p^{'}}{\partial x_j}} + \overline{\frac{u_j^{'}}{\rho} \frac{\partial p^{'}}{\partial x_i}} \rightarrow 6 \text{ unknowns}$$

This illustrates the closure problem of turbulence, i.e. the issue of establishing a sufficient number of equations for all the unknowns. In a nutshell, when NS equations were time-averaged, RANS was obtained. RANS looks similar to NS equations with an additional Reynolds Stress term. So, 6 unknowns were introduced while doing the time avergaing without the addition of new equations. As a result, we went for a quest to find extra equations to solve for those 6 extra unknowns. For that we took moment of the N-S equations of the form $\left[\overline{u_i^{'} N(u_j) + u_j^{'} N(u_i)} = 0\right]$. This led to the Reynolds Stress equation. Thus we got 6 new equations to solve for the Reynolds stress but in doing so, we also introduced 22 new unknowns!

The closure problem occurs because of the non-linearity of the NS equations. As we take higher and higher moments, we generate additional unknowns at each level. At no point will this procedure balance our unknowns/equations ledger.

This is where turbulence modeling comes to our rescue. The function of turbulence modeling is to devise approximations for the unknown correlations in terms of flow properties that are known so that a sufficient number of equations exists. In making such approximations, we close the system. There exists a whole different classes of turbulence models. However, only eddy-viscosity based turbulence models are studied in this work which will be our topic of discussion in the next chapter.

2.5 Summary

Turbulence is a saptially complex and highly chaotic fluid flow phenomenon. However, it can be modeled mathematically and NS equations are its governing equations. Since it is highly chaotic, we cannot predict deterministically all the instantaneous flow properties. So, we resort to statistical analysis. A very simple statistical tool called Reynolds Decomposition is used to obtain RANS from NS equations which are now our governing equations. We solve them to get the mean flow properties. But while solving RANS, we have a problem; we have more unknowns than equations. This is the closure problem which is also discussed in this chapter.

Chapter 3

Turbulence Modeling

In Chapter 2, it was seen that on performing Reynolds time averaging for getting conservation equations for the mean quantities, we arrived at the equations called the RANS equations. They look similar to the NS equations except the extra Reynolds stress tensor term. So, in order to compute all mean-flow properties of the turbulent flow under consideration, we need a prescription for computing the Reynolds stresses $-\rho\overline{u_i' u_j'}$ and turbulence models provide that description. For a turbulence model to be good and useful, it

* must have wide applicability
* must be accurate
* must be simple and
* must be economical to run

Following are the classes of turbuence models :

* RANS based models
  - Linear Eddy viscosity models
    * Algebraic models
    * One and two-equation models
  - Non-linear eddy viscosity models and algebraic stress models
  - Reynolds stress transport models
* Large eddy simulations
* Detached eddy simulations and other hybrid models
* Direct numerical simulations

Only RANS based eddy viscosity models will be touched in this chapter.

For eddy viscosity models, it all started with Boussinesq in 1877 when he first introduced the concept of eddy viscosity in an attempt to develop a mathematical description of turbulent stresses. He linked the Reynolds stresses with the eddy viscosity, called as the Boussinesq eddy viscosity hypothesis, by mimicing the molecular gradient-diffusion process. Eddy-viscosity unlike the molecular viscosity is not the property of the fluid but it depends upon the flow.

One thing to note here is the amazing historical fact that Boussinesq hypothesis was proposed in 1872 in a meeting of French Academy of Science and was published in 1877. How is it possible that Joseph Boussinesq proposed in 1872 a closure for an equation that would be written more than 20 years later (Reynolds averaging was proposed by Reynolds in 1895)!!

3.1 Boussinesq eddy viscosity hypothesis

Boussinesq eddy viscosity hypothesis lies at the heart of the eddy viscosity turbulence models. Boussinesq postulated that momentum transfer caused by turbulent eddies can be modeled with an eddy viscosity ν_t. This is in analogy with how the momentum transfer caused by the molecular motion in a gas can be described by a molecular viscosity. The Boussinesq assumption states that the Reynolds stress tensor, $-\rho\tau_{ij}$, is proportional to the trace-less mean strain rate tensor, S_{ij}, and can be written in the following way:

$$-\rho\overline{u_i'u_j'} = 2\mu_T S_{ij} - \frac{2}{3}\mu_T \frac{\partial U_k}{\partial x_k}\delta_{ij} - \frac{2}{3}k\delta_{ij} \tag{3.1}$$

where k is the turbulence kinetic energy (TKE). The third term is included so as to balance the normal Reynolds stresses. For incompressible flows, we have

$$\frac{\partial U_k}{\partial x_k} = 0 \tag{3.2}$$

For the one-equation model, we neglect the third term. So, the Boussinesq hypothesis becomes

$$-\rho\overline{u_i'u_j'} = 2\mu_T S_{ij} \tag{3.3}$$

3.2 Various Linear eddy-viscosity turbulence models

The Linear eddy viscosity models makes use of the Boussinesq eddy viscosity hypothesis presented in the previous Section 3.1. Depending upon the number of transport equations needed to be solved to compute the eddy viscosity, these models are divided into zero (or algebraic), one or two-equation models.

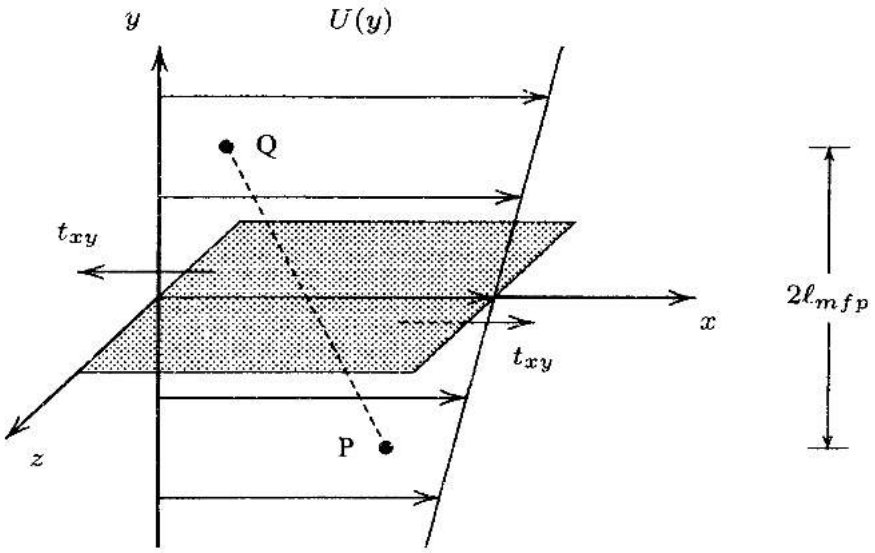

Figure 3.1: Shear-flow schematic [3]

3.2.1 Prandtl's mixing length hypothesis

The basic premise of Prandtl's mixing length hypothesis is the analogy of turbulent motion with the molecular motion. To understand the analogy, consider a shear flow with average velocity $\mathbf{U} = U(y)\mathbf{i}$ as shown in the Figure (3.1).

Now, because of the molecular motion across the plane $y = 0$, a shear stress τ_{xy} will be produced. From Kinetic theory of gases, taking the average molecular velocity as the thermal velocity v_{th} and l_{mfp} as the mean free path, the shear stress resulting from the molecular transport of momentum in a perfect gas is given by

$$t_{xy} = \frac{1}{2}\rho v_{th} l_{mfp} \frac{dU}{dy} \tag{3.4}$$

or we can write

$$t_{xy} = \mu \frac{dU}{dy} \tag{3.5}$$

where μ is the fluid viscosity defined by

$$\mu = \frac{1}{2}\rho v_{th} l_{mfp} \tag{3.6}$$

Equations [3.4] and [3.6] are the foundation stones for Prandtl's mixing length hypothesis. The Mixing length hypothesis put forth by Prandtl in 1925 is a method attempting to describe momentum transfer by turbulence Reynolds stresses within a newtonian fluid boundary layer by means of an eddy viscosity. It considers a simplified model for turbulent motion in which fluid particles coalesce into lumps that cling together and move as a unit. It further assumes that in a shear flow such as that depicted in Figure (3.1), the lumps retain their x-directed momentum

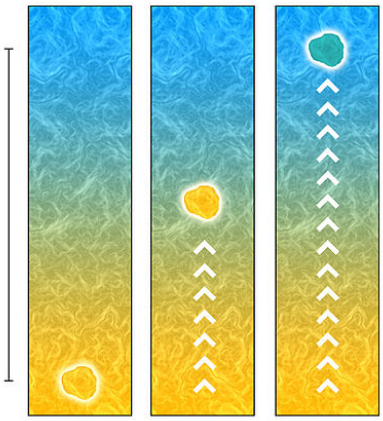

Figure 3.2: A Lump of fluid particles travelling a distance equal to the mixing length shown by the bar on the left [13]

for a distance in y-direction, l_{mix}, called as the mixing length. That is to say the mixing length is a distance that a lump of fluid particles will keep its original characteristics before dispersing them into the surrounding fluid. Thus, the concept of mixing length is analogous to the concept of mean free path. The Figure (3.2) shows one such lump of fluid particles.

In analogy to the molecular momentum transport process with Prandtl's lump of fluid replacing the molecule and l_{mix} replacing l_{mfp}, using Equation (3.4) we can write

$$\rho\tau_{xy} = \frac{1}{2}\rho v_{mix} l_{mix} \frac{dU}{dy} \qquad \Rightarrow \qquad \tau_{xy} = \frac{1}{2} v_{mix} l_{mix} \frac{dU}{dy} \tag{3.7}$$

To specify v_{mix}, on dimensional grounds Prandtl further postulated that

$$v_{mix} = constant \cdot l_{mix} \left| \frac{dU}{dy} \right| \tag{3.8}$$

Now, as l_{mix} is not a physical property of the fluid, we can absorb the constant in Equation [3.8] and the factor $\frac{1}{2}$ in Equation [3.7] in the mixing length and by analogy to Equations [3.4] and [3.6], Prandtl's mixing length hypothesis leads to

$$\tau_{xy} = \nu_T \frac{dU}{dy} \tag{3.9}$$

where ν_T is the **kinematic eddy viscosity** given by

$$\nu_T = l_{mix}^2 \left| \frac{dU}{dy} \right| \tag{3.10}$$

For specifying l_{mix}, Prandtl postulated that for flows near solid boundaries the mixing length is proportional to distance from the surface. The mixing length is different for each flow and must be known in advance to obtain a solution.

3.2.2 Cebeci-Smith (1967) Algebraic Model

Developed in 1967 by Cebeci and Smith, this is a two layer model with ν_T given by separate expressions in each layer as

$$\nu_T = \begin{cases} \nu_{T_i} & , \quad y \leq y_m \\ \nu_{T_o} & , \quad y > y_m \end{cases} \tag{3.11}$$

where y = normal distance from the nearest solid boundary and $\nu_{T_i} = \nu_{T_o}$ for $y = y_m$.

Inner layer

$$\nu_{Ti} = l_{mix}^2 \left[\left(\frac{\partial U}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial x} \right)^2 \right]^{1/2} \tag{3.12}$$

$$l_{mix} = \kappa y \left[1 - e^{-y^+/A^+} \right] \tag{3.13}$$

where

$$y^+ = \frac{u_\tau y}{\nu} \tag{3.14}$$

u_τ is the friction velocity in the log layer.

Outer layer

$$\nu_{T_o} = \alpha U_e \delta_v^* F_{Kleb}(y; \delta) \tag{3.15}$$

where $F_{Kleb}(y; \delta)$ = Klebanoff intermittency function given by

$$F_{Kleb}(y; \delta) = \left[1 + 5.5 \frac{y}{\delta}^6 \right]^{-1} \tag{3.16}$$

δ = BL thickness , U_e = BL edge velocity, δ_v^* = Velocity thickness given by

$$\delta_v^* = \int_0^\delta \left(1 - \frac{U}{U_e} \right) dy \tag{3.17}$$

Closure Coefficients

$$\kappa = 0.40, \qquad \alpha = 0.0168, \qquad A^+ = \left[1 + y \frac{dP/dx}{\rho u_\tau^2} \right]^{-1/2} \tag{3.18}$$

3.2.3 Baldwin-Barth (1990) One equation model

It is the simplest complete model of turbulence as it involves no adjustable function or coefficients. It is derived from the k-ϵ model. It involves transport equation for the turbulent Reynolds no. $\tilde{R_T}$. The model is as follows.

Kinematic Eddy Viscosity

$$\nu_T = c_\mu \nu \tilde{R_T} D_1 D_2 \tag{3.19}$$

Turbulence Reynolds Number

$$\begin{aligned}\frac{\partial}{\partial t}(\nu\tilde{R_T}) + U_j\frac{\partial}{\partial x_j}(\nu\tilde{R_T}) = (C_{\epsilon 2}f_2 - c_{\epsilon 1})\sqrt{\nu\tilde{R_T}P} + \left(\nu + \frac{\nu_T}{\sigma_\epsilon}\right)\frac{\partial^2}{\partial x_k \partial x_k}(\nu\tilde{R_T}) \\ - \frac{1}{\sigma_\epsilon}\frac{\partial \nu_t}{\partial x_k}\frac{\partial}{\partial x_k}(\nu\tilde{R_T})\end{aligned} \tag{3.20}$$

Closure Coefficients and Auxiliary Relations

$$c_{\epsilon 1} = 1.2,\ \ c_{\epsilon 2} = 2, \qquad C_\mu = 0.09,\ \ A_o^+ = 26, \qquad A_2^+ = 10 \tag{3.21}$$

$$P = \nu_T\left[\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - \frac{2}{3}\frac{\partial U_k}{\partial x_k}\frac{\partial U_k}{\partial x_k}\right] \tag{3.22}$$

$$D_1 = 1 - e^{-y^+/A_o^+} \qquad D_2 = 1 - e^{-y^+/A_2^+} \tag{3.23}$$

$$\begin{aligned}f_2 = \frac{C_{\epsilon 1}}{C_{\epsilon 1}} + \left(1 - \frac{C_{\epsilon 1}}{C_{\epsilon 1}}\right)\left(\frac{1}{\kappa y^+} + D_1 D_2\right)\cdot \\ \left[\sqrt{D_1 D_2} + \frac{y^+}{\sqrt{D_1 D_2}}\left(\frac{D_2}{A_o^+}e^{-y^+/A_o^+} + \frac{D_1}{A_2^+}e^{-y^+/A_2^+}\right)\right]\end{aligned} \tag{3.24}$$

3.2.4 The Turbulence Energy Equation

Turbulence kinetic energy (TKE) is the mean kinetic energy per unit mass associated with eddies in a turbulent flow. The value of TKE directly represents the strength of the turbulence in the flow. TKE can be produced by fluid shear, friction or buoyancy. It is then transferred down the turbulence energy cascade, and is dissipated by viscous forces. So, TKE is a very important quantity in turbulence. Also, the two-equation turbulence models like k-ω and k-ϵ are based upon TKE. In addition, to arrive at the SA equation most of the analogies are drawn with the TKE equation. So, its worth studying the TKE equation.

We determine k by taking the trace of Reynolds-stress tensor as

$$\tau_{ii} = -\overline{u_i' u_i'} = -2k \tag{3.25}$$

Thus, the trace of Reynolds-stress tensor is proportional to the TKE. So, TKE equation is derived by taking the trace of the Reynolds stress Equation [2.22]. The TKE is given by

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \tau_{ij} \frac{\partial U_i}{\partial x_j} - \epsilon + \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \tag{3.26}$$

where σ_k is the closure coefficient.

Meaning of various terms of TKE equation

$$LHS = \frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} \tag{3.27}$$

The 1^{st} term represents the local change in k and the 2^{nd} term represents the convection of k. Together they represent the rate of change of k if we follow a turbulent eddy.

$$\tau_{ij} \frac{\partial U_i}{\partial x_j} \Rightarrow \text{Production term}$$

The production term represents the rate at which TKE is transferred from the mean flow to the turbulence.

$$\epsilon = \nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}} \Rightarrow \text{Dissipation}$$

Dissipation represents the rate at which TKE is converted into thermal internal energy.

$$\left(\nu + \tfrac{\nu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \Rightarrow \text{Molecular + Turbulent Diffusion}$$

The above term represents the diffusion of TKE due to molecular and turbulent transport processes.

3.2.5 A glimpse at the k-ϵ model

The standard k- ϵ by Launder-Sharma (1974) is by far the most popular of the two-equation turbulence models. For this model, the kinematic eddy viscosity is defined as :

$$\nu_T = C_\mu \frac{k^2}{\epsilon} \tag{3.28}$$

The model involves two transport equations ; one for the TKE given by Equation [3.26] and the second for turbulent dissipation ϵ given by,

$$\frac{\partial \epsilon}{\partial t} + U_j \frac{\partial \epsilon}{\partial x_j} = C_{\epsilon 1} \frac{\epsilon}{k} \tau_{ij} \frac{\partial U_i}{\partial x_j} - C_{\epsilon 2} \frac{\epsilon}{k} \epsilon + \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_T}{\sigma_\epsilon} \right) \frac{\partial \epsilon}{\partial x_j} \right] \tag{3.29}$$

The closure coefficients and Auxiliary relations are :

$$C_{\epsilon 1} = 1.44, \;\; C_{\epsilon 2} = 1.44, \qquad\qquad C_\mu = 0.09, \;\; \sigma_k = 1, \qquad\qquad \sigma_\epsilon = 1 \tag{3.30}$$

$$\omega = \frac{\epsilon}{C_\mu k} \qquad\qquad and \qquad\qquad l = \frac{C_\mu k^{3/2}}{\epsilon} \tag{3.31}$$

where ω is the specific dissipation of the TKE and l is the turbulence length scale.

3.3 Summary

To deal with the issue of closure problem, turbulence modeling is required. Various turbulence models are available to provide relations for the unknown correlations, but here only eddy-viscosity turbulence models are described. All the eddy-viscosity models are based on the Boussinesq eddy-viscosity hypothesis, which relates the Reynolds stress with the strain rate tensor via turbulent eddy viscosity ν_T. Thus, the aim of an eddy-viscosity model is to determine ν_T. Prandtl in his mixing length hypothesis provided a means to determine ν_T. Depending on the number of equations that are solved to determine ν_T, we have zero, one or two-equation turbulence models. Cebeci-Smith (1967) algebraic model, Baldwin-Barth (1990) one equation model and k-ϵ two-equation model are presented in this chapter. Also, the meanings of various terms of the TKE equation are discussed.

Chapter 4

Development of Spalart-Allmaras one equation turbulence model

4.1 Motivation to study SA one equation turbulence model

Eddy viscosity models described in the previous chapter, while being the simplest of turbulence models, are more than adequate in solving most of the aerodynamic computational problems. When dealing with zero equation or algebraic turbulence models, one runs into the problem of using boundary layer thicknesses which are not properly defined and the inadequate prediction of shock-boundary layer interactions (SBLI). Other phenomenon that zero-equation models have trouble dealing with are massively separated flows. Also, since zero-equation models are not local, i.e., equation at one point depends on the solution at other points, they cannot be used with unstructured grids. The bottom line for zero-equations model is that they are very simple to code, they are inexpensive and they only work well for relatively simple problems.

While two equation models ensure the simplest complete closure and are more mathematically sophisticated, they too have their disadvantages. For the higher computational effort, they provide no significant advantage over one-equation models for the prediction of shock-boundary layer interactions or separation from smooth surfaces. In terms of higher computational effort, these model require much finer grids near wall, have much stronger source term which degrade the convergence, and demand non-trivial upstream and free-stream conditions for the turbulence variables. Wall functions are usually introduced which further complicate these models.

These arguments do not prove that one-equation models are the best eddy-viscosity models. In fact a compromise between the one-equation and two-equation turbulence models is often done to solve a particular turbulent flow problem.

So one equation models are the field of interest to us. The one equation SA model has been chosen to study in this work because of its many advantages over other fellow class of one equation turbulence models and also algebraic and two-equation models. For e.g. unlike all the

algebraic models like Baldwin-Lomax(1978), Johnson-King(1985), Cebecci-Smith(1967) and one-equation models like Bradshaw, Ferris & Atwell(1967), Nee-Kovasznay(1969), Michel-tree, Salas and Hassan (1990), it is local. Hence, it is compatible with grids of any structure, i.e. structured as well as unstructured grids, and Navier-Stokes solvers in 2D and 3D. Also, the accuracy of the results obtained is more compared to other one-equation models. It can be calibrated on 2D mixing layer, wakes and flat plate boundary layers. Also, it yields satisfactory predictions of B.L. in pressure gradients. Also, unlike two-equation models, trivial values for the free-stream turbulence variables can be used. It can be used for complex flows such as high-lift systems or wing body junctions. However, it cannot yield good results in case of homogeneous turbulence. But thats not a drawback as such because it is not meant to be universal.

This one equation turbulence model developed by Spalart and Allmaras in 1992 is basically intended for aerodynamic flows. It is an evolution of the Nee-Kovasznay model [1]. Nee and Kovasznay in 1969 developed a simple phenomenological mathematical model for turbulent shear flows which governs the variation of the turbulent viscosity. Rather than deriving the model from the first principles, they developed it by making an educated guess concerning the various terms based on corresponding physical mechanisms such as diffusion, production and dissipation. Following the footsteps of Nee and Kovasznay, Spalart and Allmaras developed a transport equation for the turbulent kinematic viscosity ν_t using :

* Empiricism
* Dimensional analysis
* Galilean invariance
* Selective dependence on the molecular viscosity

As a matter of fact, Spalart and Allmaras took Nee and Kovasznay's model one step further by re-inventing their near-wall destruction term so as to accurately capture the log layer near the wall.

Spalart and Allmaras in [2] discussed four versions of the model for different flow situations :

1^{st} version which is the basic version is for **"Free shear flows at high Re no."** also called as the **inviscid model**.

2^{nd} version is for **"Flows near the wall at high Re no."**.

3^{rd} version is for **"Flows near the wall at low Re no."** also called as the **complete viscous model**.

4^{th} version is for **"Laminar region and tripping"** to get a control over the laminar regions of the shear layers.

Each version is calibrated for a certain flow situation and for each new physical effect, new terms are added and calibration of the model is done accordingly.

In this work, SA model without the trip term is studied as most of the flows encountered in the applications relevant to us are fully developed turbulent flows. The following subsections will discuss the development of the complete viscous model withot the trip term. The rationale behind adding and calibrating each term in the transport equation for eddy viscosity is discussed.

4.2 Development of the model

The central idea which lies at the heart of development of the SA model is the fact that evolution of any scalar quantity, like ν_t, subject to the conservation laws is given by the following differential equation :

$$\frac{\partial \nu_t}{\partial t} + \nabla \cdot \phi(\mathbf{x}, t) = s(\mathbf{x}, t) \tag{4.1}$$

where $\phi(\mathbf{x}, t)$ is the flux of ν_t and $s(\mathbf{x}, t)$ is the source or sink term which combines the effects that create or destroy ν_t.

Now, the flux $\phi(\mathbf{x}, t)$, i.e. the rate of transport of ν_t, takes into consideration the changes in ν_t due to its convection $\phi_C(\mathbf{x}, t)$ by the mean velocity field $\mathbf{U}$ and diffusion $\phi_D(\mathbf{x}, t)$ due to its uneven distribution. The convective flux can be rewritten as

$$\phi_C(\mathbf{x}, t) = \mathbf{U}\nu_t \tag{4.2}$$

As a result, the transport equation for ν_t takes the following form :

$$\frac{\partial \nu_t}{\partial t} + \nabla \cdot (\mathbf{U}\nu_t) + \nabla \cdot \phi_D = Production - Destruction \tag{4.3}$$

In tensor notation, the above equation can be written as

$$\frac{D\nu_t}{Dt} \equiv \frac{\partial \nu_t}{\partial t} + U_j \frac{\partial \nu_t}{\partial x_j} = Production + Diffusion - Destruction$$

where Diffusion = $-\nabla \cdot \phi_D$

Most of the terms in the RHS are formed mainly by drawing an analogy of ν_t with the turbulent energy. The basic premise for drawing this analogy is that ν_t may be regarded as the ability of the turbulent flow to transport momentum and hence it can be directly related with the turbulent energy.

The rationale behind the selection of a particular form of the Production, Diffusion and Destruction term is explained based on the following arguments.

1. **Physics** : This explains why a particular form is chosen for each term using the arguments based on the physics of each of the physical mechanisms of Production, Diffusion and Destruction. In most of the cases, analogies of ν_t are drawn with the turbulent energy to arrive at a particular form. The terms thus formed decribes the turbulence characteristics of the fluid flow which is modeled.

2. **Scalar form** : This argument puts a constraint that each of the terms in the equation must be scalar as the governing equation is a transport equation for a scalar quantity ν_t.

3. **Dimensional homogeneity** : This puts a constraint that each of the terms in the equation must be dimensionally homogeneous.

4. **Galilean Invariance** : This argument states that as all the conservation laws assumes the same forms in all of the inertial reference frames, the equation of the SA model and hence each of its terms must assume the same form in any of the inertial frames.

4.2.1 The material derivative of ν_t

The LHS of the transport equation represents the material or substantial derivative of ν_t given by :

$$\frac{D\nu_t}{Dt} = \frac{\partial \nu_t}{\partial t} + U_j \frac{\partial \nu_t}{\partial x_j} \tag{4.4}$$

1. **Physics** : The 1^{st} term i.e. $\dfrac{\partial \nu_t}{\partial t}$ represents the local change, i.e. a change at a fixed point in space, in ν_t. The 2^{nd} term i.e. $U_j \dfrac{\partial \nu_t}{\partial x_j}$ represents the convection of ν_t.

2. **Scalar form** : The material derivative of ν_t is scalar.

3. **Dimensions** : The dimensions of the material derivative are $[L^2T^-2]$. So, all the terms on RHS must have the same dimensions of $[L^2T^-2]$.

4. **Galilean invariance** : The material derivative is Galilean invariant i.e. to say that the form of the material derivative of ν_t does not change under Galilean transformation from reference frame 1 given by (x, y, z, t) to reference frame 2 given by ($x^{'}$, $y^{'}$, $z^{'}$, $t^{'}$). Suppose that the reference frame 2 is moving w.r.t. reference frame 1 at a constant velocity a, then the Galilean transformation is given by :

   $x^{'} = x - at$, $y^{'} = y$, $z^{'} = z$ & $t^{'} = t$ and the velocity transforms to $U^{'} = U - a$, $V^{'} = V$ & $W^{'} = W$.

   The partial derivative between the two reference frame transforms as :
   $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial x^{'}}$, $\dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial y^{'}}$, $\dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial z^{'}}$ and $\dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial t^{'}} - a\dfrac{\partial}{\partial x^{'}}$

Using the above results, the material derivative transforms to :

$$\begin{aligned}\frac{D\nu_t}{Dt} &= \frac{\partial \nu_t}{\partial t} + U\frac{\partial \nu_t}{\partial x} + V\frac{\partial \nu_t}{\partial y} + W\frac{\partial \nu_t}{\partial z} \\ &= \left(\frac{\partial}{\partial t'} - a\frac{\partial}{\partial x'}\right)\nu_t + (U' + a)\frac{\partial \nu_t}{\partial x'} + V'\frac{\partial \nu_t}{\partial y'} + W'\frac{\partial \nu_t}{\partial z'} \\ &= \frac{\partial \nu_t}{\partial t'} + U'\frac{\partial \nu_t}{\partial x'} + V'\frac{\partial \nu_t}{\partial y'} + W'\frac{\partial \nu_t}{\partial z'} \\ &= \frac{D\nu_t}{Dt'}\end{aligned} \tag{4.5}$$

Hence, the material derivative of ν_t does not change its form in two different inertial frames. This shows that it is Galilean invariant.

It is to be noted that in all of the following arguments, wherever it is mentioned that a particular term is Gallilean invariant, it is meant to be taken that after doing the Galilean transformation similar to that as shown by the above exercise, the form of that particular term remains the same.

4.2.2 Production term

1. **Physics** : The production term of turbulent energy equation shows that the turbulent energy must increase monotonically with the magnitude of the deformation tensor $\frac{\partial U_i}{\partial x_j}$. Hence, by analogy, ν_t must increase monotonically with $\frac{\partial U_i}{\partial x_j}$. Also, ν_t must increase with the increasing level of turbulent agitation and hence with ν_t itself. Now, as the deformation tensor is a 2^{nd} order tensor and any 2^{nd} order tensor can be decomposed into a symmetric and an anti-symmetric form, we can write :

$$\frac{\partial U_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) \tag{4.6}$$

The 1^{st} term of the RHS, i.e. the symmetric form, represents the mean strain rate tensor, i.e.,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{4.7}$$

whereas the 2^{nd} term, i.e. the anti-symmetric form, represents the vorticity tensor, i.e.,

$$\Omega ij = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) \tag{4.8}$$

Now, as the SA model is basically meant for aerodynamic flows and in such flows turbulence is found only where the vorticity is, the vorticity tensor is selected for the production

term. However, the whole deformation tensor or the mean strain rate tensor can also be used in the production term.

2. **Scalar form** : Now, as the production term must have the vorticity tensor $\frac{\partial U_i}{\partial x_j}$, and as the term must be scalar, scalar norm, denoted by S, of the vorticity tensor is used.

$$\therefore S = \sqrt{\Omega_{ij}\Omega_{ij}} \tag{4.9}$$

Likewise, $\sqrt{S_{ij}S_{ij}}$ or $\sqrt{\frac{\partial U_i}{\partial x_j}\frac{\partial U_i}{\partial x_j}}$ can also be used.

3. **Dimensional homogeneity** : The formed production term $S\nu_t$ has the dimesions of $[L^2T^{-2}]$ which are consistent with the dimensions of the term on LHS.

4. **Galilean invariance** : The deformation tensor and hence the vorticity tensor is Galilean invariant. As a result, the production term is Galilean invariant.

Now, as all the requirements for the formulation of production term are satisfied, the final form of the Production term is given by :

$$Production = c_{b1}S\nu_t \tag{4.10}$$

where c_{b1} is the constant which gives flexibilty to tune the production term so as to match the pre-established simulation or the experimental results.

4.2.3 Diffusion term

1. **Physics** : The diffusion term is of the form : $-\nabla \cdot \phi_D$, where ϕ_D is the flux of ν_t due to its diffusion.

   Since diffusion like processes are driven by the gradients of the concentration field, the diffusion term must have gradient of ν_t.

   Drawing an analogy with the Fourier's law of heat conduction and Fick's law of mass diffusion, the general form of flux of any scalar quantity F due to diffusion can be given by :

$$\phi_D = -D_F \nabla F \tag{4.11}$$

   where D_F is the coefficient of diffusion for the quantity F. Here, ν_t is transportable quantity i.e. $F = \nu_t$. Now, as the turbulent motion diffuses by itself, the coefficient of diffusion is assumed to be ν_t itself. Hence, the diffusion term takes the form : $\nabla \cdot (\nu_t \nabla \nu_t)$. A turbulent Prandtl number σ is introduced in the diffusion term so as to have a better control over it. As a result, the diffusion term becomes

Diffusion term : $\nabla \cdot ([\nu_t/\sigma]\, \nabla \nu_t)$.

The above form of the diffusion term conserves the integral of ν_t. However, it is not a necessary condition that integral of ν_t be conserved as the prevalent turbulence models like standard k - ϵ uses diffusion terms that are not conservative. So, going by this analogy, a non-conservative term, like that in the standard K - ϵ model, is added. So, the diffusion term becomes

Diffusion term : $\frac{1}{\sigma}\left[\nabla \cdot (\nu_t \nabla \nu_t) + c_{b2}\left(\nabla \nu_t\right)^2\right]$

where c_{b2} is a constant.

2. **Scalar form, Dimensional homogeneity and Galilean invariance** : The form presented above is scalar and has dimensions of $[L^2T^{-2}]$. Hence, it is dimensionally homogeneous with other terms in the equations. Also, it is Galilean invariant as well.

In general, the transport of conserved quantities from regions of high concentration into regions of low concentration may be caused by random molecular motion or turbulence. Molecular diffusion represents the natural tendency of a physical system towards an equilibrium, whereas turbulent diffusion is due to unresolved eddies that enhance the macroscopic mixing rate. The corresponding mathematical models look the same but the coefficients differ by orders of magnitude. In what follows in the subsequent sections, both molecular and turbulent mixing will be referred to as diffusion. A note is to be taken that in the SA model for high Re no. near wall flows, only the turbulent mixing is considered i.e. diffusion due to ν_t. However, for the SA model for low Re no. near wall flows, the molecular mixing is also considered along with the turbulent mixing.

4.2.4 Destruction term

Following sections gives the formulation of the destruction term as applied to two cases. The first case is for near wall flows at high Re. number whereas the second case is for near wall flows at low Re no.

Destruction term for near wall flows with high Reynolds number

For high Re no. flows, the near wall behaviour of the flow is descibed by the law of wall and log layer is the portion of the turbulent BL where the law of wall accurately represents the velocity. So, for modeling high Re no. flows near the wall, log layer needs to be accurately produced. Also, skin friction coefficient is another important parameter which needs to be accurately calculated.

In order to produce a log layer and calculate the skin friction coefficient accurately a destruction term is used in the SA model. The destruction term remains active only near the wall and far away it goes to zero due to the way it has been formulated. It represents the destruction of the

turbulent kinematic viscosity due to high viscous dissipation in the proximity of solid walls. The modeled destruction term should be able to do two things :

1. Accurately produce the log layer
2. Accurately predict the skin-friction coefficient

The Destruction term is formulated based on the following arguments :

1. **Physics** : The rate of decay of turbulent energy to a very rough approximation is inversely proportional to the square of the energy itself , i.e.,

$$\frac{d\overline{u'^2}}{dt} \propto -\left(\overline{u'^2}\right)^2 \tag{4.12}$$

   Drawing an analogy of the turbulent kinematic viscosity with the turbulent energy, the decay of ν_t can be given by

$$\frac{d\nu_t}{dt} \propto -\nu_t^2 \tag{4.13}$$

$$\frac{d\nu_t}{dt} = -constant * \nu_t^2 \tag{4.14}$$

2. **Dimensional homogeneity, Scalar form and Galilean invariance** : Now, inorder to make the constant of proportionality a non-dimensional universal constant and the destruction term dimensionally homogeneous with other terms in the equation, square of a length scale d is introduced such that the destruction term becomes

$$\textbf{Destruction term :} - c_{b1}\left(\frac{\nu_t}{d}\right)^2 \tag{4.15}$$

   where d is the distance from the wall and c_{w1} is an universal dimensionless constant. The dependence of the decay term on the distance from the wall is quite necessary to account for the high rate of dissipation near the wall. The destruction term thus formulated is scalar and is Galilean invariant. However, when it is taken this way, it accurately produces a log layer but the skin friction coefficient calculated is not accurate and is too low. This indicates that the destruction term decays too slowly in the outer region of the BL. So, to calculate skin friction accurately as well, a non-dimensional function f_w is introduced.

Choosing proper fw function The choice of f_w is inspired by Prandtl's mixing length hypothesis where the mixing length plays a major role near the wall. SA introduced a non-dimensional argument :

$$r = \frac{l^2}{\kappa^2 d^2} \tag{4.16}$$

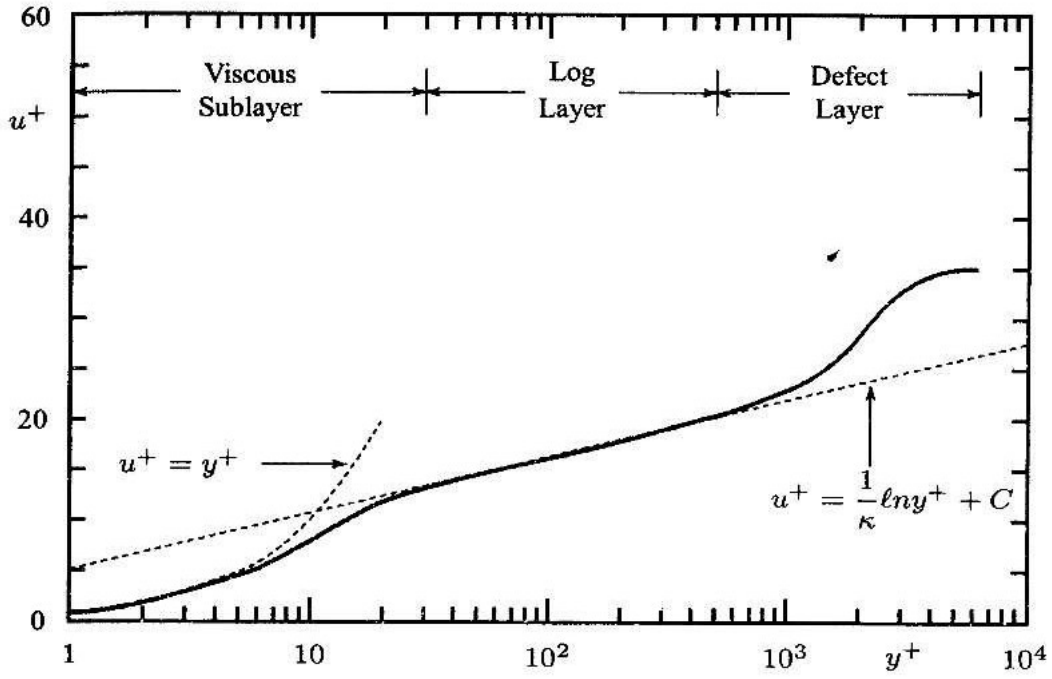

Figure 4.1: Typical velocity profile for a Turbulent Boundary layer [3]

where l = mixing length and κ is the $Kármán$ constant. Now, from the Prandtl's mixing length hypothesis,

$$l = \sqrt{\frac{\nu_t}{S}} \tag{4.17}$$

$$\therefore r = \frac{\nu_t}{S\kappa^2 d^2} \tag{4.18}$$

Both r and f_w equal 1 in the log layer and decrease in the outer region. Now, in a classical log layer with friction velocity u_τ, we have

$$S = \frac{u_\tau}{\kappa d} \quad and \quad \nu_t = u_\tau \kappa d \tag{4.19}$$

$$\begin{aligned} \therefore Destruction \;\; term &= -c_{w1} f_w \left(\frac{\nu_t}{d}\right)^2 \\ &= -c_{w1} f_w \left(\frac{u_\tau \kappa d}{d}\right)^2 \\ &= -c_{w1} f_w u_\tau^2 \kappa^2 \end{aligned} \tag{4.20}$$

Any dimesionally correct function of (ν_t, d, S) that reduces to $-c_{w1} f_w u_\tau^2 \kappa^2$ in a log layer can be chosen for the destruction term.

In the model, f_w is defined as

$$f_w(r) = g\left[\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right]^{1/6} \quad and \quad g = r + c_{w2}(r^6 - r) \tag{4.21}$$

Here g is used as the limiter which makes f_w to saturate to 2 for values of r beyond 1.2.

Destruction term for near wall flows with low Reynolds number

For the model discussed earlier for near wall flows with high Re no., the eddy viscosity ν_t equals $\kappa y u_\tau$ in the log layer but not in the buffer layer and viscous sublayer. So, to model the buffer layer and viscous sub-layer, $\tilde{\nu}$ is introduced which equals ν_t except in the turbulent BL. A quantity $\chi = \frac{\tilde{\nu}}{\nu}$ is defined so that $\tilde{\nu}$ retains its log layer profile i.e. $\tilde{\nu} = \kappa y u_\tau$ upto the wall. $\tilde{\nu}$ behaves linearly near the wall.

The near wall viscous effects in the buffer layer and the viscous sub-layer is modeled by a damping function of the form

$$\nu_t = \tilde{\nu} f_{v1} \quad and \quad f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3} \tag{4.22}$$

c_{v1} is a constant to be chosen for matching the results obtained using this model with that of pre-established simulation and experimental results. The function f_{v1} brings the effect of viscosity right upto the wall. Also, the production term is modified as

$$\tilde{S} = S + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2} \quad and \quad f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}} \tag{4.23}$$

f_{v2} is constructed so that $\tilde{S}$ would maintain its log layer behaviour i.e. $\tilde{S} = \frac{u_\tau}{\kappa y}$ all the way to the wall. Other quantities involved in the inviscid model are also redefined in terms of $\tilde{\nu}$ and $\tilde{S}$ instead of ν_t and S respectively.

A viscous diffusion term, consistent with the Dirichlet boundary condition at the wall, $\tilde{\nu} = 0$, is added together with the turbulent diffusion term. Thus, the final form of the equation is :

$$\frac{D\tilde{\nu}}{Dt} = c_{b1}\tilde{S}\tilde{\nu} + \frac{1}{\sigma}\left[\nabla \cdot ((\nu + \tilde{\nu})\nabla\tilde{\nu}) + c_{b2}(\nabla\tilde{\nu})^2\right] - c_{w1} f_{w1} \left(\frac{\tilde{\nu}}{d}\right)^2 \tag{4.24}$$

4.2.5 Calibration of the constants

SA model has some constants and non-dimensional functions. The procedure for evaluating and defining the appropriate form for these constants and non dimensional functions respectively is called calibration. In the process of calibration, constants and functions are defined with the help of experimental and numerical results of the type of a flow that is to be modeled. The basic assumptions made for formulating each term are also helpful to recognize the range of constants. In the previous sections, the form for each of the non-dimensional functions is explained and now the constants needs to be determined.

The calibration of the constants c_{b1}, c_{b2} and σ for the inviscid model is done using the peak stress values in two dimensional mixing layers and wakes. Peak shear stress $= 0.01(\Delta U)^2$ in

mixing layer and $0.06(\Delta U)^2$ in wake where $\Delta U =$ peak velocity difference. Since 3 constants are there which can be tuned to match with 2 values of the peak shear stress, one constant, i.e., σ is chosen freely and the other two constants i.e. c_{b1} and c_{b2} are tuned during the numerical simulations. Based on the observations in the numerical simulation results, SA suggested the values of the constants to be : $\sigma = 2/3$, $c_{b1} = 0.1355$ and $c_{b2} = 0.6220$

The value of the $Kármán$ constant κ is taken to be 0.41. The constant c_{w1} is calculated from establishing an equilibrium between the production and diffusion terms and the destruction term as a result of which the condition on c_{w1} becomes

$$c_{w1} = \frac{c_{b1}}{\kappa^2} + \frac{1 + c_{b2}}{\sigma}$$

c_{w3} is taken to be 2. $c_{w2}$is calibrated to match the skin friction coefficient in a flat plate boundary layer. From the simulation results, c_{w2} is calibrated to 0.3. The constant c_{v1} is taken to be 7.1. This completes the calibration of the model.

4.2.6 Choosing the freestream values of ν_t

SA model accepts zero values of ν_t in the free stream. One of the biggest advantages of this model over the Baldwin-Barth model and many other 2-equation models is that it is insensitive to non-zero free-stream values of ν_t provided they are much smaller than the values in the turbulent region. The production term depends on the gradients of the velocity field and inside the BL and thin Shear layers (TSL), velocity gradients are very high. Thus, the production of ν_t will be more inside the BL and TSL's compared to the outside regions. Hence small free stream values of ν_t would increase more only inside the BL and TSL's and not outside them. ν_t will grow to such values inside the BL and TSL that the non-linear terms like diffusion and destruction terms will become active there.

4.3 Summary

Although zero-equation models are the simplest, they are not local and do not give accurate results for SBLI and for conditions where flow separation occurs. The two-equation models also do not show any significant advantages over one-equation models for such computations. A sacrifice in terms of computational accuracy can always be made for computational effort required. So, one-equations turbulence models are studied in this work, especially SA model because of its many advantages over other fellow one-equation models. An in-depth presentation of the development of SA model is made term-by-term. The final version of the SA model presented here gives fairly accurate results for incompressible and weakly compressible flows.

Chapter 5

Compressible formulation of the SA model

In the original SA paper [2], the SA model was calibrated for free shear flows and flows near the wall with high and low Reynolds number. It gives good results for incompressible flows. But in cases of flows with high Mach numbers, e.g. supersonic or hypersonic flows, the original form is not suitable to use. To add the compressibility effects, we have to modify the original form of the SA model. Also, depending on the requirements and conditions of the flow and computations, the model can be calibrated by adding new terms or modifying the already existing terms. After the original model was proposed, several researchers have modified or corrected the SA model depending upon their computation requirements. Below sections present the compressible form of the SA model and some other modifications which are relevant to our applications.

5.1 Compressible form of the SA model

In case of flows where compressibility effects are important, we must account for density and temperature fluctuations in the flow field in addition to velocity and pressure fluctuations. Also, we need to introduce conservation of energy and an equation of state along with conservation of mass and momentum. For getting density and temperature fluctuation effects, if we follow conventional Reynolds time averaging, then we end up with equations having such correlations, like triple correlations between $\rho^{'}$, $u_i^{'}$ and $u_j^{'}$, for which suitable closure approximations are very hard to establish. So, in order to reduce the complexity of the process of establishing appropriate form of the time-averaged equations, Favre in 1965 introduced density-weighted averaging also called as **Favre averaging** and the resulting equations are called Favre-averaged NS equations (FANS). The mass-averaged velocity,$\tilde{u_i}$, is defined as

$$\tilde{u_i} = \frac{1}{\overline{\rho}} \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \rho(\mathbf{x},\tau) u_i(\mathbf{x},\tau) d\tau \tag{5.1}$$

where $\overline{\rho}$ is the conventional Reynolds-averaged density. For Favre-averaging, various flow properties are decomposed as follows.

$$\begin{aligned} u_i &= \tilde{u_i} + u_i^{''} \\ \rho &= \overline{\rho} + \rho^{'} \\ p &= P + p^{'} \\ h &= \tilde{h} + h^{''} \\ e &= \tilde{e} + e^{''} \\ T &= \tilde{T} + T^{''} \\ q_j &= q_{L_j} + q_j^{'} \end{aligned} \tag{5.2}$$

where h = specific enthalpy, e = specific internal energy, T = temperature and q_j = heat flux.

Some important results for Favre averaging

$$\begin{aligned} \overline{\rho}\tilde{u_i} = \overline{\rho u_i} &= \overline{\rho} U_i + \overline{\rho^{'} u_i^{'}} \\ \overline{\rho u_i^{''}} &= 0 \\ \overline{u_i^{''}} &\neq 0 \end{aligned} \tag{5.3}$$

5.1.1 Favre-averaged Equations

Conservation of Mass

The conservation of mass equation for compressible flows is given by,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \tag{5.4}$$

Decomposing the flow variables and taking the mass average we get,

$$\overline{\frac{\partial(\overline{\rho} + \rho^{'})}{\partial t}} + \overline{\frac{\partial(\overline{\rho} + \rho^{'})(\tilde{u_i} + u_i^{''})}{\partial x_i}} = 0 \tag{5.5}$$

Making use of the properties of Reynolds averaging and Favre averaging given by Equation [5.3], we get

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\overline{\rho}\tilde{u_i}) = 0 \tag{5.6}$$

Equation [5.6] is the Favre-averaged Mass conservation equation.

Conservation of Momentum

The conservation of momentum equation is given by,

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i) = -\frac{\partial p}{\partial x_i} + \frac{\partial t_{ji}}{\partial x_j} \tag{5.7}$$

Again following the same procedure as that followed for arriving at Favre-averaged Mass-conservation equations, i.e. decomposing the flow variables as given by Equation [5.2] and then taking mass-average, we get the Favre-averaged Momentum Equation given by,

$$\frac{\partial}{\partial t}(\overline{\rho}\tilde{u}_i) + \frac{\partial}{\partial x_j}(\overline{\rho}\tilde{u}_j\tilde{u}_i) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\overline{t_{ji}} - \overline{\rho u_j'' u_i''}\right] \tag{5.8}$$

Energy Conservation Equation

The energy conservation equation is given by,

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{1}{2}u_i u_i\right)\right] + \frac{\partial}{\partial x_j}\left[\rho u_j\left(h + \frac{1}{2}u_i u_i\right)\right] = \frac{\partial}{\partial x_j}(u_i t_{ij}) - \frac{\partial q_j}{\partial x_j} \tag{5.9}$$

Again following the same procedure as that followed for arriving at Favre-averaged Mass and Momentum conservation equations, we get the Favre-averaged Energy Equation given by,

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\overline{\rho}\left(\tilde{e} + \frac{1}{2}\tilde{u}_i\tilde{u}_i\right) + \frac{\overline{\rho u_i'' u_i''}}{2}\right] + \frac{\partial}{\partial x_j}\left[\overline{\rho}\tilde{u}_j\left(\tilde{h} + \frac{1}{2}\tilde{u}_i\tilde{u}_i\right) + \tilde{u}_j\frac{\overline{\rho u_i'' u_i''}}{2}\right] \\ &= \frac{\partial}{\partial x_j}\left[-q_{L_j} - \overline{\rho u_j'' h''} + \overline{t_{ji}u_i''} - \overline{\rho u_j''\frac{1}{2}u_i'' u_i''}\right] + \frac{\partial}{\partial x_j}\left[\tilde{u}_i\left(\overline{t_{ij}} - \overline{\rho u_i'' u_j''}\right)\right]\end{aligned} \tag{5.10}$$

Also, the equation of state given by $p = \rho RT$ on Favre-averaging becomes

$$P = \overline{\rho}R\tilde{T} \tag{5.11}$$

All terms appearing in these equations are closed except for those resulting from turbulent fluctuations. In particular, the unclosed terms are as follows:

1. the Reynolds stress tensor $(-\overline{\rho u_j'' u_i''})$
2. Turbulent Heat flux $(\overline{\rho u_j'' h''})$
3. Molecular diffusion of turbulence KE $(\overline{t_{ji}u_i''})$
4. Turbulent transport of turbulence KE $(\overline{\rho u_j''\frac{1}{2}u_i'' u_i''})$, and
5. Turbulence KE $(\frac{1}{2}\overline{\rho u_i'' u_i''})$

Common Closure Approximations

The above unclosed terms in the FANS equations are closed using the following closure approximations.

Reynolds-stress tensor : The Reynolds stress tensor appears in both the Favre-averaged momentum [5.8] and energy [5.10] equations. It is closed using Boussinesq approximation given below.

$$\overline{\rho}\tau_{ij} = -\overline{\rho u_j'' u_i''} = 2\mu_T S_{ij} \tag{5.12}$$

where μ_T is the eddy viscosity.

Turbulence Heat-flux The most commonly used closure approximation for the turbulent heat flux vector, q_{T_j}, follows from the analogy between momentum and heat transfer, given by

$$q_{T_j} = \overline{\rho u_j'' h''} = -k_T \frac{\partial \tilde{T}}{\partial x_j} \tag{5.13}$$

where k_T is the eddy thermal conductivity. Now

$$k_T = \frac{\mu_T c_p}{Pr_T} \tag{5.14}$$

where Pr_T is the turbulent Prandtl number. Thus, in terms of Favr-averaged enthalpy, the closure approximation is

$$q_{T_j} = -\frac{\mu_T}{Pr_T} \frac{\partial \tilde{h}}{\partial x_j} \tag{5.15}$$

Turbulence KE Generally for SA model, TKE is neglected i.e k = 0 as we don't have any means to compute it.

Molecular Diffusion and Turbulent transport These terms appears both in Favre-Averaged Energy equation and the Favre-Averaged TKE (See [3]). As SA model neglects TKE, its a general practice that these terms are also neglected.

Using the above discussed closure approximation, the Equations [5.6], [5.8] and [5.10] becomes

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\overline{\rho}\tilde{u}_i) = 0 \tag{5.16}$$

$$\frac{\partial}{\partial t}(\overline{\rho}\tilde{u}_i) + \frac{\partial}{\partial x_j}(\overline{\rho}\tilde{u}_j\tilde{u}_i) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2(\overline{\mu} + \mu_T)\tilde{S}_{ij}\right) \tag{5.17}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\overline{\rho}\left(\tilde{e} + \frac{1}{2}\tilde{u}_i\tilde{u}_i\right)\right] + \frac{\partial}{\partial x_j}\left[\overline{\rho}\tilde{u}_j\left(\tilde{h} + \frac{1}{2}\tilde{u}_i\tilde{u}_i\right)\right] \\ &= \frac{\partial}{\partial x_j}\left[-q_{L_j} - q_{T_j}\right] + \frac{\partial}{\partial x_j}\left[2(\overline{\mu} + \mu_T)\tilde{S}_{ij}\tilde{u}_i\right] \end{aligned} \tag{5.18}$$

In the above FANS equations, the only unknown is the eddy viscosity. SA model gives a way to calculate it. Below sections describe the compressible forms of the SA model.

5.1.2 Compressible form by Allmaras et al.

For taking into consideration the compressibility effects, Allmaras [4] suggests a conservation form of the SA model by combining the SA model equation with mass conservation equation given by,

$$\overline{\rho} * (\text{SA model}) + \tilde{\nu} * (\text{mass-conservation}) = 0 \tag{5.19}$$

Now,

$$\text{SA model} = \frac{\partial \tilde{\nu}}{\partial t} + \tilde{u}_i \frac{\partial \tilde{\nu}}{\partial x_i} - P + D - \frac{1}{\sigma}\nabla \cdot [(\overline{\nu} + \tilde{\nu})\nabla\tilde{\nu}] - \frac{c_{b2}}{\sigma}(\nabla\tilde{\nu})^2 = 0 \tag{5.20}$$

where $\overline{\nu}$ is the Favre-averaged kinematic molecular viscosity, P = Production term and D = Destruction term, and

$$\text{mass-conservation} = \frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho}\tilde{u}_i}{\partial x_i} = 0 \tag{5.21}$$

Combining the above two equations, Equation [5.19] becomes

$$\begin{aligned} 0 &= \overline{\rho} * \left[\frac{\partial \tilde{\nu}}{\partial t} + \tilde{u}_i \frac{\partial \tilde{\nu}}{\partial x_i} - P + D - \frac{1}{\sigma}\nabla \cdot [(\overline{\nu} + \tilde{\nu})\nabla\tilde{\nu}] - \frac{c_{b2}}{\sigma}(\nabla\tilde{\nu})^2\right] \\ &\quad + \tilde{\nu} * \left[\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho}\tilde{u}_i}{\partial x_i}\right] \\ &= \overline{\rho}\frac{\partial \tilde{\nu}}{\partial t} + \overline{\rho}\tilde{u}_i \frac{\partial \tilde{\nu}}{\partial x_i} - \overline{\rho}(P - D) - \frac{\overline{\rho}}{\sigma}\nabla \cdot [(\overline{\nu} + \tilde{\nu})\nabla\tilde{\nu}] - \frac{c_{b2}}{\sigma}\overline{\rho}(\nabla\tilde{\nu})^2 \\ &\quad + \tilde{\nu}\frac{\partial \overline{\rho}}{\partial t} + \tilde{\nu}\frac{\partial \overline{\rho}\tilde{u}_i}{\partial x_i} \end{aligned} \tag{5.22}$$

Now,

$$\overline{\rho}\frac{\partial \tilde{\nu}}{\partial t} + \tilde{\nu}\frac{\partial \overline{\rho}}{\partial t} = \frac{\partial \overline{\rho}\tilde{\nu}}{\partial t} \tag{5.23}$$

Also,

$$\overline{\rho}\tilde{u}_i \frac{\partial \tilde{\nu}}{\partial x_i} = \frac{\partial}{\partial x_i}(\overline{\rho}\tilde{u}_i\tilde{\nu}) - \tilde{\nu}\frac{\partial(\rho\tilde{u}_i)}{\partial x_i} \tag{5.24}$$

Now, if ϕ is a scalar field and **F** is a vector field, then

$$\nabla \cdot (\phi\mathbf{F}) = \nabla\phi \cdot \mathbf{F} + \phi\nabla \cdot \mathbf{F} \tag{5.25}$$

Using the above product rule, we can write

$$\overline{\rho}\nabla \cdot [(\overline{\nu} + \tilde{\nu})\nabla\tilde{\nu}] = \nabla \cdot [\overline{\rho}(\overline{\nu} + \tilde{\nu})\nabla\tilde{\nu}] - (\overline{\nu} + \tilde{\nu})\nabla\overline{\rho} \cdot \nabla\tilde{\nu} \tag{5.26}$$

Using the above equations, Equation [5.22] becomes

$$\begin{aligned} 0 &= \frac{\partial \overline{\rho}\tilde{\nu}}{\partial t} + \nabla \cdot (\overline{\rho}\tilde{u}_i\tilde{\nu}) - \overline{\rho}(P - D) - \frac{1}{\sigma}\nabla \cdot [\overline{\rho}(\overline{\nu} + \tilde{\nu})\nabla\tilde{\nu}] \\ &\quad + \frac{1}{\sigma}(\overline{\nu} + \tilde{\nu})\nabla\overline{\rho} \cdot \nabla\tilde{\nu} - \frac{c_{b2}}{\sigma}\overline{\rho}(\nabla\tilde{\nu})^2 \end{aligned} \tag{5.27}$$

Thus, the final compressible form of the SA model which needs to be solved along with the FANS equations is,

$$\frac{\partial \overline{\rho}\tilde{\nu}}{\partial t} + \nabla \cdot (\overline{\rho}\tilde{u}_i\tilde{\nu}) = c_{b1}\overline{\rho}\tilde{S}\tilde{\nu} + c_{w1}f_w\overline{\rho}\left(\frac{\tilde{\nu}}{d}\right)^2 - \frac{1}{\sigma}\nabla \cdot [\overline{\rho}(\overline{\nu}+\tilde{\nu})\nabla\tilde{\nu}] + \frac{1}{\sigma}(\overline{\nu}+\tilde{\nu})\nabla\overline{\rho}\cdot\nabla\tilde{\nu} - \frac{c_{b2}}{\sigma}\overline{\rho}(\nabla\tilde{\nu})^2 \tag{5.28}$$

In the above formulation, there's a term depending on the density gradients which is not there in the original SA model. This term brings in the compressibility effects.

5.1.3 Compressible form by Catris et al.

Beyond Mach number 5, the original SA model poorly predicts the logarithmic region of the turbulent BL and skin friction coefficient in presence of large density gradients. So, in order to improve its predictions at high Mach nos., Catris et al in [5] suggested a compressible form of the SA model. The compressible form was formulated so as to get good agreement with the compressible turbulent BL. The compressibility corrections are active only when the turbulence motion is compressible i.e. for external Mach no. greater than 5. Only zero pressure gradient (ZPG) BL is considered to emphasize the compressibility effects.

The near wall region compressible equation, at high turbulent Reynolds number is:

$$\frac{D\overline{\rho}\tilde{\nu}}{Dt} = c_{b1}\overline{\rho}\tilde{S}\tilde{\nu} + \nabla\cdot\left[\frac{1}{\sigma}(\overline{\mu}+\overline{\rho}\tilde{\nu})\nabla\tilde{\nu}\right] + \frac{c_{b2}}{\sigma}\overline{\rho}(\nabla\tilde{\nu})\cdot(\nabla\tilde{\nu}) - c_{w1}f_w\overline{\rho}\left(\frac{\tilde{\nu}}{d}\right)^2 \tag{5.29}$$

Catris et al. [5] suggested that for improving the predictions of the logarithmic layer of a compressible BL in presence of large density variations, the diffusion terms in the transport equation must be modified. In their modified SA equation, the transported quantity remains $\overline{\rho}\tilde{\nu}$, but the diffused quantity becomes $\sqrt{\overline{\rho}}\tilde{\nu}$. With this taken into consideration, the SA equation [5.29] gets modified as

$$\frac{D\overline{\rho}\tilde{\nu}}{Dt} = c_{b1}\overline{\rho}\tilde{S}\tilde{\nu} + \frac{1}{\sigma}\nabla\cdot(\overline{\mu}\nabla\tilde{\nu}) + \frac{1}{\sigma}\nabla\cdot(\sqrt{\overline{\rho}}\tilde{\nu}\nabla(\sqrt{\overline{\rho}}\tilde{\nu})) + \frac{c_{b2}}{\sigma}(\nabla(\sqrt{\overline{\rho}}\tilde{\nu}))\cdot(\nabla(\sqrt{\overline{\rho}}\tilde{\nu})) - c_{w1}f_w\overline{\rho}\left(\frac{\tilde{\nu}}{d}\right)^2 \tag{5.30}$$

For this model, the model function $f_{v2} = 0$. The definitions and values of all the terms in the equation remains the same as defined in Chapter 4.

5.2 Strain-Adaptive formulation

To extend the predictive capability of the SA model in the non-equilibrium conditions, Rung et al. in [6] presented a strain adaptive formulation of the SA model. The strain-adaptive formulation involves sensitizing the production term to non-equilibrium effects. This form is meant for weakly compressible media with negligible density fluctuations. For this form of the SA model, the Boussinesq eddy-viscosity hypothesis is taken as

$$-\overline{u_i' u_j'} = \nu_t S_{ij}' - \frac{2}{3} k \delta_{ij} \tag{5.31}$$

where $-\overline{u_i' u_j'}$ is the Reynolds stress and k is the turbulent Kinetic energy given by

$$k = \frac{S^* \nu_t}{\sqrt{c_\mu}} \tag{5.32}$$

where $c_\mu = 0.09$

All other terms and model coefficients for this variant are same as the standard form of the SA model described in the Chapter 4 except for the following changes. The term

$$\frac{1}{\sigma} \nabla \cdot [(\nu + \tilde{\nu}) \nabla \tilde{\nu}]$$

is written as

$$\nabla \cdot \left[\left(\nu + \frac{\tilde{\nu}}{\sigma} \right) \nabla \tilde{\nu} \right]$$

Also,

$$\tilde{S} = S^* \left[\frac{1}{\chi} + f_{v1} \right] \tag{5.33}$$

$$r = 1.6 tanh \left[0.7 \sqrt{\frac{\rho_o}{\rho}} \left(\frac{\tilde{\nu}}{\tilde{S} \kappa^2 d^2} \right) \right] \tag{5.34}$$

where ρ_o is the free-stream stagnation density and

$$S^* = \sqrt{2 S_{ij}' S_{ij}'} \tag{5.35}$$

where

$$S_{ij}' = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{1}{3} \frac{\partial U_k}{\partial x_k} \delta_{ij} \tag{5.36}$$

S_{ij}' is the deviatoric component of the strain rate tensor. It is trace-less. It is related to the shear distortion of the turbulent eddy. The sensitization to the non-equilibrium effects comes in through a change in the production term, mainly c_{b1}. Rung et al follows Menter's aproach in defining the production term. Menter in [7] transformed standard k-ϵ model into an one-equation turbulence model by following the assumption that the turbulent shear stress is pro-

portional to the turbulent K.E., i.e.

$$-\overline{u_i' u_j'} = \nu_t S^* = a_1 k \tag{5.37}$$

where a_1 is a constant of proprtionality and its value is taken to be equal to $\sqrt{c_\mu}$. This assumption, also called as the Bradshaw's relation, is confirmed for non-equilibrium adverse pressure gradient flows by large number of experiments. Now, the eddy-viscosity ν_t is defined as

$$\nu_t = c_\mu \frac{k^2}{\epsilon} \tag{5.38}$$

where ϵ is the dissipation. Combining equations [5.37] and [5.38], we get

$$\sqrt{c_\mu} = c_\mu \frac{kS^*}{\epsilon} \tag{5.39}$$

Menter's one-equation model derived from the standard k-ϵ model contains production term of the form

$$\begin{aligned} Production &= \nu_t S^* (C_{\epsilon 2} - C_{\epsilon 1}) \left(c_\mu \frac{S^* k}{\epsilon} \right) \\ &\approx \nu_t \tilde{S} c_{b1}^{'} \end{aligned} \tag{5.40}$$

where

$$c_{b1}^{'} = (C_{\epsilon 2} - C_{\epsilon 1}) \sqrt{c_\mu} \tag{5.41}$$

Thus, the coefficient $c_{b1}^{'}$ is very crucial to the model's predictive performance for non-equilibrium flows. By substituting $C_{\epsilon 1} = 1.45$ and $C_{\epsilon 2} = 1.9$ from the standard k-ϵ model, we get $c_{b1}^{'} = 0.135$ which is in close agreement with the original SA model for which the coefficient denoted by c_{b1} is equal to 0.1355. Thus the formulation of the production term and the coefficients in it is consistent with the original SA model. As indicated in the equations [5.39] and [5.41], the model coefficients $(C_{\epsilon 2} - C_{\epsilon 1})$ and c_μ and hence $c_{b1}^{'}$ is a function of the strain rate and model coefficients. In general, both model coefficients tend to decrease with an increase of strain, which motivated Rung et al. to the following modification of the standard coefficient c_{b1} as $c_{b1}^{'}$ given by $c_{b1}^{'} = 0.1355\sqrt{\Gamma}$ where

$$\Gamma = min\left[1.25, max(\gamma, 0.75)\right] \qquad \gamma = max(\alpha_1, \alpha_2)$$

$$\alpha_1 = \left[1.01 \frac{\tilde{\nu}}{S^* \kappa^2 d^2}\right]^{0.65} \qquad \alpha_2 = max\left[0, 1 - tanh\frac{\chi}{68}\right]^{0.65}$$

The Production term thus changes from $c_{b1}\tilde{S}\tilde{\nu}$ to $c_{b1}^{'}\tilde{S}\tilde{\nu}$ and c_{w1} changes from

$$c_{w1} = \frac{c_{b1}}{\kappa^2} + \frac{1 + c_{b2}}{\sigma}$$

to

$$c_{w1} = \frac{c_{b1}^{'}}{\kappa^2} + \frac{1 + c_{b2}}{\sigma}$$

The modification $\sqrt{\Gamma}$ primarily causes a reduction of production for excessive strains via α_1.

5.3 Mixing layer compressibility correction

In compressible mixing layers, SA model is considerably improved by a correction due to Secundov [8]. A term

$$- C_5 \tilde{\nu}^2 \frac{U_{i,j} U_{i,j}}{a^2} \tag{5.42}$$

is added to $\frac{D\tilde{\nu}}{Dt}$, where a is the speed of sound and empirically $C_5 = 3.5$. It can be shown that the term is based on a Turbulent Mach number M_t. M_t is the measure of the compressibility of fluctuating turbulence. It is defined as

$$M_t^2 = \frac{2k}{a^2} \tag{5.43}$$

where k is the TKE. From mixing length theory, we have

$$\tilde{\nu} \sim v_{mix} l_{mix} \tag{5.44}$$

Also,

$$v_{mix} \sim l_{mix} S \tag{5.45}$$

where S is the magnitude of the deformation tensor $\frac{\partial U_i}{\partial x_j} = U_{i,j}$ given by

$$S = \sqrt{U_{i,j} U_{i,j}} \tag{5.46}$$

From these relations, we can write

$$\tilde{\nu} \sim l_{mix}^2 S \tag{5.47}$$

This gives,

$$l_{mix} \sim \sqrt{\frac{\tilde{\nu}}{S}} \tag{5.48}$$

Therefore,

$$v_{mix} \sim \sqrt{\tilde{\nu} S} \tag{5.49}$$

Now, TKE can be written as

$$k \sim v_{mix}^2 \tag{5.50}$$

$$\therefore k \sim \tilde{\nu} S \tag{5.51}$$

Using this, we can write M_t as

$$M_t^2 \sim \frac{\tilde{\nu} S}{a^2} \tag{5.52}$$

Thus, the correction term in Equation [5.42] can be written in terms of M_t as

$$\text{Correction term} \sim -\tilde{\nu} S M_t^2 \tag{5.53}$$

The above analysis based on the mixing length hypothesis shows that the correction term depends on the turbulent Mach number M_t. From this relation it can be seen that, correction term is nothing but the negative of the Production term of the SA model multiplied by the square of turbulent Mach number. Thus, for high M_t, this term supresses the production of ν_T and hence the turbulent fluctuations.

All other terms in the SA model remains the same. This correction can be important for studies of supersonic cavities and blunt bases such as on missiles.

5.4 Other modifications

Apart from the above mentioned modifications, many other modifications have been done for getting better predictions for different flow conditions. Shur et al. in [9] modified the SA model to account for the system rotation and streamline curvature effects for rotating and curved channel flows. Dacles Mariani et al. in [10] and [11] also made modifications to account for the rotation effects. These modifications are simpler compared to [9]. These modifications reduces eddy viscosity in regions where vorticity exceeds strain rate, such as in vortex core regions where pure rotation should not produce turbulence. They are passive in thin shear layers where vorticity and strain are very close. Another modification to the model was done by Aupoix et al. [12] and Spalart [8] for predicting rough walls.

All these modifications improve the predictive capabilities of the SA model for different flow conditions and can be used independently or in conjunction with other modifications. Thus the SA model can be tailored as per the requirements of the computations to be done.

5.5 Summary

To extend the prediction capability of the standard SA model for compressible flows, Allmaras et al. and Catris et al. have suggested some modifications which are discussed in this chapter. Modifications for non-equilibrium conditions where the strains are very high and for compressible mixing layers are discussed in brief. In the end, some other relevant modifications are also discussed in brief.

Chapter 6

Conclusion and Future work

6.1 Conclusion

In spite of its relative simplicity compared to the two-equation models like k-ϵ or k-ω, the standard SA model gives fairly good results for incompressible and weakly compressible aerodynamic flows. The sacrifice in terms of getting somewhat less computational accuracy compared to the two-equation models for reducing the computational effort is worth giving a consideration. The model's predictive powers can be effectively improved with some modifications. The compressible formulation of the model extends its predictive capability to high speed compressible flows. For flows with non-equilibrium conditions e.g. shock waves where the strain rates are very high, the strain adaptive formulation is important. A modification which takes into account the compressibility effects along with high strain rates is needed.

6.2 Future Work

As the standard SA model yields inaccurate results for supersonic and hypersonic flows which involve shock waves, as seen in SBLI and SSI cases, modifications are required to improve its predictive capability in such cases. The following work will be done in the future with an emphasize to modify the model for supersonic and hypersonic flows.

* Numerical implementation of the standard SA model and its variants to understand the effect of each term on the solution.
* Formulation of a modification which takes into account high strain rates along with high density fluctuations.
* Detailed study of the modification for compressible mixing layers.

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