The mathematical modelling of turbulent flows

N. C. Markatos

Centre for Numerical Modelling and Process Analysis, School of Mathematics, Statistics and Computing, Thames Polytechnic, Wellington Street, London SE18 6PF, UK

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This paper reviews the problems and successes of computing turbulent flow. Most of the flow phenomena that are important to modern technology involve turbulence. Apart from pure academic interest, there is therefore a practical need for designers to be able to predict quantitatively the behaviour of turbulent flows. The review is concerned with methods for such computer predictions and their applications, and describes several of them. These computational methods are aimed at simulating either as much detail of the turbulent motion as possible by current computer power or, more commonly, its overall effect on the mean-flow behaviour. The methods are still being developed and some of the most recent concepts involved are discussed.

The basic points to be made are:

- Turbulence computations are needed for practical simulations of engineering, environmental, biomedical, etc. processes.
- Some success has been achieved with two-equation models for relatively simple hydrodynamic phenomena; indeed, routine design work can now be undertaken in several applications of engineering practice, for which extensive studies have optimized these models.
- Failures are still common for many applications particularly those that involve strong curvature, intermittency, strong buoyancy influences, low-Reynolds-number effects, rapid compression or expansion, strong swirl, and kinetically-influenced chemical reaction. New conceptual developments are needed in these areas, probably along the lines of actually calculating the principal manifestation of turbulence, e.g. intermittency. A start has been made in this direction in the form of 'multi-fluid' models, and full simulations.
- Although some of the latest concepts hold promise of describing some of the most important physical consequences of turbulence, they have not yet reached a definite stage of development. From this point of view, the older and simpler methods can still be recommended as the starting point (and sometimes the finishing point) for engineering simulation.

Despite the relative novelty of the subject, the relevant material is already too much to be reviewed in a single paper. For this reason the author confines attention to what he considers the better-established or more promising models. No disrespect is therefore implied for the models that are scarcely – or not at all – mentioned. Extensive use has been made of the published literature on the topic and in particular of two recent reviews by Reynolds and Cebeci[1] and by Kumar.[2] Extensive use is also made of the work of Spalding and of the recent work of Malin. Turbulent heat and mass transport are not explicitly covered in this review; the interested reader is directed to the review by Launder.[3] Further details on turbulence models may also be found in the lecture course by Spalding.[4] The review concludes with a summary of the advantages and disadvantages of the various turbulence models, in an attempt to assist the potential user in choosing the most suitable model for his particular problem.

Key words: turbulence, mathematical modelling, field models, stress equation, multi-scale, kinetic energy, dissipation, inhomogeneities, intermittency, large-eddy, time scales, effective viscosity

Introduction

The partial differential equations for turbulent physical systems

Turbulence is the most complicated kind of fluid motion, making even its precise definition difficult. A fluid motion is described as turbulent if it is rotational, intermittent, highly disordered, diffusive and dissipative. It is generally accepted[5] that turbulence can be described by the Navier-Stokes momentum-transport equations (the second-order Chapman-Enskog approximation to the Bolzmann equation for molecular motion), which express the conservation of momentum for a continuum fluid with viscous stress directly proportional to rate of strain. This description is the simplest that can be imagined. According to this principle, the 'Eulerian' equations governing the

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dynamics and heat/mass transfer of a turbulent fluid can be written as follows, in Cartesian tensor notation, and using the repeated-suffix summation convention:

- Mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0 \tag{1}$$

- Momentum conservation for the x_i-component of velocity, u_i, in 'divergence' form:

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} + F_i \tag{2}$$

- Scalar conservation (e.g. enthalpy, h, concentration, C, etc):

$$\frac{\partial \rho \phi}{\partial t} + \frac{\partial \rho \phi u_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\Gamma_\phi \frac{\partial \phi}{\partial x_j}\right) + \frac{\partial p}{\partial t} + u_i \frac{\partial p}{\partial x_i} + S_\phi \tag{3}$$

(where for $\phi \equiv C$, the second and third terms on the rhs should be excluded).

Here F_i is the x_i-component of body force (for instance in a gravitational field $F_i = \rho g_i$ where g_i is a component of the gravitational acceleration), ρ is the instantaneious density, ϕ is a scalar quantity, Γ_ϕ is the diffusion coefficient of ϕ, S_ϕ is the volumetric source/sink term, and σ_{ij} the stress tensor components (stress due to deformation and bulk dilatation). Equation (2) applies generally, whatever the constitutive law for σ_{ij}, even if the mean velocities in turbulent flow are concerned, provided that then σ_{ij} includes apparent turbulence (Reynolds) stresses.

For Newtonian fluids, the instantaneous deformation stress is:

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_i}\right) + (\mu^b - \tfrac{2}{3}\mu)\,\delta_{ij}\frac{\partial u_l}{\partial x_l} \tag{4}$$

where δ_{ij} is the Kronecker delta, and μ^b is the bulk viscosity of the same order as μ.[6] In the most general case $\partial \sigma_{ij}/\partial x_j$ is very complicated.[7,8] In Newtonian fluids $\sigma_{ij} = \sigma_{ji}$, so σ_{ij} is a diagonally-symmetric tensor. Finally, an equation of state (single-component fluid) relates pressure to density and temperature:

$$p \equiv p(\rho, T) \tag{5}$$

where ρ, T, are the instantaneous values of density and absolute temperature, respectively.

Fortunately, viscosity does not usually affect the larger-scale eddies which are primarily responsible for turbulent mixing, with the exception of the 'viscous sublayer' very close to a solid surface. Furthermore, the effects of density fluctuations on turbulence are small if, as in the majority of practical situations, the density fluctuations are small compared to the mean density, the exception being the effect of temporal fluctuations and spatial gradients of density in a gravitational field. Therefore, one can usually neglect the direct effect of viscosity and compressibility on turbulence. It is also important to note that it is the fluctuating velocity field that drives the fluctuating scalar field, the effect of the latter on the former usually being negligible.

Equations (1)-(5) constitute the mathematical representation of fluid flows, under the assumptions[9] that the turbulent fluid is a continuum, Newtonian (equation (4)) in nature, and that the flow can be described by the Navier-Stokes (NS) equations.

For turbulent flows, equations (1)-(5) represent the instantaneous values of the flow properties. The equations for turbulence fluctuations are obtained by Reynolds decomposition which describes the turbulent motion as a random variation about a mean value:

$$\phi = \bar{\phi} + \phi' \tag{6}$$

where the mean value $\bar{\phi}$ is defined[10] as:

$$\bar{\phi} \equiv \langle \phi, w \rangle = \int_D \phi(x_i - \xi_i, t - t_1)\, w(\xi_i, t_1)\, d\,Q \tag{7a}$$

with the weighting function w normalized as:

$$\int_D w(\xi_i, t_1)\, dQ = 1 \tag{7b}$$

where Q denotes the set (ξ_i, t_1) over the domain D.

The averaging procedure can be either temporal (i.e. time-averaging) or Favre (i.e. mass weighted) averaging.[11] The latter appears promising for flows with variable density, leading to all double correlations with density fluctuations vanishing, by definition. Favre-averaged quantities, however, are not easily comparable with experimentally measured quantities, which are normally time-averaged.

The weighting function for the temporal averaging can be expressed as:

$$w = \frac{\delta(\xi_i)}{\Delta t}, \quad |t_1| < \Delta t$$
$$= 0, \quad |t_1| > \Delta t \tag{8}$$

with the averaged quantity having the form:

$$\bar{\phi} = \frac{1}{\Delta t}\int_{t_1}^{t_1 + \Delta t} \phi \, dt \text{ and } \bar{\phi}' = 0 \tag{8a), (8b)}$$

Using equations (6) and (8), equations (1)-(5) can be manipulated to derive the following equations:

Mass conservation:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\overline{\partial \rho u_j}}{\partial x_j} = 0 \tag{9a}$$

or

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \bar{\rho}\bar{u_j}}{\partial x_j} \simeq 0, \overline{\rho' u_j'} \ll \bar{\rho}\bar{u_j} \text{ (for weak density fluctuations)} \tag{9b}$$

Momentum conservation:

$$\frac{\partial \bar{\rho}\bar{u_i}}{\partial t} + \frac{\partial \bar{\rho}\bar{u_i}\bar{u_j}}{\partial x_j} + \partial \bar{\rho}\overline{u_i' u_j'}/\partial x_j$$
$$= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu S_{ij} + (\mu^b - \tfrac{2}{3}\mu) S_{ll}\delta_{ij}\right]$$
$$-\left[\frac{\partial \overline{\rho' u_i'}}{\partial t} + \frac{\partial \overline{\rho' u_i' \bar{u_j}}}{\partial x_j} + \frac{\partial \overline{\rho' u_j' \bar{u_i}}}{\partial x_j} + \frac{\partial \overline{\rho' u_i' u_j'}}{\partial x_j}\right] + f_i \tag{10}$$

where:

$$S_{ij} = \left(\frac{\partial \bar{u_i}}{\partial x_j} + \frac{\partial \bar{u_j}}{\partial x_i}\right); S_{ll} = \frac{\partial \bar{u_l}}{\partial x_l}; \frac{\partial \overline{u_l'}}{\partial x_j} = 0$$

Scalar conservation:

$$\frac{\partial \bar{\rho}\bar{\phi}}{\partial t}+\frac{\partial \bar{u}_j\bar{\rho}\bar{\phi}}{\partial x_j}+\frac{\partial \bar{\rho}\overline{u_j'\phi'}}{\partial x_j}$$
$$=\frac{\partial}{\partial x_j}\left(\Gamma_\phi\frac{\partial \bar{\phi}}{\partial x_j}\right)+\frac{\overline{Dp}}{Dt}-\left[\left(\frac{\partial \overline{\rho'\phi'}}{\partial t}+\frac{\partial \overline{u}_j\overline{\rho'\phi'}}{\partial x_j}\right)\right.$$
$$\left.+\frac{\partial}{\partial x_j}(\overline{\rho'u_j'}\bar{\phi}+\overline{\rho'u_j'\phi'})\right]+\bar{S}_\phi \qquad (11)$$

Thermal equation of state (single-component gas):

$$p\equiv R\bar{\rho}\bar{T}=R(\bar{\rho}\bar{T}+\overline{\rho'T'}) \qquad (12)$$

Equations (9)-(12) are the first step towards modelling, as the averaging process itself masks some detailed characteristics of turbulence.

Some useful concepts

Before discussing the turbulence models a very brief description of some concepts is provided. The main characteristic of turbulence is the transfer of energy to smaller spatial scales across a continuous wave-number spectrum, e.g. a 3D, nonlinear process. A useful concept for discussing the main mechanisms of turbulence is that of an 'eddy'.[12–14] An eddy can be thought of as a typical turbulence pattern, covering a range of wavelengths, large and small eddies co-existing in the same volume of fluid. The actual modes of turbulence are eddies and high-vorticity regions. By analogy with molecular viscosity, which is a property of the fluid, turbulence is often described by eddy viscosity as a local property of the fluid, the corresponding mixing length being treated in an analogous manner to the molecular mean-free path derived from the kinetic theory of gases. This description is based on erroneous physical concepts but has proved useful in the quantitative prediction of simple turbulent flows.

The eddies can be considered as a tangle of vortex elements (or lines) that are stretched in a preferred direction by mean flow and in a random direction by one another. This mechanism, the so-called 'vortex stretching', ultimately leads to the breaking down of large eddies into smaller ones. This process takes the form of an 'energy cascade'. Since eddies of comparable size can only exchange energy with one another,[12] the kinetic energy from the mean motion is extracted from the largest eddies. This energy is then transferred to neighbouring eddies of smaller scales continuing to smaller and smaller scales (larger and larger velocity gradients), the smallest scale being reached when the eddies lose energy by the direct action of viscous stresses which finally convert it into internal thermal energy on the smallest-sized eddies.[14] It is important to note that viscosity does not play any role in the stretching process nor does it determine the amount of dissipated energy; it only determines the smallest scale at which dissipation takes place. It is the large eddies (comparable with the linear dimensions of the flow domain), characterizing the large-scale motion, that determine the rate at which the means-flow kinetic energy is fed into turbulent motion, and can be passed on to smaller scales and be finally dissipated. The larger eddies are thus mainly responsible for the transport of momentum and heat, and hence need to be properly simulated in a turbulence model. Because of direct interaction with the mean flow, the large-scale motion depends strongly on the boundary conditions of the problem under consideration.

An increase in Reynolds number increases the width of the spectrum, i.e. the difference between the largest eddies (associated with low-frequency fluctuations) and the smallest eddies (associated with high-frequency fluctuations). This suggests that at high Reynolds numbers the turbulent motion can be well approximated by a three-level procedure, namely, a mean motion, a large-scale motion and a small-scale motion. This procedure, proposed by Kovasznay,[15] is frequently used for turbulent flow calculations.

Why use a turbulence model?

Turbulence is a three-dimensional, time-dependent, nonlinear phenomenon. Computer programs,[16–18] now exist which are capable of solving three-dimensional, time-dependent Navier–Stokes equations, within practical computer resources. So why not make direct computer simulations of turbulence? The reason is that turbulence is dissipated, and momentum exchanged by small-scale fluctuations; and there is no existing or foreseen computer system with large enough memory or speed to allow the resolution of the small-scale effects. Therefore, in quantitative work one is obliged to use turbulence models based on using averaged NS equations and, in addition, a set of equations that supposedly express the relations between terms appearing in the NS equations. It must be realised that the available 'transport' models pay no respect to the actual physical modes of turbulence (eddies, velocity patterns, high-vorticity regions, large structures that stretch and engulf ...) and, therefore, obscure the physical processes they purport to represent. Flow visualization experiments confirm this point and demonstrate the difficulty of precise definition and modelling. It is therefore hardly surprising that the actual physics of turbulence are nowhere to be seen in the 'transport' models; simply because nobody can see as yet how mathematics can be employed to represent them in the models. It is, however, also true that the engineering community has fortuitously often obtained very useful results by using 'transport' models; results that would have required much more man-time and experimental cost to obtain in their absence. Therefore, cautiously exercised and interpreted the turbulence models can be valuable tools in research and design despite their physical deficiencies.

Field models of turbulence

The computation of turbulent flows, essentially the search for a model of the Reynolds stresses $-\rho\overline{u_i'u_j'}$ appearing in equation (10) (the 'closure' problem) in terms of either known or calculable quantities, has been a serious problem since Reynolds' time. Until the advent of powerful computers most boundary-layer prediction methods were highly empirical and based on ordinary differential equations. The impact of computers by the mid-1960s led several workers to develop methods based on the governing partial differential equations of mean motion, incorporating turbulence transport equations. The main impact of the 1968 Stanford University specialists' conference was to legitimize pde methods, which proved to be more accurate and more general than the best integral methods. Vigorous development of pde turbulence models then followed. This review outlines the main features and advantages/disadvantages of several levels of turbulent-flow pde models. For methods based on ordinary differential equations, which still have

some uses, the comprehensive review by Reynolds and Cebeci[1] should be consulted. For modelling of turbulent heat and mass transport, see the review by Launder.[3] Modelling of turbulence has been attempted in general by five methods, as follows:[19]

- Analytical turbulence theories: these are normally developed in Fourier (wave number or frequency) space.[20-22] These theories are very complicated and have not yet been applied to complex engineering problems.
- Sub-grid scale closure models: these are based on the hypothesis that turbulent motion could be separated into large scale and sub-grid (small-scale) components such that the separation between the two does not have a significant effect on the evolution of large scales.[23] These methods, although promising, still require too much computational time to be useful for engineering applications.[24]
- Direct numerical simulation: This method attempts to simulate directly all the dynamically important scales of large Reynolds number turbulent flows. It is based on the hypothesis that direct simulation may be carried out by artificially decreasing the Reynolds number to the point where the important scales can be simulated accurately on existing computers, and that a sufficient number of large scales remain unchanged by any change in Re. This method still requires too much computational time.[19,25]
- Turbulence transport models: These are the basis of the engineers' approach, where attempts are concentrated on seeking simplified modelling of the terms governing the transport of momentum, heat, etc. These models are similar to the analytical theories in that both treat dynamical quantities as statistically-averaged fields; but they simulate only the gross features of turbulence, while analytical theories pay attention to interactions between the various scales of motion.[26]
- Two-fluid models of turbulence: The 'fragmentariness' of turbulent flows is neglected by the conventional (single-fluid) models of turbulence. It can be allowed for by the use of a multi-fluid model which permits different (arbitrarily distinguished) fluids to exist and interact in the same space. Thus, when a two-fluid model is applied to a turbulent jet, two-phase theory is applied to the flow of a single thermodynamic phase; the 'two fluids' being (1) injected fluid possessing vorticity (2) surrounding irrotational fluid. A conventional turbulence model can still be used within phase 1. Multi-fluid turbulence modelling is just beginning and appears promising.[27-29]

This review will discuss the turbulence transport models in some detail because they have been intensively developed and are essentially today's standard practice. Large-eddy simulations are still in the development stage and are serving mainly to help assess the lower-level models. Because of their long term promise they are discussed briefly. Finally, the two-fluid models are discussed. They are in their infancy but may in future inject more physical content into turbulence modelling.

Turbulence transport models

A transport model is a set of equations, additional to the averaged Navier-Stokes equations, purporting to express relations between terms appearing in those equations. The starting point of all transport models is the Reynolds equation (10), and the problem of 'closure' reduces to the modelling of the Reynolds stresses $-\overline{\rho u_i' u_j'}$, in terms of mean-flow quantities.

Simple closure models of the Reynolds stresses, $-\overline{\rho u_i' u_j'}$, normally use an eddy viscosity hypothesis based on an analogy between molecular and turbulent motions. Accordingly, the turbulence eddies are thought of as lumps of fluid, which, like molecules, collide and exchange momentum, obeying the kinetic theory of gases. Thus, in analogy with the molecular viscous stress, the Reynolds (turbulence) stresses are modelled according to:

$$-\overline{u_i' u_j'} = \nu_t\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \tfrac{2}{3}\delta_{ij}k$$

$$\equiv 2\nu_t S_{ij} - \tfrac{2}{3}\delta_{ij}k \qquad (13)$$

Here $\nu_t\,[\equiv \mu_t/\rho]$ is referred to as the turbulence or eddy (kinematic) viscosity which, in contrast to the molecular (kinematic) viscosity, $\nu\,[\equiv \mu/\rho]$, is not a fluid property but depends on the local state of the turbulence; it is assumed to be a scalar and may vary significantly from point to point and flow to flow. The symbol k in the above equation denotes the kinetic energy of the turbulent motion, expressed as:

$$k = \tfrac{1}{2}\overline{u_i' u_i'} = \tfrac{1}{2}(\overline{u_1'^2} + \overline{u_2'^2} + \overline{u_3'^2}) \qquad (14)$$

This, being a measure of normal (or diagonal) turbulence stresses, is also the turbulence equivalent of the (kinematic) static pressure $\bar{p}/\rho$ of the molecular motion, and is therefore absorbed normally in the unknown $\bar{p}/\rho$ (replacing $\bar{p}/\rho$ by $\bar{p}/\rho + \frac{2}{3}k$) if an explicit equation for k is not involved. (It should be noted that equation (13) reduces to the identity $k = k$ for $i = j$, and thus cannot be used for the determination of k.)

Even though the eddy viscosity hypothesis (equation (13)) faces some conceptual difficulties,[13] it has proved quite successful in many flow situations.[30-37] Its success results from the fact that the (only) unknown ν_t in equation (13) can be determined, to a good approximation, simply by writing:

$$\nu_t \propto \hat{V}L \qquad (15)$$

where L is a length scale characteristic of large turbulence eddies and $\hat{V}$ is a velocity scale characteristic of fluctuating velocities (also of the large eddies). It has been pointed out[24] that it is the distribution of these scales that can be well approximated in many flows.

Equation (13) defines the mean rate of strain S_{ij}. The difference between zero-equation and one-equation models, discussed below, is that in the latter ν_t, instead of being related directly to the mean-flow scales, is modelled by:

$$\nu_t = C_{\nu 1}\sqrt{k}L$$

Zero-equation models

These are models using only the pde's for the mean fields and no differential equation for turbulence quantities. The models of this group relate the turbulence shear stress uniquely to the mean-flow conditions at each point. Since they require only algebraic expressions, they have been very popular. All models of this class use the eddy-viscosity concept of Boussinesq, which for $\overline{u'v'}$ in a thin shear layer becomes:

$$-\overline{u'v'} = \nu_t \frac{\partial U}{\partial y} \qquad (16)$$

The two most familiar versions of this class are distinguished by the way in which ν_t is calculated.

(a) *The constant-eddy-viscosity model.* Trubchikov[38] and Prandtl[39] have proposed the following formula for free-jet flows:

$$\nu_t = C\delta(U_{\max} - U_{\min}) \tag{17}$$

All quantities on the right hand side of (17), except C, may be functions of the longitudinal distance x; ν_t is supposed to be uniform over any cross-section.

(b) *The mixing length hypothesis of Prandtl*[40] which was proposed for two-dimensional boundary layers:

$$\nu_t = l^2 \left| \frac{\partial U}{\partial y} \right| \tag{18}$$

The mixing length of turbulent motion, l, is analogous to the molecular mean-free path, and for jet flows it is usually taken as proportional to the jet width. Prandtl[40] determined l by postulating that this was proportional to the distance from the nearest wall. Von Kármán[41] proposed the following correlation of l with the mean velocity profile by using dimensional analysis:

$$l = \kappa \left| \frac{\partial U_1/\partial X_2}{\partial^2 U_1/\partial^2 X_2} \right| \tag{19}$$

where κ is the von Kármán constant ($\simeq 0.4$). This expression has had limited success because it yields an infinite mixing length when there is a point of inflexion ($\partial^2 U_1/\partial^2 X_2 = 0$) in the mean velocity profile, as is generally the case with flows like jets and wakes. Formulae (16) and (18) are based, it should be mentioned, on erroneous physical arguments, but can be regarded as definitions of the quantities ν_t and l which, in simple flows, are simpler to correlate empirically than $\overline{u'v'}$ itself. In most boundary-layer calculation methods the turbulent boundary layer is regarded as consisting of inner and outer regions, and the l and ν_t distributions are described by two separate empirical expressions in each region. For example, outside the viscous sublayer close to the wall, l is proportional to y in the inner region, and it is proportional to δ in the outer region. Therefore:

$$l = \begin{cases} \kappa y & y_v \leqslant y \leqslant y_i \\ \alpha_1 \delta & y_i \leqslant y \leqslant \delta \end{cases} \tag{20}$$

where y_v is the viscous sublayer thickness, $40\nu/u_\tau$, and y_i is a distance obtained from the continuity of mixing length. The empirical parameters κ and α_1 vary slightly according to the experimental data. For flows at $Re >$ 5000 they are taken as $\kappa = 0.4$ and $\alpha_1 = 0.075$. According to recent studies[42] at $Re < 5000$ α_1 is a function of Re. Similarly, according to several studies, ν_t varies linearly with y in the inner region where $-\rho\overline{u'v'}$ is nearly equal to τ_w, and is nearly constant in the outer region. The following formulae may be used:

$$\nu_t = \begin{cases} l^2 \left| \dfrac{\partial U}{\partial y} \right| & y_v \leqslant y \leqslant y_i \\ \alpha \left| \displaystyle\int_0^\infty (U_e - U)\, \mathrm{d}y \right| & y_i \leqslant y \leqslant \delta \end{cases} \tag{21}$$

with l given by (20). The 'constant' α is assumed equal to 0.0168 for $Re > 5000$, and it also varies, like α_1, with the Reynolds number when $Re < 5000$. Cebeci[43] has given a formula for α as a function of the 'strength of the wake'.

There have been numerous attempts to extend equation (20) into the viscous sublayer, by multiplying l with functions of the sublayer thickness or otherwise.[44] Van Driest[45] using an analogy with the laminar flow on an oscillating flat plate, proposed:

$$l = \kappa y\,[1 - \exp(-y/A)] \tag{22}$$

for a smooth flat-plate flow. Here A is a damping-length constant for which the best empirical choice (e.g. dimensionally correct) is $A^+\nu(\tau_w/\rho)^{-1/2}$ with A^+ being about 26. A^+ varies somewhat with pressure gradient, transpiration, etc.[1]

Abbott *et al.*[46] using an intermittent model of the sublayer, obtained an expression similar to equation (22) by considering the unsteady one-dimensional vorticity equation rather than the unsteady one-dimensional momentum equation considered by Van Driest. This model, which is developed for incompressible flows, has not yet been used and tested as thoroughly as the Van Driest model.

The eddy viscosity and mixing-length formulae, like all expressions for turbulent flows, are empirical. Over the years, several empirical corrections to these formulae have been made, to account for the effects of low Reynolds number, transitional region, mass transfer, pressure gradient, transverse curvature and compressibility. An example is the eddy-viscosity formulation of Cebeci and Smith (Ref. 1, p. 219) that accounts for all of these effects and appears to give satisfactory results. Eddy-viscosity formulations have also been generalized to 3D incompressible and compressible flows by Cebeci,[47] Cebeci and Abbott,[48] Cebeci *et al.*[49] They proposed:

$$\nu_t = \begin{cases} (\nu_t)_i = L^2|S(y)| & (\nu_t)_i \leqslant (\nu_t)_0 \\ (\nu_t)_0 = 0.0168 \displaystyle\int_0^\infty (|q_e| - |q(y)|)\, \mathrm{d}y & \\ (\nu_t)_i \geqslant (\nu_t)_0 & \end{cases} \tag{23}$$

where:

$$L = 0.4y\,[1 - \exp(-y/A)]; \quad A = 26(\nu/|S_w|)^{1/2} \tag{24}$$

Here y denotes distance normal to the surface, q the velocity vector parallel to the wall, $q = (U, W)$, and S denotes the strain vector $S = \partial q/\partial y = (\partial U/\partial y, \partial W/\partial y)$. The subscripts e and w denote the edge and the wall, respectively. The expressions in (23) have been used to compute a number of 3D flows ranging from swept infinite cylinders to arbitrary wings for incompressible and compressible flows. The governing boundary-layer equations were solved by numerical methods and comparisons made with experiment and with the predictions of Bradshaw's method.[50] *Figure 1* shows the results for a 45° infinite swept wing and *Figure 2* the results for a laterally diverging boundary layer. The experimental data in *Figure 1* were obtained by Bradshaw and Terrell[51] on the flat rear of the wing, in a region of nominally zero pressure gradient and decaying cross-flow. The experimental data in *Figure 2* were obtained by Johnston[52] in an apparatus consisting of a rectangular inlet duct from which an issuing jet impinged on an end wall from the outlet of the channel. The jet was confined on the top and bottom by flat surfaces. Other zero-equation methods based on mixing length or eddy viscosity have also been extended to 3D flows. Such a model which

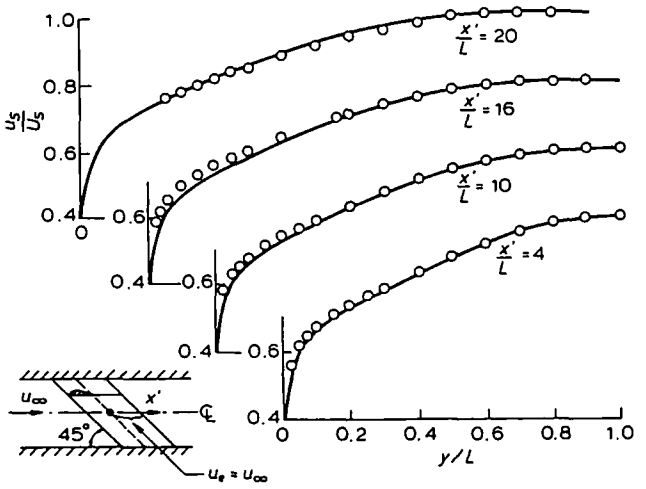


Figure 1 Predictions of a swept-wing boundary using equation (23) (0), experiments (reference 51); (——), calculations. (Taken from reference 1)

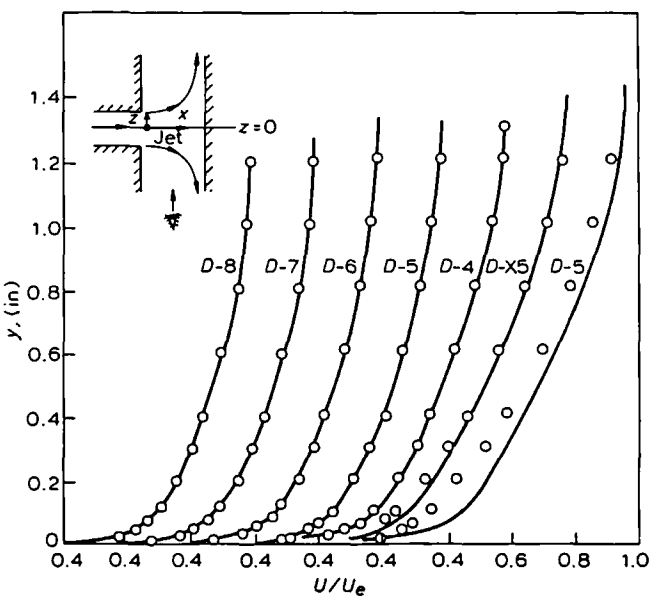


Figure 2 Predictions of a laterally diverging boundary layer using equation (23). (0), experiments (reference 52); (——), calculations. (Taken from reference 1)

appears to give satisfactory results is due to Hunt *et al*,[53] and has been adopted and used by others.[54,55]

Though several other algebraic formulae for ν_t have been proposed, they did not attract much attention because they need more *ad hoc* adjustment than the mixing-length model. Thus, Prandtl's model is still the basis of many calculations of the turbulent boundary layer which are carried out today; its merits and short-comings are therefore discussed in greater detail.

For many boundary-layer flows, Prandtl's mixing length hypothesis works surprisingly well. The spreading rate, as well as the profiles of velocity, temperature, and concentration, can be predicted satisfactorily. Unfortunately, the constants involved must vary with the problem considered. *Table 1* exemplifies this for the following jets: plane mixing layer, plane jet, radial jet and round jet; all of these are without swirl, and issue into stagnant surroundings.

The mixing length, l, divided by the jet width, δ, varies from one case to another. This lack of universality is an indication that the underlying model of turbulence lacks some of the important features of real flows. Thus, the mixing-length hypothesis implies that generation and dissipation of turbulence energy are in balance everywhere; so the convection and diffusion of turbulence energy are ignored. Only the employment of differential equations for the turbulence quantities (whereby the eddy viscosity or the associated characteristic scales of turbulent motion are calculated through the transport equations) can overcome these restrictions.

Models which employ differential equations for turbulence quantities

The eddy-viscosity hypothesis has been extended to higher levels of complexity for non-equilibrium boundary layers (where, locally, generation-dissipation $\neq 0$). Following Mellor and Herring,[56] mean-velocity-field (MVF) models include all those eddy-viscosity models (algebraic and pde) which provide information only about the mean velocity profile, in addition to the turbulence shear stresses (through equations (13) and (15)), whereas mean-turbulence-energy (MTE) models include all other (pde but not necessarily eddy-viscosity) models that provide additional information about the kinetic energy of turbulent motion. The models of Saffman and Wilcox,[57] and Nee and Kovasznay[58] are classified as MVF models since their transportable parameters (such as ν_t) are themselves models of turbulent flows and not physically realizable quantities like k. The lack of popularity of these models compared with MTE models is due to that reason. MTE models, reviewed recently by Harsha,[59] can be classified into two groups: PK models, proposed by Prandtl[60] and Kolmogorov,[61] use the eddy-viscosity hypothesis (equations (13) and (15)) with the scale of velocity fluctuations specified as $\sqrt{k}$ where k is determined through a transport equation. On the other hand, models proposed by Nevzglijadov[62] and Dryden[63] are based on the hypothesis that the shear Reynolds stresses are proportional to the turbulence kinetic energy (k) or normal Reynolds stresses:

$$\overline{u_i' u_j'} = a_1 k, \quad i \neq j \tag{25}$$

where a_1 is a dimensionless constant (≈ 0.3) or a function of y/δ where δ is the shear-layer thickness. The latter models derive a transport equation for the Reynolds stresses from the k-equation rather than from the full Navier-Stokes equations. Furthermore, because the energy in equation (25) cannot assume negative values, the use of this assumption is restricted to problems in which the shear stress varies monotonically from a finite value on one boundary to zero at the other.

One-equation models

Several models focus attention on the kinetic energy of turbulence k, as defined by equation (14). k is the dependent variable of a differential transport equation, derived

Table 1 Mixing-length constants for free jets

	Mixing layer	Plane jet	Radial jet	Round jet
l/δ	0.07	0.09	0.13	0.075

exactly from the Navier-Stokes equations, as follows (for ρ = constant):

$$\underbrace{\frac{\partial k}{\partial t}}_{\text{rate of change (I)}} + \underbrace{\overline{u_j}\frac{\partial k}{\partial x_j}}_{\text{advective transport (II)}}$$

$$= \underbrace{-\frac{\partial}{\partial x_j}\left[\overline{u_j'\left(\frac{u_i'u_i'}{2}+\frac{p'}{\rho}\right)} - 2\nu\overline{u_i' s_{ij}}\right]}_{\mathscr{D} \equiv \text{inertial diffusive transport (III)}}$$

$$\underbrace{-\overline{u_i'u_j'}\frac{\partial \overline{u_i}}{\partial x_j}}_{P_k \equiv \text{production by shear stress (IV)}} \quad \underbrace{-\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i}\right)}}_{\epsilon \equiv \text{viscous (kinematic) dissipation rate (V)}} \tag{26a}$$

or

$$\frac{Dk}{Dt} \equiv \frac{\partial k}{\partial t} + \overline{u_j}\frac{\partial k}{\partial x_j} = \mathscr{D}_k + P_k - \epsilon \tag{26b}$$

where:

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right) \tag{27}$$

is the deformation or symmetric part of the second-order fluctuating rate-of-strain tensor ($\partial u_i'/\partial x_j$). The first two terms of equation (26) are self-explanatory. Other terms involving unknown correlations need some discussion. The term $\mathscr{D}_k$(III), which represents the turbulent transport of k by diffusion, is an inertial term and vanishes when integrated over the representative volume of any flow.[20] It implies that the term neither creates turbulence energy nor destroys it; it merely moves it from place to place in configuration space, and one wave number to another in Fourier space. The viscous diffusion part, $2\nu\overline{s_{ij}u_i'}$ can be shown to be negligible at high Reynolds numbers[12] (Viscous diffusion may not be negligible outside the inviscid region, i.e. in the viscous region near solid boundaries). The rest of this term is modelled, in analogy with the eddy-viscosity hypothesis (equation (13), according to the gradient-diffusion assumption:

$$-\overline{u_j'\left(\frac{u_i'u_i'}{2}+p'\right)} \simeq -\overline{u_j'\frac{u_i'u_i'}{2}} \simeq \gamma_k\frac{\partial k}{\partial x_j}, \quad \left(\begin{matrix}\text{eddy-diffusivity}\\ \text{hypothesis}\end{matrix}\right) \tag{28}$$

where $\gamma_k (\equiv \Gamma_k/\rho)$ is the turbulence (or eddy) diffusivity of k and, similarly to ν_t, is not a property of the fluid. This may be determined from the Reynolds analogy between different transport diffusivities, which states that ν_t, describing the turbulent transport of momentum, is linearly related to γ_k, describing the turbulent transport of k, i.e.:

$$\gamma_k = \frac{\nu_t}{\sigma_k}$$

where σ_k is an empirical diffusion number. It has been pointed out,[24] on the basis of experimental observations, that even though the turbulence diffusivities may vary significantly across the flow, their ratio (i.e. $(\nu_t/\gamma_k) \equiv \sigma_k$) stays almost constant; nevertheless buoyancy may significantly influence the value of σ_k. The term P_k (IV) represents the rate of transfer (or production) of kinetic energy from the mean to the turbulent motion. It is worth mentioning here that this term appears with opposite sign as a sink term in the equation for $\frac{1}{2}\overline{u_i u_i}$, the kinetic energy of the mean motion.[12] The correlation $\overline{u_i'u_j'}$ implicit in the term P_k is modelled according to equation (13). The term ϵ(V) represents the 'average' (kinematic) dissipation rate of turbulence (kinetic) energy, i.e. the rate at which turbulence energy is destroyed by molecular viscous stresses. At large, Re, where local isotropy prevails at smaller scales of the turbulent motion, ϵ can be equivalently written as:

$$\epsilon = \nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i}\right)} \equiv 2\nu\overline{s_{ij}\frac{\partial u_i'}{\partial x_j}} \equiv 2\nu\overline{s_{ij}s_{ij}} \tag{29}$$

The modelled form of ϵ is, however, derived from the experimental results which suggest that it is governed by the energy of large eddies ($\sim k$) and their single time-scale ($\sim L_k/\sqrt{k}$). Thus, on dimensional grounds, equation (29) is simplified to:

$$\epsilon = \overline{\nu s_{ij}s_{ij}} \simeq C_D\frac{k^{3/2}}{L_k} \tag{30}$$

where C_D is an empirical constant for (viscous) dissipation. The second equality of the above equation implies that the turbulence viscous dissipation is a passive process, in the same sense that ϵ can be estimated from the large-scale eddies which do not involve viscosity. The first equality suggests that the (smallest) scale at which eddies are dissipated due to viscosity can then be determined by the dissipation rate (ϵ) and the viscosity. With equations (13) and (26)-(30), the modelled form of the k equation (26) at large Re becomes:

$$\frac{\partial k}{\partial t} + \overline{u_j}\frac{\partial k}{\partial x_j} \simeq \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right) + 2\nu_t S_{ij}\frac{\partial \overline{u_i}}{\partial x_j} - C_D\frac{k^{3/2}}{L_K} \tag{31}$$

It can be seen that if advective and diffusive transport terms are neglected, this equation reduces to the mixing-length model (equation (18)), with l and L_k being related by:

$$l = \left(\frac{C_{\nu_1}^3}{C_D}\right)^{1/4} L_K \tag{32}$$

C_{ν_1} being the proportionality 'constant' in the $\nu_t \sim L$ relation. The modelled k-equation (31) still involves L_K as an unknown. It is the specification of L_K that distinguishes various PK models. In general, the one-equation models can be classified as follows:

Models which use the eddy-viscosity concept, equation (16):

(a) Models solving the above transport equation for k. Kolmogorov,[61] Prandtl,[60] and Emmons[64] independently proposed the use of the relationship:

$$\nu_t = C_{\nu_1}\sqrt{k}L \tag{33}$$

k is to be calculated from the differential equation, and the length-scale L, in the models of this class, from an empirical algebraic expression.

Wieghardt,[60] Glushko,[65] and Beckwith and Bushnell[66] applied this model to various boundary-layer flows. Wolfshtein[67] used the model to calculate the impinging jet and Spalding[68] applied it to separated flows. The quality of the prediction depends, of course, on the realism of the expression for L; but, whenever the diffusion and convection of turbulence play a significant role, the superiority of this model over zero-equation models is clearly evident.

(b) Models solving a transport equation for ν_t. Nee and Kovasznay[58] have proposed a transport equation for ν_t; they also needed to incorporate an empirical algebraic expression for L. They applied the resulting model to a turbulent wall boundary-layer without pressure gradient and obtained good agreement with experimental results. The objection to these models is the use of a non-physical quantity as dependent variable.

Models which use the 'structural' assumption, equation (25) rather than the 'Newtonian' assumption, equation (16).

A length scale is needed, because the turbulence transport and dissipation terms in the k-equation have dimensions of (velocity)^3/(length) and are modelled as functions of the scales $\sqrt{k}$ and L. Use of equation (25) produces an equation for $\overline{u'v'}$. In both approaches L/δ is taken as an empirical function of y/δ. Townsend[69] proposed that the ratio of $\overline{u'v'}$ to k might be a universal constant. This is not always plausible. $\tau/(\rho k)$ cannot remain constant near zero-gradient regions; in practice it can change sign. Bradshaw *et al*[70] employed the 'structural' assumption to derive a shear-stress transport equation from the turbulence energy equation (ND model), and used an empirically specified bulk convection velocity for the turbulence transport terms. They were successful in making predictions for several uniform-density boundary layers near walls. *Figure 3* presents calculations by Bradshaw *et al* and experiments for flow past a flat plate with uniform u_∞. *Figure 4* presents the same information for equilibrium boundary layer with $u_\infty \propto x^{-0.15}$ and *Figure 5* for equilibrium in favourable pressure gradient with $\delta_1/\tau_{wall}\, dp/dx = -0.35$. The agreement of the Bradshaw *et al* model with experiment is very good. Other models, however, can also give agreement in these cases which is as good. The method has also been applied to a wide range of boundary-layer problems including 3D[50,71,72] and compressible flow with heat transfer;[73] also internal flows,[74] free shear layers[75] and unsteady flow.[76] For many flows, however, the underlying assumption is not realistic; often where k remains finite the shear stress vanishes, or, as in the case of round jets, when k has its maximum value. Other limitations of the model are:

- Only uni-directional k diffusion is allowed.
- The length-scale distribution must be inserted as empirical input, as for the mixing-length and Prandtl energy methods; no universal prescription is available, even for flows near walls.
- For recirculating flows, the situation is as for other one-equation models.

The Bradshaw *et al* model has in the past attracted much attention. Its supporters have pointed to the experimental data showing that sometimes τ and $\partial u/\partial y$, and $-\overline{u'k'}$ and $\partial k/\partial y$, have opposite signs; such data exist and show the limited validity of 'effective-viscosity' and 'turbulence diffusion' models. Nevertheless, the implausibilities of the Bradshaw *et al* model appear to be as severe. For flows where shear stress changes sign, Morel *et al*[77] have accounted for the sign of the shear stress profiles by considering them as two opposite boundary layers overlapped together, and then treating each layer according

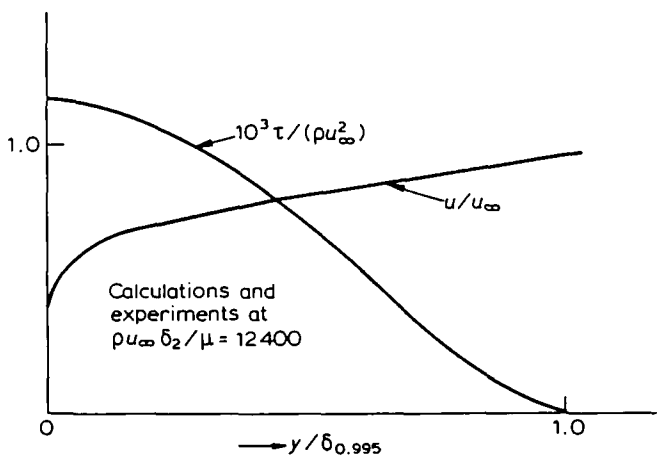


Figure 3 Flow past a flat plate with uniform u_∞. Calculations by Bradshaw *et al*. (Figure taken from reference 4)

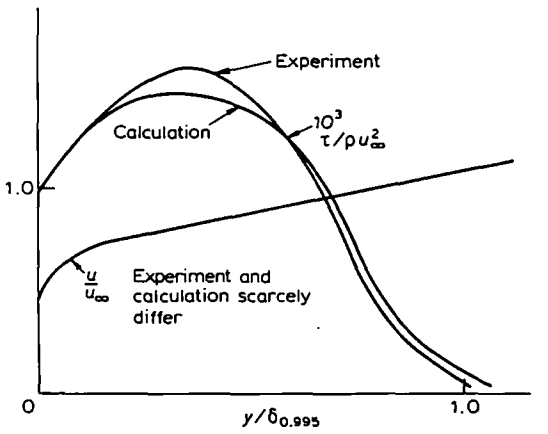


Figure 4 Equilibrium boundary layer with $u_\infty \propto x^{-0.15}$. Calculations by Bradshaw *et al*. (Figure taken from reference 4)

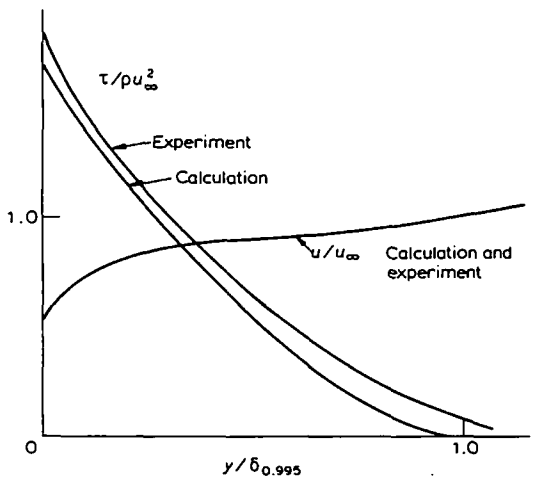


Figure 5 Equilibrium boundary layer in favourable pressure gradient. Calculations by Bradshaw. (Figure taken from reference 4)

to equation (25). This concept also fails in axi-symmetric flows,[59] where an infinite number of interacting layers has to be postulated and parameters of the model have to be modified as well. For 2D and axisymmetric flows, Harsha and co-workers[78–80] have, alternately, taken into account the sign of shear stress profiles by proposing an expression for x_j, that effectively transforms the ND (or structural) hypothesis to the PK (or Newtonian) hypothesis. Contrary to Bradshaw and co-workers[70,81] Harsha and co-workers[78,80] have therefore used the gradient-diffusion assumption for the diffusion term; the expression similar to (30) is again used for the dissipation term. Another interesting class of ND models includes the semi-integral formulation of the Bradshaw *et al* model[70] by Patel and Head[82] and that of Lee and Harsha[78] by Peter and Phares.[83] The survey by Harsha[59] has shown that, besides being numerically efficient, the integral models provide significantly better predictions than the corresponding finite-difference versions in some flows, such as in far-field wakes and jets. The numerical efficiency of the integral models is, however, not considered a substantial advantage since equally fast finite-difference procedures are also available.[59]

Finally, the one-equation model of Norris and Reynolds[44] for use in the viscous sublayer as well as in the fully turbulent regions should also be mentioned. They argued that the length scale L should do nothing special in the viscous region, but should behave like $L \simeq \kappa y$ all the way down to the wall. However, they assumed that the turbulence transport is suppressed by the presence of the wall and took:

$$\nu_t = C_1\sqrt{k}L\,[1 - \exp(-C_2\sqrt{k}y/\nu)] \qquad (37)$$

where C_1, C_2 are constants. A similar approach was adopted by Wolfshtein.[84] The latter allowed L to depart from κy, in the viscous region, but used $\mathscr{D} = C_3 k^{3/2}/L$ for dissipation (Norris and Reynolds used $\mathscr{D} = C_3(k^{3/2}/L)(1 + C_4/(\sqrt{k}L/\nu))$. When cast in comparable form, the constants used by Wolfshtein and by Norris and Reynolds are similar.

All one-equation models require the input of a length-scale distribution based on experimental information and hope. Information about these distributions is available for simple flows (although not complete); but it is not available for elliptic flows. For this reason, most researchers have abandoned these models in favour of two- or even more-equation models. For some flows of interest, it may still be easier to specify the length-scale distribution than to compute it with a pde. This would be particularly true if the length scale should really be governed by the global features of the flow through an integro-differential equation. Therefore, the would-be user of a turbulence model is advised not to disregard them without some preliminary thought.

Two-equation models

General. All the models considered so far have necessitated the use of an algebraic length-scale specification; and experience has shown that this specification must vary with the boundary conditions. There is little hope of achieving universality for the empirical inputs until L, or some combination of L and k, is itself determined from an independent differential equation. Kolmogorov[61] made the first proposal of this kind; he introduced a pde for the 'frequency' $f = k^{1/2}/L$. The Kolmogorov energy-frequency model represents a major advance in turbulence computations, as it permits the length scale to be predicted (at the cost of solving an additional pde and of supplying further empirical constants). Rotta[85] derived an exact equation for L (related to the integral scale) from the Navier-Stokes equations. Harlow *et al*[86–88] postulated an equation for L, and later replaced it by one for the dissipation of k. They applied their model to turbulent pipe flow and obtained fair agreement with experimental results.

The groups at Imperial College and at Stanford both experimented with *ad hoc* transport equations for L with no real success. However, these and other groups have achieved greater success using equations for $k \sim w$, $k \sim \epsilon$ and $k \sim kL$ where w represents the 'vorticity' fluctuations square, and ϵ the eddy dissipation rate.[89–92,96] Rodi and Spalding[93] used a $k \sim kL$ model based on Rotta's model, and employed the eddy-viscosity concept and the relationship for ν_t, which was introduced much earlier by Kolmogorov, Prandtl and Emmons. They succeeded in predicting the behaviour of free jets with agreement with the experiment lying within the experimental accuracy. Markatos[94,95] used the $k \sim \epsilon$ model for recirculating flows, also with relative success.

The model that has attracted most attention from the research community employs as the second pde that for the 'isotropic dissipation' $\mathscr{D}$, based on the exact transport equation for $\mathscr{D}$, which follows from differentiating the x_i-component Navier-Stokes equation with respect to x_j, multiplying by $\partial u_i/\partial x_j$ and averaging. At high turbulence Reynolds numbers, the difference between ϵ and the 'isotropic dissipation' $\mathscr{D}$ is small, and one can use ϵ for convenience. Of course the ϵ-equation implies a transport equation for L, using the transport equation for k. Since neither the terms in Rotta's length-scale equation nor those in the ϵ equation have ever been measured, the reason for the greater usefulness of the ϵ equation is not entirely clear. It can be shown, however, that modelling turbulence transport of L by an eddy viscosity gives implausible results in the inner layer of a wall flow, and most researchers have abandoned the L equation rather than the eddy-viscosity concept. The alternative to eddy viscosity is the bulk-convection hypothesis of Townsend,[97] in which turbulence energy and other quantities are supposed to be transported by the turbulence at a velocity that depends on the energy distribution but not on the local gradient. This may be a more realistic model than gradient transport, for transport by the large eddies. However, there is as yet no quantitative model for the behaviour of the bulk-convection velocity.[1] An adversely critical discussion of gradient-transport concepts in turbulence is given by Corrsin.[98]

Mathematical analysis. It has been emphasised[99] that, for any substantial improvement in the performance of the PK models over mixing-length models, a transport equation of L_k seems essential. Otherwise, any algebraic length-scale specification must vary with the boundary conditions and there is little hope of achieving universality for the empirical inputs. The length scale L_k is usually not chosen as a proper dependent variable for the L_k-equation. Instead, a combination of k and L_k, having the form:

$$z = k^m L_k^n \qquad (35)$$

is chosen as the dependent variable, such that it can be interpreted physically. The most popular forms of z are:

$z = k^{1/2}/L_k \equiv f$ (turbulence frequency of energy containing eddies)[61]

$z = k/L_k^2 \equiv w$ (time-average square of the vorticity fluctuations)[91–93]

$z = k^{3/2}/L_k \equiv \epsilon$ (dissipation rate of turbulence energy)[86-89, 103]

$z = kL_k$ (energy ~ length-scale product).[93,100,101]

Recently, Lin and Wolfshtein[102] proposed an equation for the length scale in terms of tensor volume of turbulence, defined as the volume integral of the two-point correlation:

$$V_{ij} = \iiint_Q (\overline{u_{iA}' u_{jB}'}/\tfrac{2}{3} k_{AA})\, dQ, k_{AA} \equiv \overline{u_{iA}' u_{jA}'}/2$$

where Q is the total volume surrounding the point A.

It has been shown[24] that the various modelled transport equations for z differ mainly in diffusion and 'secondary' source (as a near-wall correction term which is zero for free flows) terms. The variable $z = \epsilon$ is preferred normally because it does not require a secondary source term[24] and a simple gradient diffusion hypothesis is fairly good for the diffusion term.[89] An explicit transport equation for ϵ also eliminates the use of any empirical expression like equation (30).

The $k \sim \epsilon$ model is the only one, when compared with the $k \sim kl$ and $k \sim w$ models, that permits the 'turbulence Prandtl number' σ_z to have a reasonable value (= 1.3), which will fit the experimental data for the spread of the various entities at locations far from walls, without modification of any constants. Of course, the above 'superiority' of the ϵ equation is very dubious; because it may be that some of the 'constants' should not be constants, and perhaps also the true behaviour of turbulence requires that the gradients of more than one turbulence property drive diffusional effects. Until further theoretical or experimental evidence of this becomes available, it seems better to stand by the simplest formulations. There is no reason, however, to suppose that, if an equal amount of attention were given to it, another two-equation model would not perform as well, or even better than the $k \sim \epsilon$ model.

An exact transport equation for ϵ has been derived from the NS equations independently by Davidov[103] and by Harlow and Nakayama,[86] which for incompressible flows is:

$$\underbrace{\frac{D\epsilon}{Dt}}_{\text{I}}$$

time rate of change and advection

$$= \underbrace{-2\nu \frac{\partial \overline{u_i}}{\partial x_m}\left(\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_m'}{\partial x_j}} + \overline{\frac{\partial u_j'}{\partial x_i}\frac{\partial u_j'}{\partial x_m}}\right)}_{\text{II}} \underbrace{- 2\nu \frac{\partial^2 \overline{u_i}}{\partial x_m \partial x_j}\overline{u_k' \frac{\partial u_i'}{\partial x_j}}}_{\text{III}}$$

generation by mean motion

$$\underbrace{-2\nu \overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_j}\frac{\partial u_m'}{\partial x_j}}}_{\text{IV}} \underbrace{- 2\left(\nu \overline{\frac{\partial^2 u_i'}{\partial x_m \partial x_j}}\right)^2}_{\text{V}}$$

generation by self stretching of vortex elements | viscous dissipation

$$\underbrace{-\nu\left[\frac{\partial}{\partial x_m}\overline{u_m'\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \frac{\partial}{\partial x_m}\overline{\frac{\partial p'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} - \frac{\partial^2 \epsilon}{\partial x_m^2}\right]}_{\text{VI} \quad \text{VII} \quad \text{VIII}} \tag{36}$$

diffusion

The interpretation of this equation does not look straightforward when ϵ is taken as the rate of viscous dissipation of the turbulence energy. Its modelling can, however, be performed by analogy with Chou's vorticity-decay equation for the mean-squared fluctuating vorticity ($\overline{\omega_i \omega_i}$) which closely resembles the ϵ-equation.[104] At high Reynolds numbers:

$$\epsilon = 2\nu \overline{s_{ij} s_{ij}} \cong \nu \overline{\omega_i \omega_i} \tag{37}$$

where s_{ij} is the symmetric part of the fluctuating strain rate $\partial u_i/\partial x_j$.

Unlike ϵ, which is a passive scalar, the fluctuating vorticity ω_i may be considered as an active scalar property of the small-scale turbulence, because most of the turbulence energy is associated with the large-scale motion whereas most of the vorticity is associated with the small-scale motion.[12] With these concepts, the ϵ-equation seems easier to understand in terms of $\overline{\omega_i \omega_i}$. Its most direct interpretation,[105,106] however, is an equation for the rate at which turbulence kinetic energy is transferred across the spectrum from large eddies (or low wave numbers) to smaller and smaller eddies (or higher and higher wave numbers). In the light of these remarks, the modelled ϵ-equation is of the form:[105]

$$\frac{D\epsilon}{Dt} \simeq C_\epsilon \frac{\partial}{\partial x_m}\left(\frac{k}{\epsilon}\overline{u_m' u_j'}\frac{\partial \epsilon}{\partial x_j}\right) + \frac{\epsilon}{k}(C_{\epsilon 1} P_k - C_{\epsilon 2}\epsilon) \tag{38}$$

where C_ϵ, $C_{\epsilon 1}$, $C_{\epsilon 2}$ are empirical parameters. Upon rearranging equations (31) and (38) the $k \sim \epsilon$ model is as follows:

$$\frac{Dk}{Dt} \equiv \frac{\partial k}{\partial t} + \overline{u_j}\frac{\partial k}{\partial x_j} \simeq \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial}{\partial x_j}\right) + P_k - \epsilon \tag{39}$$

$$\frac{D\epsilon}{Dt} \equiv \frac{\partial \epsilon}{\partial t} + \overline{u_j}\frac{\partial \epsilon}{\partial x_j} \simeq \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_j}\right) + \frac{\epsilon}{k}(C_{\epsilon 1} P_k - C_{\epsilon 2}\epsilon) \tag{40}$$

$$P_k = -\overline{u_i' u_j'}\frac{\partial \overline{u_i}}{\partial x_j} \simeq \nu_t\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right)\frac{\partial \overline{u_i}}{\partial x_j} = 2\nu S_{ij}\frac{\partial \overline{u_i}}{\partial x_j} \tag{41}$$

and

$$\nu_t \simeq C_{\nu 1}\sqrt{k} L_k \simeq C_{\nu 1}\frac{k^2}{\epsilon} \tag{42}$$

The empirical parameters normally used in these k-ϵ models are:[89]

$C_\mu \equiv C_{\nu 1} C_D\ (C_D = 1.0)$	$C_{\epsilon 1}$	
(from equilibrium shear layers) 0.09	(from equilibrium wall layers) 1.44	
$C_{\epsilon 2}$	σ_k	σ_ϵ
(from decay of grid turbulence) 1.92	1.0	1.3

(43)

with C_μ and $C_{\epsilon 2}$ modified sometimes (e.g. axisymmetric jets), according to Refs. 99 and 107.

$C_{\nu 1}$ has been further improved, especially for thin (or weak) shear flows (e.g. far field jets and wakes) where turbulence is not necessarily in equilibrium, by simulating $C_{\nu 1}$ as an empirical function of $\overline{P_k/\epsilon}$:[107]

$$C_{\nu 1} = 0.09 f_2(\overline{P_k/\epsilon}) \tag{43a}$$

with

$$\overline{P_k/\epsilon} = \int_{y_I}^{y_E} \overline{u_1' u_2'} \frac{P_k}{\epsilon} y^j \, dy \Big/ \int_{y_I}^{y_E} \overline{u_1' u_2'} \, y^j \, dy \tag{43b}$$

where y_I, y_E refer, respectively. The flow area is y^j dy so in boundary of the shear layer. P_k/ϵ is weighted with $u_1'u_2'$, so that regions with large $\overline{u_1'u_2'}$ have a dominating influence on the determination of $C_{\nu 1}$. y^j dy is the flow area, so in plane flows $j = 0$ and in axisymmetric flows $j = 1$. The graphical forms of f_2 are obtainable from Launder *et al.*[99] The improved empirical parameters of the k-ϵ model now read as:

$C_{\nu 1}$	$C_{\epsilon 1}$	$C_{\epsilon 2}$	σ_k	σ_ϵ
$0.09 f_2(\overline{P_k/\epsilon})$	1.43	1.94	1.0	1.3

(44)

for planar flows, and for axisymmetric flows with $C_{\nu 1}$ and $C_{\epsilon 2}$ modified according to:

$$C_{\nu 1} = 0.09 f_2(\overline{P_k/\epsilon}) - 0.0534 f_1 \tag{45a}$$

$$C_{\epsilon 2} = 1.94 - 0.1336 f_1 \tag{45b}$$

with the retardation parameter f_1 defined in references 99 and 107.

The empirical parameters have also been suitably adapted so as to make the k-ϵ model applicable for low-*Re* flows[108] and for laminarizing boundary layers.[109]

To understand the limitations of the two-equation PK models the assumptions, all or most of which are implicit in them are listed below:

- Nearly homogeneous turbulence. This is the first condition for the use of:

$$\overline{u'v'} = -C_{\nu 1} \sqrt{k} L_k \frac{\partial u}{\partial y}$$

  in the shear-stress source terms.
- High Reynolds numbers. This is the second condition for the use of the above expression.
- Similarity of the spectral distribution of the turbulence quantities.
- Diffusion is of the gradient type. For simplicity, constant effective Prandtl numbers σ_z are used.
- Dissipation of k can be expressed by $\epsilon = C_D k^{3/2}/L_k$ with C_D as a universal constant;[110] the viscous-diffusion terms are negligible.

The assumptions are so many that not all of them can be valid for the flows of practical interest. It must therefore be expected that the constants are not truly universal but functions of characteristic flow parameters.

The researchers of turbulence modelling have insisted that their two-equation models first describe properly the decay of isotropic turbulence, and then have worried about the behaviour of their models in homogeneous shear flows where the transport terms vanish.[1] It is therefore assumed that the same empirical input is adequate for both types of flow. For further discussion of this, see Comte-Bellot and Corrsin,[111] Lumley and Khajeh-Nouri[112] and Reynolds.[113] It seems most desirable to model the source of ϵ by reference to experiments in nearly homogeneous flow, where the transport would not confuse the issue. There are two types of such flows, those involving pure strain and those involving pure shear. Tucker and Reynolds[114] and Marechal[115] studied the pure strain case; Champagne *et al*[116] and Rose[117] studied homogeneous shearing flows. For their use in modelling, see Reynolds.[113]

The form of the models presented so far has by implication adopted the notion of a scalar turbulence viscosity. This assumption forces the principal axes of $\overline{u_i'u_j'}$ and the mean-strain rate S_{ij} to be aligned. This is true in pure strain, but not true in any flow with mean vorticity. In practice, this supposition has proved perfectly adequate in 2D flows without swirl, where only one stress component exerts much influence on the flow development. In flows with swirl, however, and indeed in 3D flows generally, the measured flow distribution can be predicted in detail only by choosing a different level of viscosity for each active stress component. Such an attempt is described in detail by Launder[118] and Rodi;[107,123] applications of the procedure have been reported by Launder and Ying[119,120] to the flows in square-sectioned ducts, by Rodi[107] to obtain the normal-stress profiles in some free-shear flows, and by Koosinlin and Lockwood[121] to the calculation of flows near rotating cones and discs. The starting point of this procedure in deriving the relevant stress-strain formulae is the exact equation for the transport of Reynolds stress. Details on approximating the diffusive, dissipative and redistributive terms in the equation are given by Launder *et al*.[122] What is especially important in this context is that the approximation of neither the dissipative nor the redistributive terms contain gradients of stress components. The essence of 'algebraic' stress modelling then resides in reducing the differential stress equation to an algebraic set of equations among the Reynolds stresses, the turbulence energy, the energy dissipation rate and mean velocity gradients:

$$\overline{u_i'u_j'} = f\left(\overline{u_p'u_q'}, k, \epsilon, \frac{\partial U_l}{\partial X_m}\right) \tag{46}$$

This is mainly done by neglecting the convection and diffusion terms, or more generally by assuming that the transport of $\overline{u_i'u_j'}$ is proportional to the transport of k.[123] By this means, most of the basic features of the full stress model are retained. k, ϵ appearing in equation (46) may be found from the pair of pdes (39) and (40); this is the simpler practice. Alternatively, one may use the values of $\overline{u_i'u_j'}$ obtained from (46) to replace $\nu_t(\partial U_i/\partial X_j + \partial U_j/\partial X_i)$ which appears in the generation terms of these equations; this would be a more consistent practice and probably a more accurate one too. This modelling lies between the PK and stress-equation modelling trying to combine the economy of the former with the 'universality' of the latter, especially in accounting for anisotropic and wall effects.[24]

Saffman[124] tried the following constitutive equation:

$$\overline{u_i'u_j'} = \frac{k}{3}\delta_{ij} - 2\nu_t S_{ij} - Cl^2(S_{il}\Omega_{lj} + S_{jl}\Omega_{li}) \tag{47}$$

where Ω_{ij} is the rotation tensor, C is a constant, and the first two terms are the same as in the scalar definition. In a two-equation model l would be expressed in terms of k and ϵ.

Equation (47) does produce the right sort of normal-stress anisotropy in shear flows, but the new terms do not alter the shear stress and hence (47) works no better than the scalar definition of eddy viscosity.

Two-equation models also fail to predict the return to isotropy after the removal of strain, or the isotropizing of grid-generated turbulence.[111] This failure arises because of the need for a constitutive equation for the $\overline{u_i'u_j'}$. To use the $k \sim \epsilon$ model in an inhomogeneous flow, Jones and Launder[108,109] used a gradient-transport model for the diffusive fluxes of ϵ, that involves an additional constant = 0.77. Lumley[125] showed that the diffusive flux of dissipation should depend also on the gradients in turbulence energy, and vice versa; therefore two gradient terms should be introduced in the diffusive fluxes of k and ϵ.[1]

Another difficulty with using the ϵ-equation as the second model equation has escaped the model developers and has been pointed out by Reynolds and Cebeci.[1] This arises from term VII of equation (36), the pressure gradient-velocity gradient term in the diffusion. Since the pressure field depends explicitly on the mean velocity field, mean velocity gradients can explicitly generate ϵ-transport. This could be an extremely important effect, especially near a wall. The omission of this consideration may be an important deficiency in all ϵ-equation models studied to date.

Other two-equation models include the Saffman-Wilcox[57] and the Wilcox-Traci[126] models. Instead of an ϵ-equation they use an equation for a 'pseudo-vorticity', a typical inverse time-scale of the energy-containing eddies.[127] In addition, they use the k-equation with appropriate modelling of the production and 'isotropic dissipation' terms. The production term as modelled in reference 57 is inconsistent with the $\overline{u_i'u_j'}$ transport equation but Reynolds and Cebeci[1] have commented that this may in fact be a strength. This model of the production is based on the experimental fact that the structure of the turbulence in the wall region of a boundary layer is essentially independent of the strain rate, and hence the production should be proportional to k. Hence the Saffman-Wilcox model is a blend of the 'Newtonian' and 'structural' alternatives, equations (16) and (25) (see also reference 56). Recently, Wilcox and Rubesin[128] have demonstrated that the Saffman-type production terms[92] can introduce errors, and they have reintroduced the unmodelled terms. A good description of the Wilcox-Rubesin model and the work at NASA Ames is given in reference 129.

In spite of the above difficulties with models based on constitutive equations for $\overline{u_i'u_j'}$, their simplicity (and therefore computational economy) makes them attractive. Two-equation models have been extensively studied and applied with various levels of success. Other models (e.g. stress-equation models, large-eddy simulation etc) also have problems. Therefore, there may still be space for further development of two-equation models. Two interesting proposals for improving the 'universality' of the $k \sim \epsilon$ model have been made by Pope[33] and Hanjalic and Launder.[130] The former introduced a 'vortex-stretching generation' term in the ϵ-equation, and correctly predicted both plane and round jets with the same set of constants. The latter introduced a preferential-irrotational-straining generation' term in the ϵ-equation (guided by the multi-scale $k \sim \epsilon$ work, see section on the multi-scale k-ϵ model) and obtained improved results for the round jet and for boundary layers in adverse pressure gradients. These proposals deserve further study. Other attractive propositions appear to be the use of equation (46) (as followed by Rodi's group at Karlsruhe,[131] or equation (47) with two-equation models; since they allow for anisotropy of the normal stresses, they can be used for predicting turbulent boundary layer separation, buoyant flows, rotating flows and flows with strong streamline curvature, which cannot be predicted properly by the commonly used scalar eddy-viscosity assumption.

Stress-equation models

In turbulent shear flows, the energy is usually first produced in one component and then transferred to the others by turbulence processes.

In complex flow situations such as in recirculating flows, the MTE models based on local isotropy of turbulence (or one velocity scale, $\sqrt{k}$) are sometimes inadequate to represent the local state of turbulence. The heated asymmetric plane jet presents simultaneously two counter-gradient diffusion zones, one for the dynamical field, the other for the thermal field.[207] Effective viscosity and diffusivity concepts cannot be retained for this type of flow. These deficiencies may be overcome by mean-Reynolds-stress (MRS) models which explicitly employ transport equations for the individual Reynolds stresses, $-\overline{u_i'u_j'}$, (and for $\overline{u_j'T'}$), each representing a separate velocity scale. An exact MRS-equation, derived from the Navier-Stokes equations by Chou[104] is (for incompressible flows):

$$\frac{D\overline{u_i'u_j'}}{Dt} = \left(\frac{\partial}{\partial t} + \overline{u_m}\frac{\partial}{\partial x_m}\right)\overline{u_i'u_j'} \equiv P_{ij} + \pi_{ij} + \mathscr{D}_{ij} - \epsilon_{ij} \tag{48}$$

where:

$$P_{ij} = -\left(\overline{u_i'u_m'}\frac{\partial \overline{u_j}}{\partial x_m} + \overline{u_j'u_m'}\frac{\partial \overline{u_i}}{\partial x_m}\right) \quad \text{(generation tensor)} \tag{48a}$$

$$\pi_{ij} = +\overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} \quad \text{(pressure strain 'redistribution tensor')} \tag{48b}$$

$$\mathscr{D}_{ij} = -\frac{\partial}{\partial x_m}\left[\overline{u_i'u_j'u_m'} + \overline{\frac{p'}{\rho}(u_i'\delta_{jm} + u_j'\delta_{im})} - \nu\frac{\partial}{\partial x_m}(\overline{u_i'u_j'})\right] \quad \text{(diffusion)} \tag{48c}$$

$$\epsilon_{ij} = 2\nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_j'}{\partial x_m}} \quad \text{(viscous dissipation tensor)} \tag{48d}$$

The radically new feature of the MRS-equation is the pressure-strain 'redistribution' term (π_{ij}) which, having zero trace (putting $i = j$), does not appear in the exact k-equation (26a). This suggests that the pressure-strain term only serves to redistribute the turbulence energy among its components (when $i = j$) and to reduce the shear stresses (when $i \neq j$), thus tending to make the turbulence more isotropic. The unknown correlations appearing in the MRS-equation are either determined by a transport equation or else are expressed in terms of second-order correlations ($\overline{u_i'u_j'}$) themselves; the latter procedure, closing the

MRS-equation at its own level, is often referred to as 'second-order closure'. The production term needs no modelling. There have been extensive efforts[24,132] aimed at modelling the remaining terms. Modelled forms commonly used for the various terms of the MRS-equation (48) can be summarized as:

$$\pi_{ij} \equiv \pi_{ij,1} + \pi_{ij,2} \qquad \text{(pressure-strain term)} \quad (49)$$

where:

$$\pi_{ij,1} \equiv \text{func}(\overline{u_i' u_j' u_m'}) \simeq -C_1 k b_{ij} / \mathscr{T}_k \qquad \text{(return to isotropy hypothesis[85])} \quad (50)$$

with b_{ij} defined as anisotropy of $\overline{u_i' u_j'}$:

$$b_{ij} \simeq (\overline{u_i' u_j'}/k - \tfrac{2}{3}\delta_{ij} k) \quad (50a)$$

$\mathscr{T}_k$ defined as characteristic time scale of energy containing eddies:

$$\mathscr{T}_k \simeq k/\epsilon \quad (50b)$$

and C_1 as an empirical parameter given by:[133,134]

$$C_1 \simeq \begin{cases} 1.5 \\ 1.8 \end{cases} \quad (50c)$$

and

$$\pi_{ij,2} \equiv \text{func}(\overline{u_i' u_j'}, \overline{u_i' u_l'}, \dots) \frac{\partial \overline{u_l}}{\partial x_m} \simeq -C_2(P_{ij} - \tfrac{2}{3}\delta_{ij} P_k), \qquad \text{(quasi-isotropy approximation[135])} \quad (51)$$

with P_{ij} defined by equation (48a), P_k defined as the shear production of k:

$$P_k = \tfrac{1}{2} P_{ii} \quad (52a)$$

and C_2 as an empirical parameter is given by:[133]

$$C_2 \cong 0.6 \quad (52b)$$

Diffusion term:

$$\mathscr{D}_{ij} \cong c_s \frac{\partial}{\partial x_m}\left(\frac{k}{\epsilon}\overline{u_l' u_m'} \frac{\overline{\partial u_i' u_j'}}{\partial x_l}\right) \qquad \text{(gradient-diffusion assumption[136])}$$

$$c_s \cong 0.25 \quad (53)$$

Viscous dissipation term:

$$\epsilon_{ij} \cong \tfrac{2}{3}\delta_{ij}\epsilon \qquad \text{(local isotropy assumption at large } Re\text{)} \quad (54)$$

With equations (49)-(54), the MRS equation (48) yields the MRS-model for non-buoyant, incompressible, free flows:

$$\frac{D\overline{u_i' u_j'}}{Dt} - \mathscr{D}_{ij} = -C_1 \frac{\epsilon}{k}\overline{u_i' u_j'} + \tfrac{2}{3}\delta_{ij}(C_1\epsilon + P_k - \epsilon) + (1 - C_2)(P_{ij} - \tfrac{2}{3}\delta_{ij} P_k) \quad (55)$$

(Wall effects (and free-surface effects) are accounted for via 'wall corrections' to the pressure-strain term[131,133,136,137]): An eddy viscosity is not needed to derive $\overline{u_i' u_j'}$ but may be used in modelling the 'transport' $\mathscr{D}_{ij}$. It is very clear from experiments that, at high Reynolds number, the small-scale dissipative structures are isotropic. Hence all workers now use equation (54). The redistribution term has been the subject of most controversy and experimentation.[1] In a flow without any mean strain, this term is responsible for the return to isotropy. In deforming flows, however, the situation is much more complicated. Guidance is provided by the exact equation for the fluctuating pressure, which is a Poisson equation containing a source term in two parts. One part involves the mean deformation explicitly, and its contribution to the redistribution term can be obtained for homogeneous fields in terms of the Fourier transform of the velocity field.[113] Models have been proposed for this.[105] The remaining part of the redistribution term should not change instantly when the mean deformation is changed, and hence should not depend explicitly on the mean deformation. The best model for this is Rotta's and has been followed by Launder and his co-workers, and others. Application of these models indicates a much slower return to isotropy than indicated in experiments.[138,139] However, different components return at decidedly different rates. It can be concluded that current stress models will not perform very well in handling the return to isotropy. They may, however, work well in flows dominated by other effects.

Inhomogeneities greatly complicate the 'redistribution' modelling. In a wall region, a complex integral model is really needed for such flows. This is a very unsatisfactory aspect of present stress-equation modelling and an area for future developments. In addition to modifications in 'redistribution', inhomogeneities require modelling of the 'transport' $\mathscr{D}_{ij}$. The gradient diffusion model is usually employed (see also Hanjalic and Launder).[105] Reynolds and Cebeci[1] remarked that $\mathscr{D}_{ij}$ contains one pressure-velocity term (equation 48c), and since p' will have a part that depends explicitly on the mean velocity gradients, it does seem that $\mathscr{D}_{ij}$ should also be explicitly linear in the mean gradients.

Three approaches have been used in stress-equation modelling; see Donaldson *et al.*[140-142] Launder *et al*[131,134] and Lumley *et al.*[112,143-146] The earlier work (Donaldson[140]) involved specification of the length scale and use of the Norris and Reynolds[44] expression to determine ϵ. Hanjalic and Launder[105] used the ϵ-equation model in conjunction with the $\overline{u_i' u_j'}$ equations. Current attempts made towards the refinement of the MRS-model equation (55) are now summarized. Leslie[147] reported that Lumley's proposition[143] of treating C_1 as a function of b_{ij} does not make any significant improvement in equation (50). With regard to $\pi_{ij,2}$ Leslie[147] has argued that the model of Launder *et al*[134] is not general enough to handle quasi-homogeneous anisotropic turbulent flows (such as those of a strong-shear type). Although a nonlinear model for $\pi_{ij,2}$ is not suggested an algebraic stress model[123] is recommended[147] for the estimation of anisotropies (particularly for the component $\pi_{23,2}$) in such flows.

A recent application of second-moment turbulence closure refers to the asymmetrical heated plane jet.[208] Except for the level of the streamwise heat flux on the high velocity side of the flow, no notable differences were found in the comparison of predictions with measurements.

Stress-equation models are still in a state of development and it is felt that it will be some time before these models are sufficiently well-developed to perform better than simpler models for engineering problems. Improvements in the modelling of the pressure interaction and diffusion processes would still be of interest. Such models

are, however, important as the starting point for deriving algebraic stress relations, and should be pursued further.

Multi-scale $k \sim \epsilon$ model

The turbulence transport models discussed employ almost the same modelled transport equation for ϵ and may differ in the treatment of the Reynolds shear stresses. This is because the engineer's interest in the dissipation processes has been simply to account for the surplus energy, unaccounted for by k or any other variable governing the production process, in terms of a variable ϵ (referred to as energy transfer rate into the dissipation range). Hanjalic and Launder[106] have argued, following Pope and Whitelaw[148] for recirculating flows, and Launder and Morse[149] for round jets in stagnant surroundings, that since even the higher order closures in MRS models show the same poor agreement as the simple two-equation isotropic eddy-viscosity models, the ϵ-equation (being common to all these models) is the probable source of error in predicting an incorrect turbulence energy level.

From energy considerations, it is probable that the two inherent assumptions in the ϵ-equation, namely, (1) the local equilibrium of the rate of transfer of turbulence energy (implying that the local rate of energy transfer across the spectrum is unchanged and is equal to the local rate of dissipation), giving rise to a single characteristic turbulence scale (i.e. ϵ/k) and (2) the unchanged spectrum shape from one flow to another, may not necessarily be true in complex flow situations. It has been pointed out[132] that although the more elaborate Reynolds stress model of Lin and Wolfshtein[150] provides transport equations for the individual components of ϵ_{ij}, it calculates only the single scale. Following Kolmogorov[110] (who postulated that the turbulence spectrum comprises independent production, inertial and dissipation ranges), Hanjalic and Launder[106] have, therefore, proposed a multi-scale model in which separate transport equations are solved for the turbulence energy transfer rates across the spectrum. Using *Figure 6* (taken from their paper) the model is mathematically expressed as:

$$\frac{Dk_P}{Dt} \simeq \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_{k_P}}\frac{\partial k_P}{\partial x_j}\right) + P_k - \epsilon_P \tag{56}$$

$$\frac{Dk_I}{Dt} \simeq \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_{kI}}\frac{\partial k_I}{\partial x_j}\right) + \epsilon_P - \epsilon_I \tag{57}$$

$$\frac{D\epsilon_P}{Dt} \simeq \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_{\epsilon_P}}\frac{\partial \epsilon_P}{\partial x_j}\right) + \frac{\epsilon_P}{k_P} \times \left(C_{1P}P_k - C_{3P}\nu_t\epsilon_{ijn}\epsilon_{lmn}\frac{\partial \overline{u_i}}{\partial x_j}\frac{\partial \overline{u_l}}{\partial x_m}\right) - C_{2P}\frac{\epsilon_P^2}{k_P} \tag{58}$$

$$\frac{D\epsilon}{Dt} \simeq \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_j}\right) + C_{1I}\frac{\epsilon\epsilon_P}{k_I} - C_{2I}\frac{\epsilon^2}{k_I} \tag{59}$$

$$\nu_t \simeq \left(C_\nu \frac{kk_P}{\epsilon_P}\right)^* \tag{60}$$

* This form has been obtained from simplifying the MRS equation by considering normal Reynolds stresses proportional to k and by taking the time scale for the pressure strain to be k_P/ϵ_P.

and with the following sets of coefficients and functions:

C_{1P}	C_{3P}	C_{2P}		
2.1	0.8	$\left[2-0.5\frac{(k_P-k_I)}{k}\right]$		
C_{1I}		C_{2I}	C_ν	$\sigma_{kP}=\sigma_{\epsilon P}=\sigma_\epsilon$
$\left[0.29\left(1+\frac{3\epsilon_P}{\epsilon}\right)\right]$		1.2	0.1	1.0

(61)

Here k_P and k_I are, respectively, the turbulence kinetic energy in the production and dissipation ranges, P_k is the rate at which turbulence energy is produced (or extracted) from the mean motion, ϵ_P is the rate at which energy is transferred out of the production range, ϵ_I is the rate at which energy is transferred into the dissipation range from the inertial range and ϵ is the rate at which turbulence energy is dissipated (i.e. converted into internal energy).

The above equations have been formulated by assuming that all the turbulence energy is generated up to wave numbers q_P (production range) and is contained at the maximum up to q_I (inertial range), and there is negligible time lag between energy entering the dissipation range and its being destroyed, i.e:

$$\epsilon_I \simeq \epsilon \tag{62}$$

The additional term (with coefficient C_{3P}) in the ϵ_P-equation has been introduced to improve the performance of the multi-scale model in shear flows. This term is interpreted as increasing the effect of normal strains relative to the shear strains or, in other words, promoting the higher rates of dissipation for irrotational than for rotational strains. It should be noted that, for normal strains, this term provides no contribution (leading to higher ϵ_P and ϵ) since $\epsilon_{ijm} = 0$ for $i = j$ or m. The multi-scale k–ϵ model becomes more realistic than the conventional (single-scale) k–ϵ model by the presence of ϵ_P (in place of P_k) in the dissipation-rate (ϵ) equation, simply because in flows where the turbulence production (P_k) is suddenly switched off, $\epsilon(\triangleq \epsilon_I)$ is not expected to decrease immediately. This model has been shown to improve the predictions of the

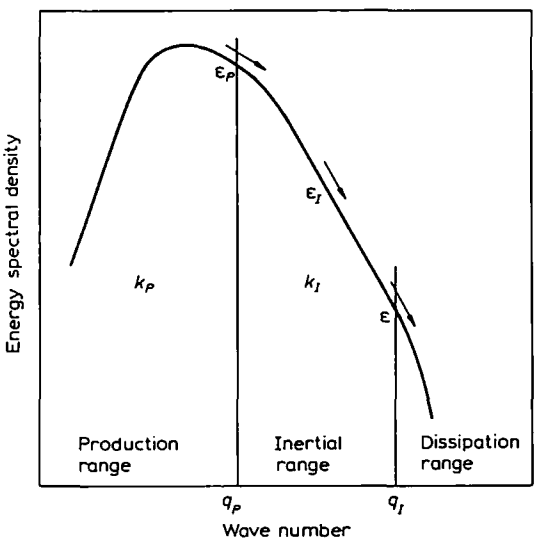


Figure 6 Partitioned energy spectrum. (Taken from reference 106)

single scale k–ϵ models in plane and round turbulent jets.[106] In general, the model (due to its multi-scale nature) is expected to be superior to the other existing single-scale models in predicting shear flows far away from local equilibrium. It has been pointed out that its poor performance in predicting plane wakes[106] can be improved if the effects of spectrum shape on the empirical coefficients are also properly accounted for. A useful evaluation of the multi-scale model was carried out by Fabris *et al.*[151]

Recent developments and applications of two-equation models

Another model which, in the author's opinion, deserves more attention than it has been given so far is the $k \sim w$ model of Spalding,[90] of which the second calculated quantity is w, a measure of the vorticity fluctuations. This model was shown by Spalding[152] and Gibson and Spalding[153] to yield predictions of 2D turbulent flows which were in good agreement with experimental data. However, it possessed the defect (shared also with the k–kl model of Rodi and Spalding,[93] and Ng and Spalding[101]) of requiring a modification of one of its constants for near-wall flows. This defect was regarded at the time as sufficient reason for allowing the k–w model (and the k–kl one) to be overshadowed by the $k \sim \epsilon$ model of Harlow and Nakayama,[86] which was then being refined by Jones and Launder;[109] for the $k \sim \epsilon$ model appeared to require no near-wall adjustment. During the last few years, experience with the k–ϵ model has not been consistently favourable, its success being uneven, and some early users of the k–ϵ have now returned.[91] Saffman[92] published an independently derived model having several points in common with the k–w model; and he has, wisely in the author's opinion, continued work on it subsequently. The rocket-plume fraternity in the USA still prefer the k–w model. It is said to fit the experimental data better than the $k \sim \epsilon$ model.[91]

The k–w model has in its favour a strong physical appeal. Experimental evidence has accumulated which points strongly to the vorticity fluctuations, resulting from the breakdown of vorticity sheets into less regular structures, as being central to the mechanism of turbulence, whereas the dissipation processes (which ϵ represents) are far less prominent. Finally, Spalding has recently[91,154] introduced a small modification to the k–w model which, while being quite general, completely solves the near-wall problem (This modification has also been introduced independently by Wilcox and Traci[126] into Saffman's model (see also reference 128)). The objectionable feature of the earlier version of the k–w model was its inclusion of a term $kw^{-1}y^{-2}$, which became very important when the distance, y, from a bounding wall was small. How should the fluid 'know about' its distance from a wall? And the inclusion of the wall-distance calculation in the equation-solution scheme was troublesome. Spalding[154] recognized that $|\text{grad}(kw^{-1})^{1/2}|$ would probably serve just as well, and that it would be free from the wall-distance problem. Thus, the w-equation now reads:

$$\frac{Dw}{Dt} = \rho^{-1}\left[\text{div}\left(\frac{\mu_t}{\sigma_w}\text{grad } w\right) + S_w\right] \tag{63}$$

where the left-hand side represents time-dependence and bulk-transport terms and the right-hand side represents the turbulence-diffusion transport and the 'source', which is expressed as:

$$S_w = k^{-1}w(C_3P_k - C_2\rho kw^{-1/2}) + C_1\mu_t(|\text{grad }\Omega|)^2 - C_4\rho w^{3/2}(|\text{grad}(kw^{-1})^{1/2}|)^{C_5} \tag{64}$$

where the last term is the novelty on which the new k–w model is founded. Ω is the major component of the local time-mean vorticity vector of the mean motion, regardless of direction. In tensor form the time-mean vorticity is defined as follows:

$$\Omega = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j} \tag{65}$$

where ϵ_{ijk} is the alternating tensor.

The revised model involves nine constants. Of these, seven have been established by earlier work ($\sigma_k = 1.0$, $\sigma_w = 1.0, C_\mu = 0.09, C_D = 1.0, C_1 = 3.50, C_2 = 0.17$, $C_3 = 1.04$). Of the two remaining constants, C_5 was guessed to equal 2.0, implying that the new term is proportional to the square of the gradient of the tubulence length scale and therefore independent of its sign. The constant C_4 was determined by reference to the Von Karman constant $\kappa = 0.435$ in the 'logarithmic law of the wall, as being equal to 2.97. The above model has been investigated by Ilegbusi and Spalding[154] and Malin and Spalding.[155] The first investigation has concerned itself with flows which are the same as those reported on by Spalding[152] and Gibson and Spalding,[153] namely: abrupt enlargement of pipe diameter (data from Krall and Sparrow[156] and Zemanick and Dougall[157]); boundary layers with longitudinal pressure gradient (data from Bradshaw[158]); plate with intense mass transfer (data from Moffatt and Kays[159]); plane free turbulent jet (data from Bradbury[160]); and several more.

Figures 7–15 present some results of this investigation. *Figure 7* refers to the experimental data of Zemanick and Dougall[157] for an air flow in a pipe downstream of an orifice, with expansion ratio of 0.54. The figures demonstrate satisfactory agreement between predicted and experimental Nusselt numbers. The variation of maximum Nusselt number suggests that the predicted behaviour exhibits a slightly smaller sensitivity to Reynolds number than do the measurements. *Figure 8* refers to the Klebanoff data for a smooth flat plate, as processed for the AFOSR-IFP Stanford conference by Coles and Hirst,[161] and presents distributions of turbulence kinetic energy and Reynolds shear stress. The agreement is satisfactory, indeed probably within the margin of experimental error over most of the range. *Figures 9 and 10* refer to the flat plate experiments of Bradshaw,[158] in which a mild and more severe adverse pressure gradient were imposed. They show typical predictions of turbulence energy and the shear stress compared with the experimental data. The agreement is satisfactory for the shear stress but not for the turbulence energy.

Figure 11 shows the predicted and measured variation with the length Reynolds number of the Stanton number, for the experimental situation of Moffat and Kays.[159] The experiment is for heat transfer from a flat plate having a porous surface, through which air can be sucked from, or blown into, the boundary layer. The results show that agreement is satisfactory even at strong suction rates. The latter success is a result of the use of a sub-layer resistance factor that ensures the asymptotic limit ($St \to -\dot{m}/\rho_G U_G$) is approached for these cases.[154] For strong blowing, the experimental Stanton number tends to zero; this behaviour is quite adequately predicted.

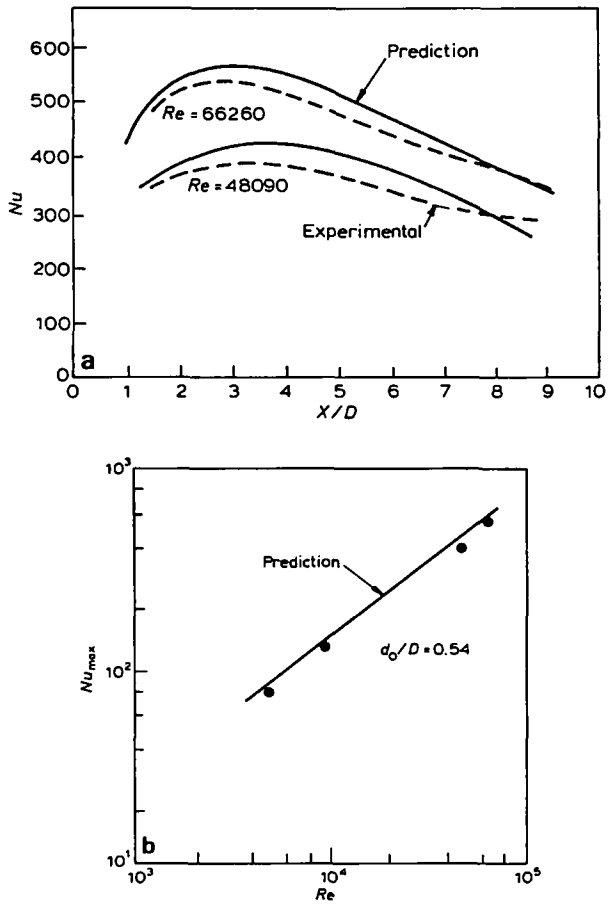


Figure 7 (a) Nusselt number distribution for $d_0/D = 0.54$ (Pipe-expansion data of Zemanick and Dougall, 1970). (Taken from reference 154). (b) Variation of maximum Nusseld number with Reynolds number $d_0/D = 0.54$. (Pipe-expansion data from Zemanick and Dougall 1970). (Taken from reference 154)

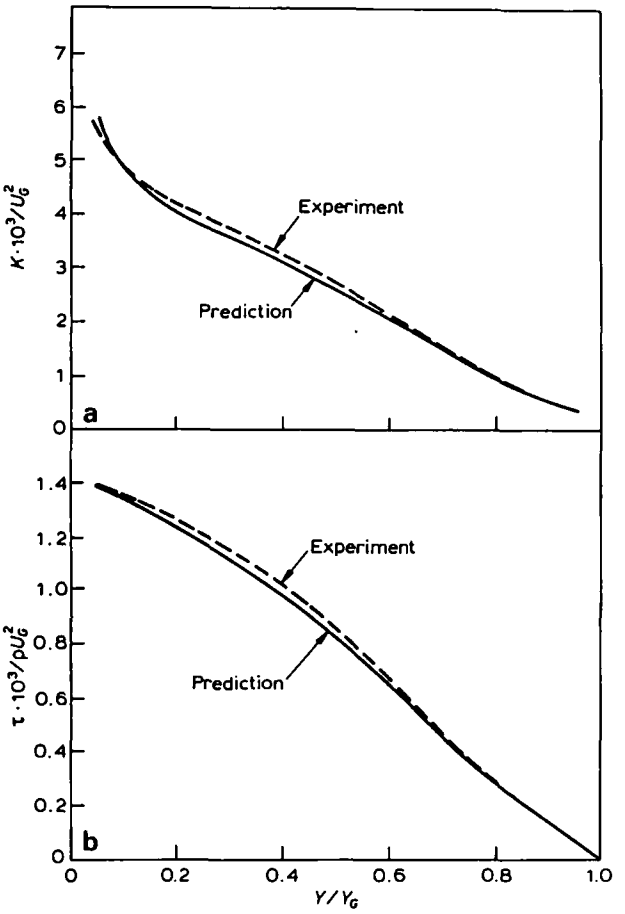


Figure 8 Distributions on flat plate of (a) turbulence kinetic energy; (b) Reynolds shear stress. (Experimental data from Klebanoff, 1984). (Taken from reference 154)

Figures 12-15 refer to the experimental situation of Bradbury,[160] which is for a plane jet issuing into slow-moving surroundings. The predictions seem to be in fair agreement with the measurements, except for the turbulence-energy distribution where the maximum predicted energy is some 7% above Bradbury's measurements.[160] However, the predicted and measured profile shapes are very similar. Finally *Figure 15* compares the predicted[155] and measured turbulence energy budgets for the self-similar jet, indicating the highest discrepancies in the production and then in the diffusion terms. The above results demonstrate that the k-w model can be accepted as a serious contender for general adoption. The studies of Saffman-Wilcox-Rubesin-Traci,[124, 126, 128] particularly for compressible and low-Re flows, tend also to confirm the above opinion.

Its current main rival is the $k \sim \epsilon$ model that has been exercised extensively with varying levels of success. For comparisons with experiment of predictions based on the $k \sim \epsilon$ model, see elsewhere.[35, 36, 162-166] Results obtained with the $k \sim \epsilon$ model for some complex engineering situations are presented in *Figures 16-21*. It should be pointed out that in several cases of practical interest (see Conclusions) the standard models do not perform well. In these cases *ad hoc* modifications to the 'constants' or additional (semi-theoretical) terms are required to produce better agreement between model predictions and

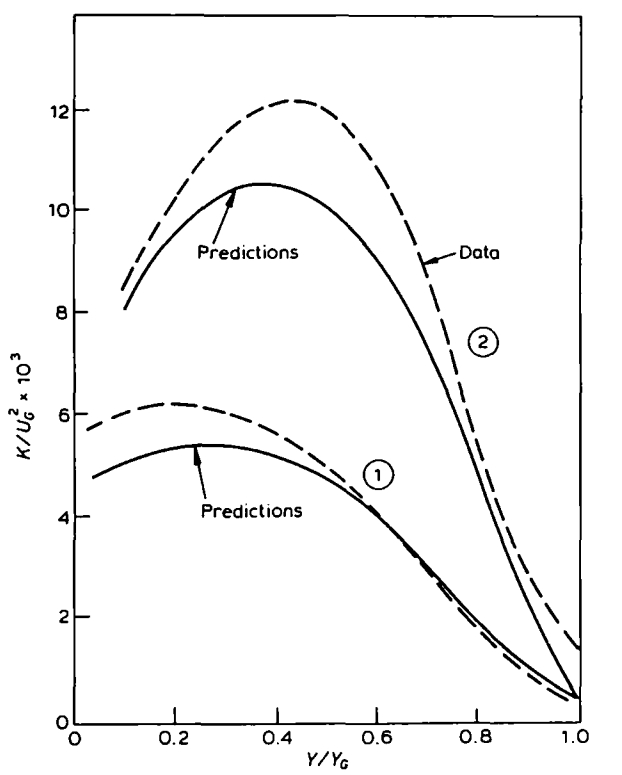


Figure 9 Predicted and measured turbulence energy on a flat plate with (1) mild and (2) severe pressure gradients. (Data from Bradshaw, 1965). (Taken from reference 154)

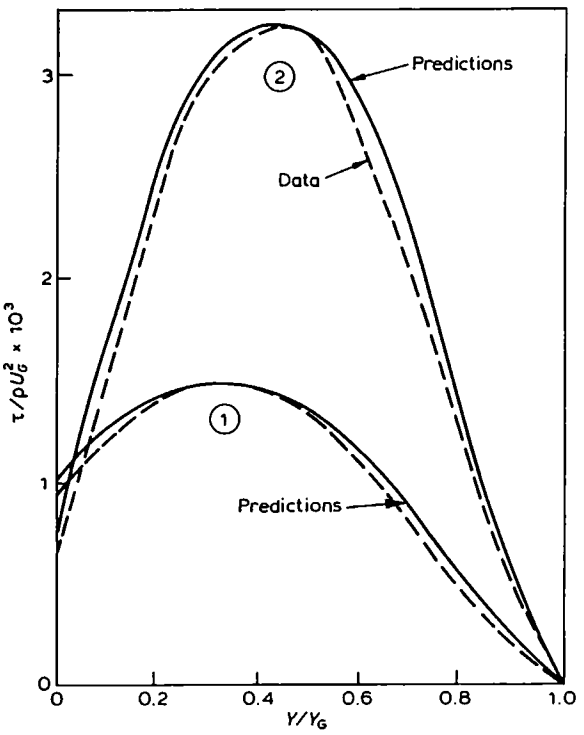


Figure 10 Predicted and measured shear stress on a flat plate with (1) mild and (2) severe pressure gradients. (Data from Bradshaw, 1965). (Taken from reference 154)

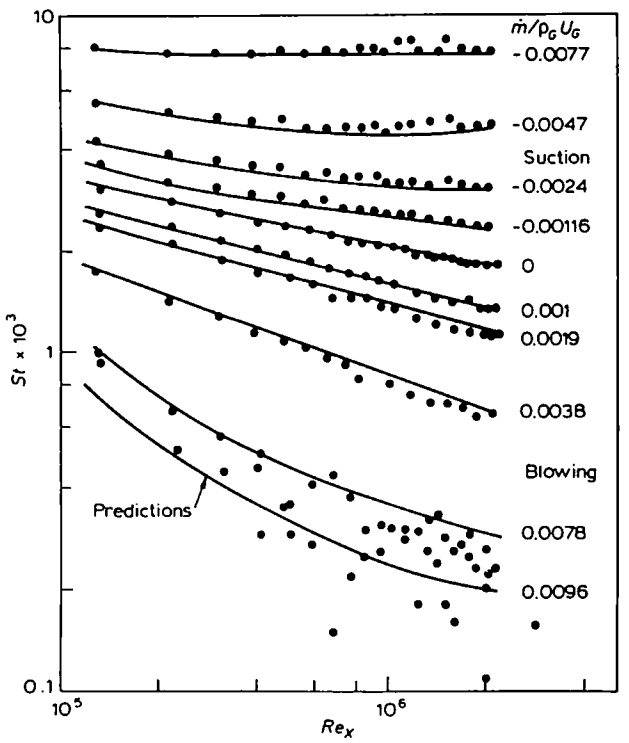


Figure 11 Predicted and measured variation with length Reynolds number of Stanton number for heat transfer to a porous flat plate with low temperature ratio and various degrees of suction and blowing. (Data from Moffat and Kays, 1968). (Taken from references 154)

experiment. Two examples of situations where such modifications to the $k \sim \epsilon$ model are necessary are (a) buoyancy-dominated flows and (b) flows subjected to rapid compression or expansion. For the former cases,

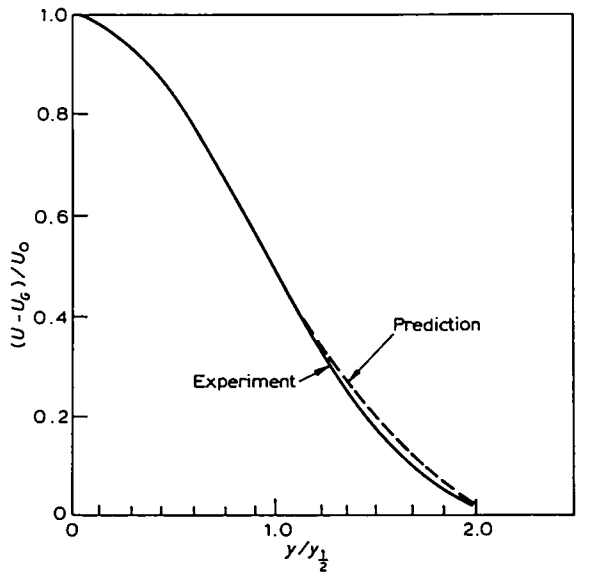


Figure 12 Axial mean velocity ($X/h = 70$). (Plane-jet data from Bradbury, 1965). (Taken from reference 154)

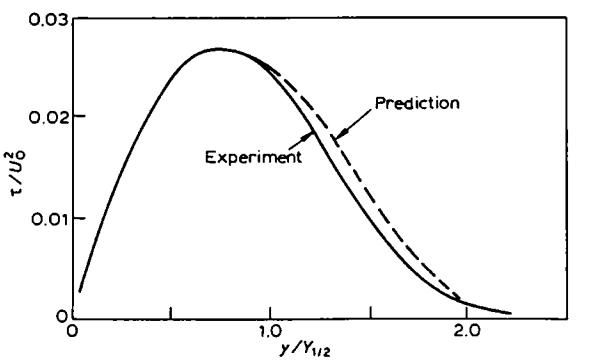


Figure 13 Shear stress ($X/h = 70$). (Plane-jet data from Bradbury, 1965). (Taken from reference 154)

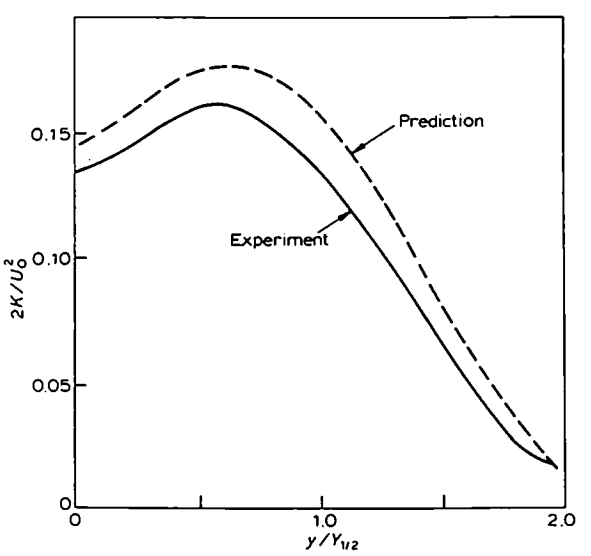


Figure 14 Turbulence kinetic energy ($X/h = 70$). (Plane-jet data from Bradbury, 1965). (Taken from reference 154)

the required modifications have been proposed in reference 24 and have been applied with relative success in, for example, references 35 and 36 (see also *Figures 16 and 17*). For the latter cases, modifications have been proposed[167] and have been generalized and applied,[168,169] with particular emphasis to flows in internal combustion engines. *Figure 21* compares the predicted swirl velocity in a 2D diesel engine[169] with the experiments of Ricardo Consultants.[170] It is clear that the modification to the $k \sim \epsilon$ model[167-169] led to a significant improvement of the predictions. Indeed, it has been reported[168] that

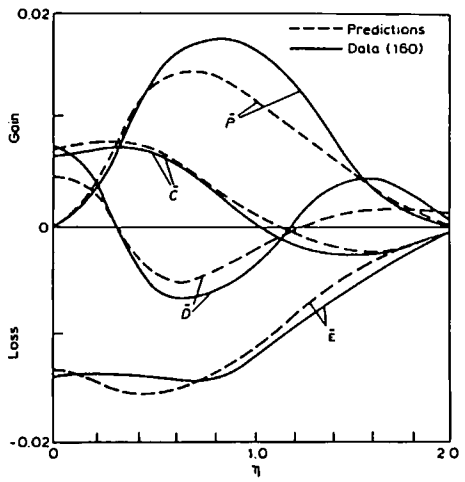


Figure 15 Plane jet, turbulent-kinetic-energy balances in self-similar region. (Taken from reference 155)

when the original model was used inside a piston-engine geometry with 8:1 compression ratio, a sharp increase in turbulence length was predicted, resulting in a length scale up to several times larger than the cylinder clearance height. By contrast, the modified turbulence model predicted a physically more plausible behaviour.

Other recent work refers to progress made in calculating developing flow in a square duct, with reasonable accuracy by means of length-scale models[171-172] and $k \sim \epsilon$ models.[173-175] Recent $k \sim \epsilon$ type predictions of fully-developed square duct flows are reported in reference 176 (see also reference 177 for algebraic stress modelling).

A critical review of closure approximations for two-equation models has been made.[178] Using a combination of singular perturbation methods and numerical computations, Wilcox[178] demonstrated that:

- conventional $k \sim \epsilon$ and $k \sim w$ formulations are generally inaccurate for boundary layers in adverse pressure gradient
- using 'wall functions' tends to mask the models' shortcomings
- a more suitable choice of dependent variables exists which is far more accurate for adverse pressure gradient ($k \sim f$).

Some other recent interesting reviews can be found in references 179-186.

Large-eddy and full simulations

There is a fundamental difficulty in the above general approach to turbulence modelling. One would like to

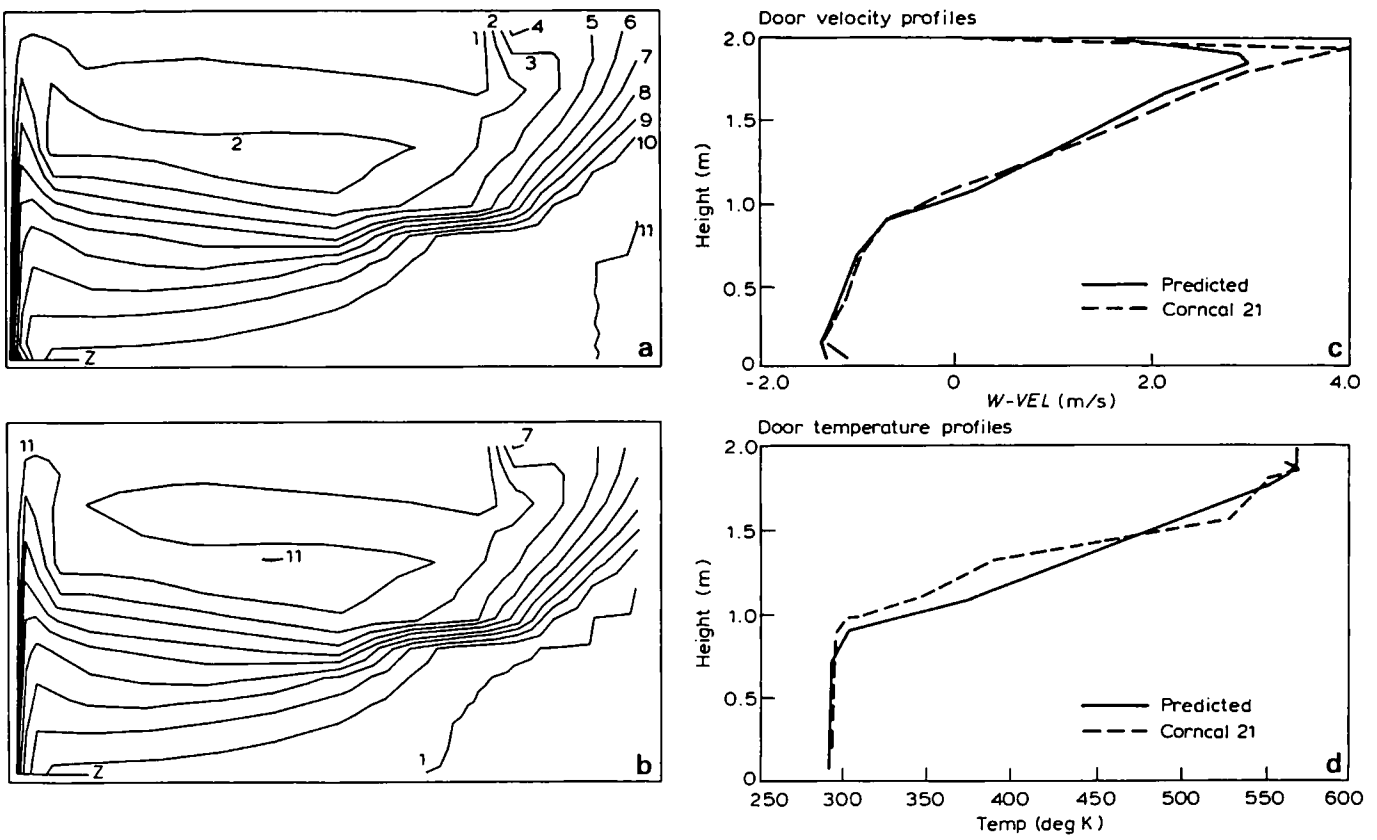


Figure 16 3D fires in enclosures (Taken from reference 36). (a) distribution of oxidant (10 contours in range 0.44-1.0); (b) distribution of combustion products (10 contours in range 0.001-0.55); (c) comparison of velocity profiles at door (symmetry plane). (——), predictions; (– – –), experiment; (d) comparison of temperature profiles at door (symmetry plane), (——), predictions; (– – –), experiment

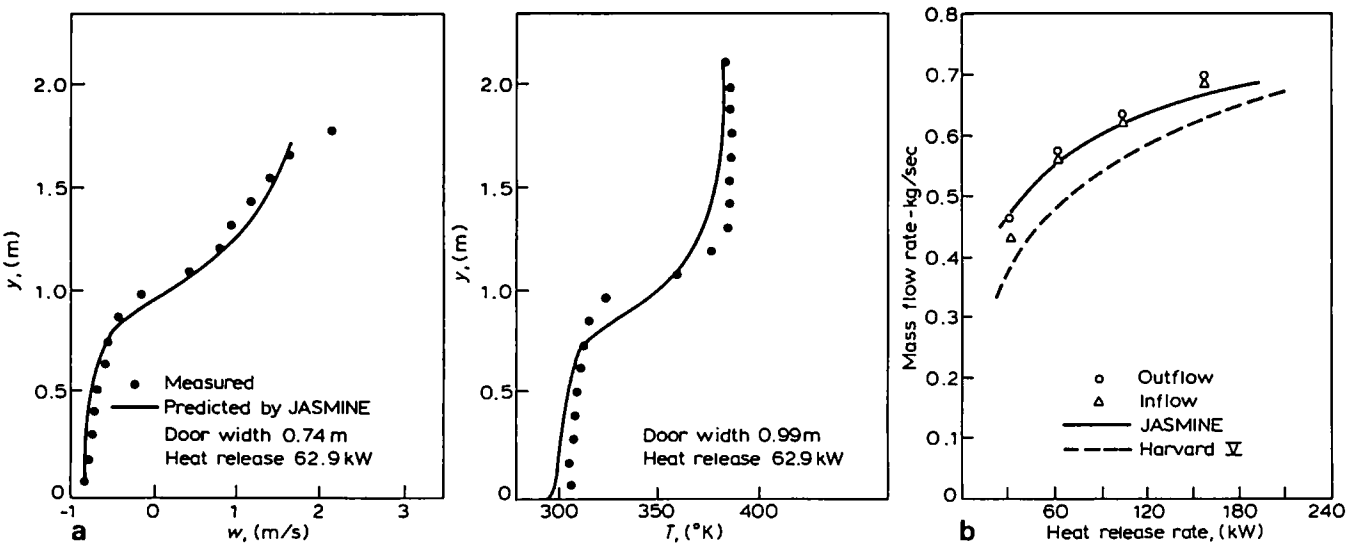


Figure 17 Smoke flow in 3D enclosures. (Taken from reference 35). (a) doorway velocity and corner temperature profiles; (b) comparison of Steckler's data with predictions of field and zone models

model only terms that respond on time scales short compared to that of the computed quantities. It is well known that the small scales respond to change much faster than the large scales, and hence it is reasonable to express a quantity dominated by small scales, such as ϵ, as a function of quantities dominated by large scales, such as $\overline{u_i' u_j'}$. However, terms like the 'transport' one have time scales of the order of those of $\overline{u_i' u_j'}$, and therefore one does not really expect an equilibrium relationship to exist between the transport and $\overline{u_i' u_j'}$. In general, higher order statistical quantities take longer to reach steady state than lower order statistics. Any model obtained by truncation at some statistical order would suffer from this difficulty. One really needs to truncate at some level of scale, and thereby take advantage of the fact that the smaller scales do adjust faster to local conditions. Then, by truncating at smaller scales, there is at least some hope of convergence, which is very difficult when truncation takes place at higher and higher orders of statistical quantities that have comparable time scales. The large-eddy simulation is a method implementing such a scale-truncation approach.

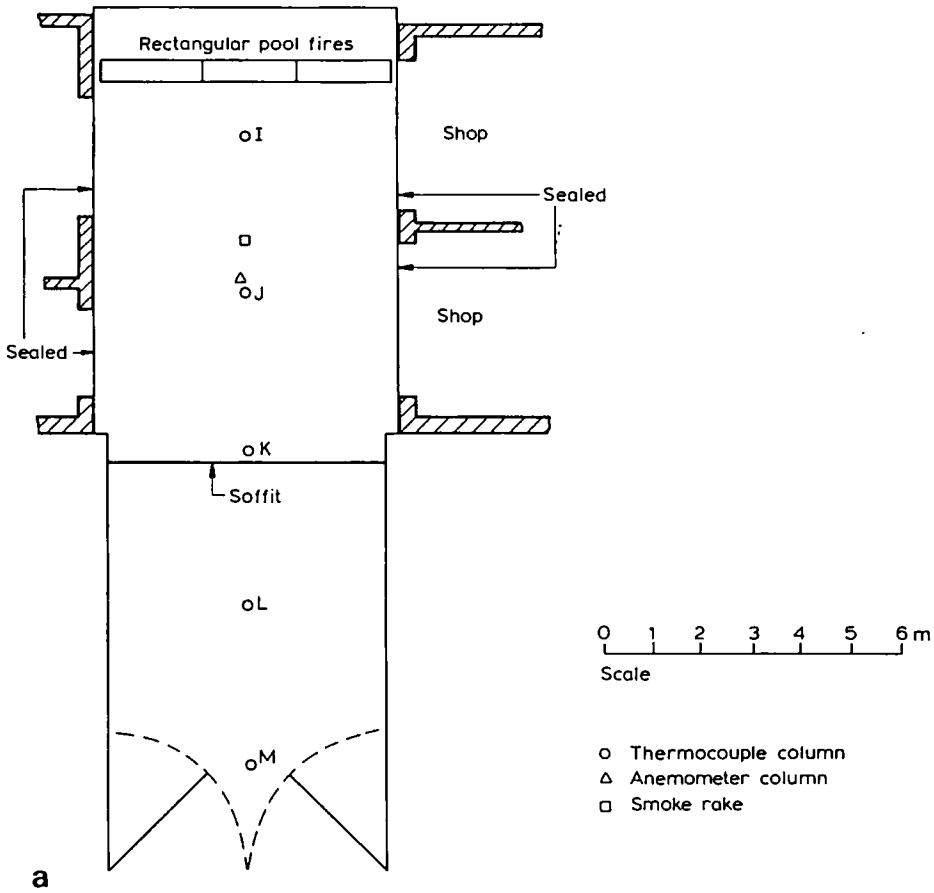
The idea is to do a 3D time-dependent numerical computation of the large-scale turbulence, and model the smallest scales that will always be impossible to compute.

In 1973 the Stanford group began a programme of development of this method, in cooperation with the NASA-Ames Laboratory. The first contribution was made by Leonard (reference 152, p. 237) who introduced spatial filtering (see also reference 138). When this operation is applied to the Navier–Stokes equation, and an expansion is carried out, an equation arises that contains the 'sub-grid-scale Reynolds stresses', and a stress-like term resulting from the filtering of the nonlinear terms (the 'Leonard stresses'). Kwak and Reynolds[138] applied the method to the isotropic decay problem, adjusting the sub-grid scale eddy viscosity to obtain the proper rate of energy decay. They also simulated the flow of reference 114 and found the main features of the experiments by using only 16 x 16 x 16 grid, and 5 min of CDC7600 for 120 time steps. Shaanan *et al*[139] used a conventional staggered grid approach that is second-order accurate and does not require the explicit inclusion of the Leonard stresses. They also reproduced the main features of the experiments of reference 116 using a 16 x 16 x 16 grid.

Another group involved in the development of this method is at Queen Mary College, London.[187–188]. They began with one dimensional Burgers turbulence, the only case in which it is possible to simulate the whole of a high Reynolds number turbulence field, and thus to check the process of subgrid modelling.[187] They have also done work on homogeneous flows with satisfactory results.[188] The technique of full simulation, is only available for transition flows. Once turbulence is fully developed, the range of eddy sizes is too great to be represented on any computer, and this is true even at the lowest Reynolds numbers at which 'proper' wall-bounded turbulence can be said to exist. Attempts at such full simulations are currently made at Stanford University and are very useful in guiding turbulence modelling. Ferziger[189] presented such work recently, and reported on using 128^3 grids and many hours of a supercomputer, CRAY X-MP, to simulate flows at low Reynolds numbers. These computer requirements are clearly outside practical limits; but such attempts are extremely useful in validating the lower-level models and should continue.

'Two-fluid' models of turbulence

The thinking which underlies current turbulence models has its roots in the notions of Boussinesq,[190] who introduced the idea of an effective viscosity, and Prandtl,[40] who conceived of turbulence mixing phenomena as being very similar to those treated by the dynamical theory of gases. This thinking has been immensely fruitful; but it leaves out of account features which turbulent fluids possess and assemblies of gaseous molecules do not. These features are vorticity and its major consequences.[27] The mathematical structure of conventional turbulence theory reflects only the unstructured diffusion of the molecular-collision process; large-structure formation and growth, and fine-structure creation and stretching, are nowhere to be found. The above facts led Spalding,[27] amongs others, to the development of the 'two-fluid' theory, briefly described below.



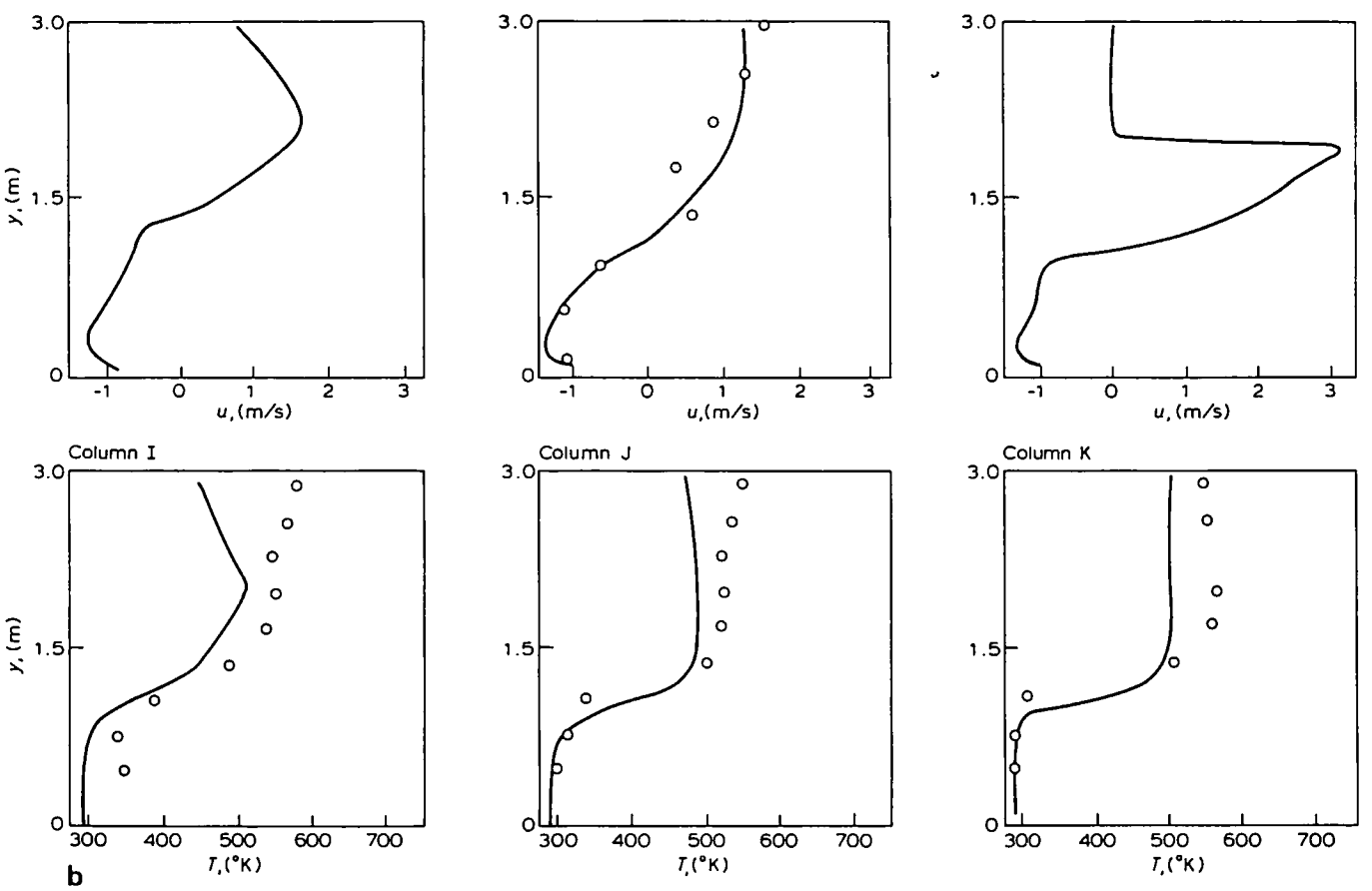


Figure 18 Smoke flow in enclosures (Taken from reference 205). (a) Plan of mall showing instrumentation; (b) predicted and measured velocity and temperature profiles for $D = 1$ m

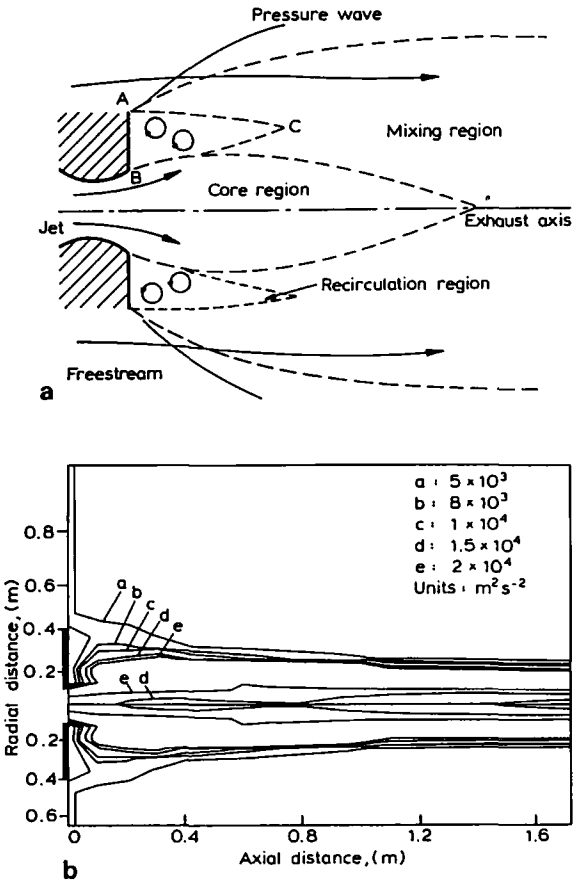


Figure 19 Flow and combustion in base-wall region of a rocket exhaust plume. (Taken from reference 204). (a) flow field around missle base wall; (b) contours of turbulence kinetic energy (m^2s^{-2}) for conditions of case 2, base wall present and free stream Mach number $M_u = 2.9$

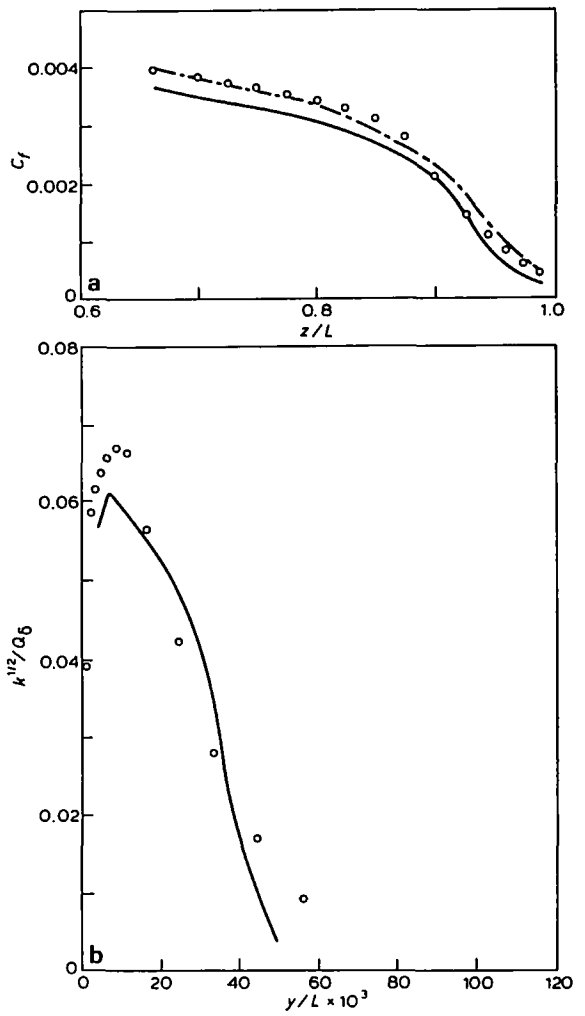


Figure 20 Flow around bodies of revolution. (Taken from reference 166). Q_δ is total velocity at boundary layer edge. (a) viscous-flow, full-body: distribution of skin friction coefficient along hull (b) viscous-flow, half-body: lateral variation of turbulent kinetic energy at $z/L = 0.96$. (000), predictions; (——), Patel experiment (reference 206); (— · — · —) Thompson (206)

The origins of two-fluid-model ideas are to be found back in the 1940s and 1950s, particularly in the context of turbulent combustion.[191-196]

Two-fluid model analysis

In the laboratory 'conditional-sampling' techniques have been developed which discriminate between the turbulent and nonturbulent zones of a turbulent shear flow. With these techniques the experimentalist is able to perform measurements of flow variables by averaging separately over the turbulent and nonturbulent parts of the flow. These measurements are referred to as conditional-sampling techniques[209] and supplement the usual unconditioned measurements obtained by conventional time-averaging. The two-fluid model described below results in the prediction of the intermittency and of the conditional flow variables within the turbulent and nonturbulent zones of the flow. This model was proposed by Spalding[27] and then developed by Malin[28,29,210] for the prediction of intermittency in free turbulent shear flows, and by Markatos and Pericleous[211] for turbulent combustion. It should be remembered that the model is still under development, and therefore the formulation that follows should not necessarily be regarded as final. There have been a number of studies concerned with the prediction of intermittent turbulent flows. The first studies appear to be those of Libby,[197,212] who presented a theoretical model which comprised equations for the intermittency and the conditioned flow variables. The basis of this work was the postulate of a conservation equation for the intermittency function, with an unknown source term representing the creation of turbulent fluid. Dopazo,[200] and later Duhamel[213] showed how to derive the exact equations for the intermittency and conditioned-averaged flow variables, thus avoiding the above postulate. The most advanced model of this type at present is probably the one developed by Kollman and his co-workers.[198,199,214] They developed both first-order and second-order closure models for intermittent turbulent shear flows, based on the equations derived by Dopazo.[200] The model contains the equations for the intermittency factor and the zone-averaged velocities, together with either a first-order or second-order closure model for the turbulent-zone stress tensor. This model was applied to the prediction of jets, mixing layers and boundary layers with some success. The Spalding[27] model has similarities with the models of Libby and Kollmann, but it is built on an analogy between intermittent flows and two-phase flows, rather than on any specific and rigorous closure of conditioned-averaged transport equations. The paper of Spiegel[215] must also be mentioned as a partial anticipator of the model described below.

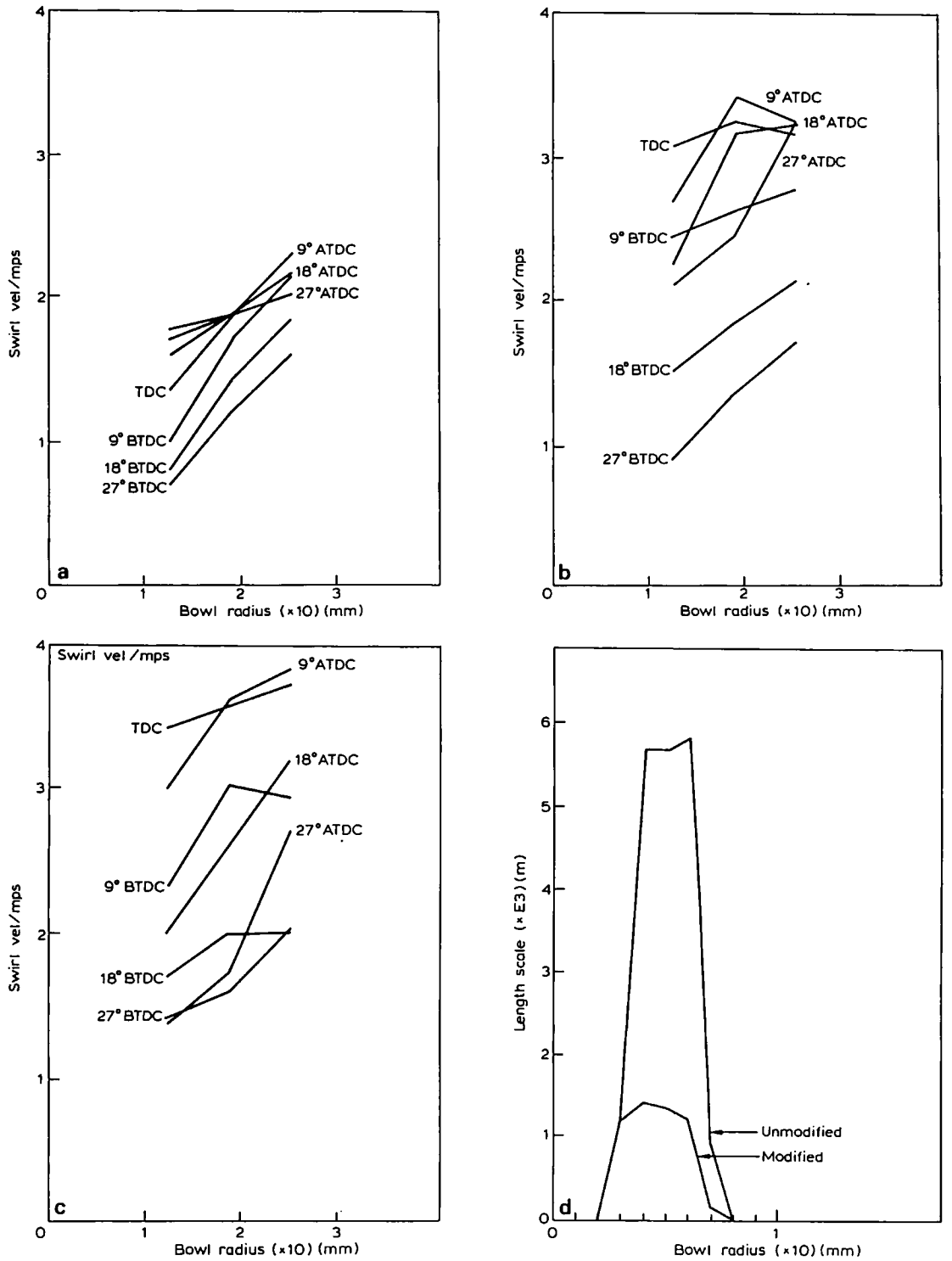


Figure 21 Swirl velocity (taken from reference 169). (a) Standard $k \sim \epsilon$; (b) modified $k \sim \epsilon$; (c) measurements (reference 170); (d) length scale at TDC. mps is mean piston speed (8.19 m s^{-1})

It is supposed that two-fluids share occupancy of space. The proportions of time during which each can be expected to occupy a particular location are called the volume fractions. There are many ways in which the two fluids could be distinguished e.g. by reference to their temperatures, chemical composition or mean velocities. When intermittency is to be computed, it is convenient to define the fluids as 'turbulent' and 'nonturbulent'. This definition is adopted in what follows and subscripts 1 and 2 denote the turbulent and nonturbulent fluids, respectively. The intermittency factor is interpreted as being the 'volume fraction' or 'presence probability' of the turbulent fluid.

The equations governing the motion of the turbulent and nonturbulent fluids are presented here in terms of Cartesian tensors (see also references 210 and 211).

The mean continuity equation of the turbulent fluid is:

$$(\phi_1 U_i)_{,i} = (\nu_t/\sigma_\gamma \phi_{1,i})_{,i} + \dot{E}''' \qquad (66)$$

and the mean global continuity is:

$$\phi_1 + \phi_2 = 1 \tag{67}$$

The mean momentum equations of the turbulent fluid are:

$$\begin{aligned}(\phi_1 U_j U_i)_{,j} = &-\rho^{-1}\phi_1 p_{,i} + (\phi_1 \nu [U_{i,j} + U_{j,i}])_{,j} \\ &- (\phi_1 \overline{u_i u_j})_{,j} + (\phi_1 \nu_t/\sigma_\gamma \cdot U_i \phi_{1,j})_{,j} \\ &+ V_i \dot{E}''' - F_i\end{aligned} \tag{68}$$

The corresponding equation for the nonturbulent fluid is:

$$\begin{aligned}(\phi_2 V_j V_i)_{,j} = &-\rho^{-1}\phi_2 p_{,i} + (\phi_2 \nu [V_{i,j} + V_{j,i}])_{,j} \\ &+ (\phi_2 \nu_t/\sigma_\gamma \cdot V_i \phi_{2,j})_{,j} - V_i \dot{E}''' + F_i\end{aligned} \tag{69}$$

The mean temperature equation of the turbulent fluid is:

$$\begin{aligned}(\phi_1 U_j T_1)_{,j} = &(\phi_1 \lambda T_{1,j})_{,j} - (\phi_1 \overline{u_i t'})_{,j} - \dot{Q}'''/\rho C_p \\ &+ (\phi_1 \nu_t/\sigma_\gamma T_1 \phi_{1,j})_{,j} + T_2 \dot{E}'''\end{aligned} \tag{70}$$

The corresponding equation for the nonturbulent fluid is:

$$\begin{aligned}(\phi_2 V_j T_2)_{,j} = &(\phi_2 \lambda T_{2,j})_{,j} + \dot{Q}'''/\rho C_p \\ &+ (\phi_2 \nu_t/\sigma_\gamma \cdot T_2 \phi_{2,j})_{,j} - T_2 \dot{E}'''\end{aligned} \tag{71}$$

In equations (66)-(71), the subscripts 1 and 2 denote the turbulent and nonturbulent fluids. Partial derivatives are represented by a subscript consisting of a comma and an index, U_i are the velocity components of the turbulent fluid, V_i are the velocity components of the nonturbulent fluid, ϕ_1 is interpreted as the intermittency factor (i.e. the probability of finding turbulent fluid in a field presenting the fragmentariness observed in turbulent flows), T_1 is the temperature of the turbulent fluid and T_2 is the temperature of the nonturbulent fluid. Fluctuating velocity components are denoted by lower-case letters and fluctuating temperature by t'.

In the continuity equation (66), the term $\dot{E}'''$ represents the rate of entrainment of nonturbulent fluid by turbulent fluid per unit volume and the gradient-diffusion term acts as a turbulence flux for the turbulent fluid. Therefore, this term is responsible for the creation of turbulent fluid; and it appears as source in the ϕ_1 -equation and as sink in the ϕ_2 -equation. Here, σ_γ is an empirical diffusion coefficient, given the value 1.55 and ν_t is akin to a turbulent-zone eddy viscosity. For thin turbulent shear layers, Malin[28] has computed ν_t from the 'mixing-length-type' formulation proposed by Spalding.[27] In future work, ν_t can be computed from a conventional two-equation turbulence model such as the one described below.

In the momentum equations (68) and (69), F_i is the mean friction force per unit mass that the turbulent fluid exerts upon the nonturbulent fluid in the i-direction. The other interfluid source term involves the entrainment $\dot{E}'''$; it represents therefore the average entrainment of momentum through the turbulent/nonturbulent interface. The turbulence stresses $-\overline{u_i u_j}$ and fluxes $-\overline{u_i t'}$, which appear in equations (68) and (70) respectively, are calculated with the aid of a turbulence model. The term $\dot{Q}'''$, which appears in the temperature equations (70) and (71), is intended to represent the heat transferred per unit volume by conduction at the turbulent/nonturbulent interface. The term involving $\dot{E}'''$ represents the mean entrainment of heat through the interface into the turbulent zone.

Time-averaged flow variables are recovered from the conditioned variables via the following relation:

$$\Phi = \phi_1 \Phi_1 + \phi_2 \Phi_2 \tag{72}$$

where Φ denotes any flow variable, such as velocity or temperature, and the subscripts 1 and 2 denote the turbulent and nonturbulent fluids.

In order to close the above set of equations (66)-(71), modelling assumptions are required for $-\overline{u_i u_j}$, $-\overline{u_i t'}$, $\dot{E}'''$, F_i and $\dot{Q}'''$. The approximations adopted by Malin[29] for these quantities are described below.

The turbulent stresses $\overline{u_i u_j}$ and fluxes $\overline{u_i t'}$ are calculated from the eddy-viscosity and diffusivity relations by replacing the mean-flow variables, which appear in these relations, with the corresponding turbulent fluid variables. The turbulent-zone eddy viscosity ν_t may then be computed from a suitable turbulence model; for example Malin[29] has used a modified form of the two-equation $k \sim \epsilon$ model. For use with the two-fluid model, this takes the following form:

$$\begin{aligned}(\phi_1 U_j k)_{,j} = &(\phi_1 \nu_t/\sigma_k \cdot k_{,j})_{,j} + (\phi_1 \nu_t/\sigma_\gamma \cdot k \phi_{1,j})_{,j} \\ &+ \phi_1 (P_K - \epsilon) + C_k \phi_1 \phi_2 P_K\end{aligned} \tag{73}$$

$$\begin{aligned}(\phi_1 U_j \epsilon)_{,j} = &(\phi_1 \nu_t/\sigma_\epsilon \epsilon_{,j})_{,j} + (\phi_1 \nu_t/\sigma_\gamma \cdot \epsilon \phi_{1,j})_{,j} \\ &+ \phi_1 |C_{1\epsilon}\epsilon/k \cdot P - C_{2\epsilon}\epsilon^2/k| \\ &+ C_\epsilon \phi_1 \phi_2 P_K\end{aligned} \tag{74}$$

where ν_t and P_K are calculated using turbulent-zone variables rather than the conventional mean-flow ones. The standard values of the model constants are retained, but in addition, the constants C_k and C_ϵ are given the value of 2.5 as recommended by Malin.[29]

The equations for k and ϵ each contain two source terms. The first appears in the standard ($k \sim \epsilon$) model, except that it has been multiplied by ϕ_1 to account for intermittency. The second term in each equation is an extra production included so as to account for an assumed production at the turbulent/nonturbulent interface.

The model for the volumetric entrainment of nonturbulent fluid is given by:

$$\dot{E}''' = \rho k_m \phi_1 \phi_2 c/L' \tag{75}$$

where $k_m = 0.7$, c is the local absolute value of the mean relative velocity between the two fluids and L' is the length scale characterizing interaction processes between the two fluids. This length scale is computed from the local values of k and ϵ through the expression.

$$L' = C_\mu^{3/4} k^{3/2}/\epsilon \tag{76}$$

This closure assumption relates this length scale to one which is representative of the large energy-containing eddies. A further implication is that the entrainment rate can be only positive, i.e. that turbulent fluid cannot enter the nonturbulent category. Since it is known that turbulence can disappear completely, this assumption is at variance with the facts. This defect may not be as serious as it sounds, in most cases.

The interfluid friction forces are given by:

$$F_i = K_f \dot{E}'''(U_i - V_i)/K_m \tag{77}$$

where $K_f = 0.05$ and $K_m = 0.35$ as recommended by Malin.[29] The implication of the friction terms which appear in the momentum equations, is that momentum is imparted to the slower-moving fluid at the expense of the loss experienced by the faster-moving fluid. This effect is additional to the transfer of momentum which is associated with the mass transfer by entrainment.

The conductive heat transfer rate across the turbulent/nonturbulent interface is modelled by:

$$\dot{Q}''' = K_H C_p \dot{E}'''(T_1 - T_2)/K_m \qquad (78)$$

where K_H is an empirical constant given the value of 0.1 recommended by Malin,[29] and C_p is the specific heat at constant pressure. Some results obtained using the above model are presented in *Figures 22 and 23* for the self similar plane wake of a circular cylinder. Within the turbulent fluid, the intermittent form of the $k \sim \epsilon$ model is used, together with Rodi's empirical function for C_μ. In the calculations, the turbulent heat flux is calculated from the eddy-diffusivity relation with $\sigma_t = 0.5$.

The task was to predict the velocity and temperature fields in the turbulent wake with intermittency taken into account. Both the unconditioned and conditioned flow variables were computed.

Comparisons were made with unconditioned and conditioned experimental data, mainly from the investigations of Townsend[201] and Fabris.[203] *Figure 22* compares computed and measured similarity profiles of the intermittency factor and streamwise velocity defects. It is seen that the calculated intermittency profile is in satisfactory agreement with the measurements of La Rue and Libby,[202] and less so with the data of Fabris.[203] It can also be seen that the calculations give excellent agreement with the measured turbulent-zone-averaged velocity profile. There is, however, a quantitative disagreement between the measured and predicted values of the nonturbulent velocity profile. It is important to note that the model predicts correctly that the turbulent fluid moves slower than the nonturbulent fluid at all cross-stream positions, as was observed experimentally by Fabris.[203] Both prediction and experiment indicate that the turbulent fluid moves slower than the average flow in the intermittent region of the wake. In the core of the wake, the flow is fully-turbulent (because the intermittency factor is unity) and so in this region both the turbulent and average flow move with the same velocity.

The computed similarity profile of the turbulent zone-averaged temperature is in very good agreement with the measured profile, as may be seen on inspection of *Figure 23*. Measurements of the temperature of the nonturbulent fluid were not reported in the experiments. This was because the temperature was used to mark the turbulence, and thus the experimenters implicitly assumed that the nonturbulent fluid remained at ambient. Other conditional-sampling studies indicate that there is a finite conductive heat-transfer through the interface.[29] It turns out, however, that the predictions of the thermal characteristics of the

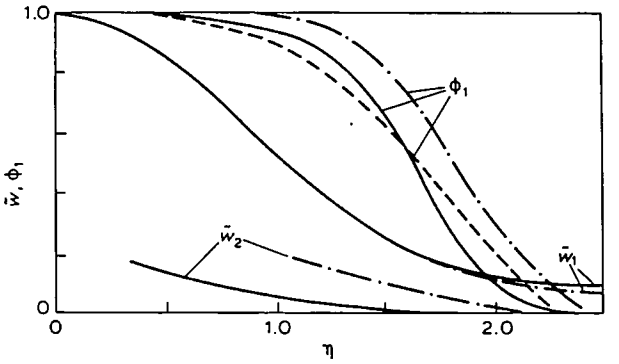


Figure 22 Plane wake. Comparison of measured and predicted similarity profiles of intermittency and conditional streamwise velocities. η stands for $y/y_{1/2}$, ϕ_1 for intermittency and $\bar{w}_i$ for $(w_\infty - w_i)/\Delta w_{mean}$. Δw_{mean} is w_∞ minus centre-line velocity and $y_{1/2}$ is y at the location where Δw is $\frac{1}{2}\Delta w_{mean}$. (——), predictions; (— · —), experiment, reference 203; (— — —), experiment, reference 202

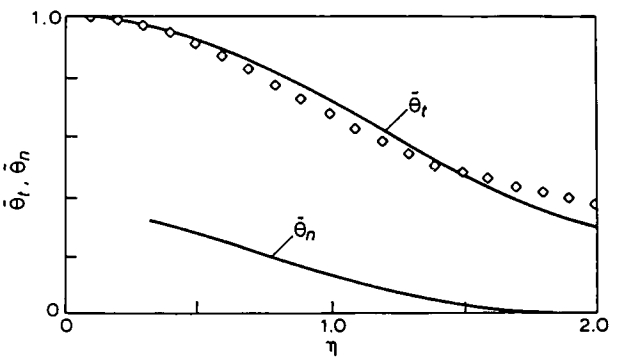


Figure 23 Plane turbulent wake. Comparison of measured and predicted similarity profiles of conditional temperatures (taken from reference 29). $\bar{\theta} \equiv (T - T_\infty)/\Delta T_m$ where ΔT_m is maximum temperature excess. (——), predictions; (◇), experiments, reference 203. Subcripts, t = turbulence, n = nonturbulence

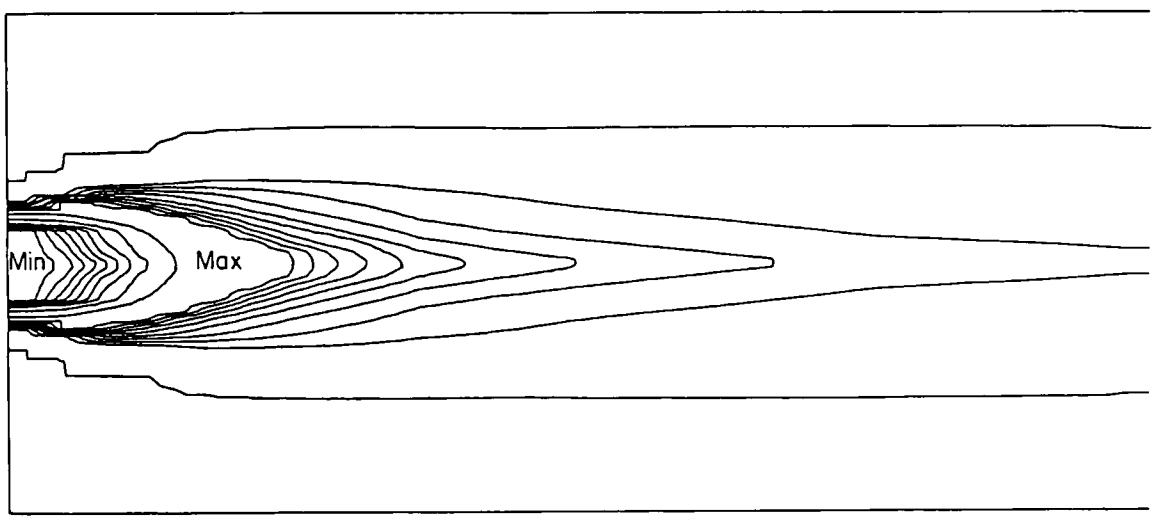


Figure 24 Product mass fraction 0, (0.1), 1. (Taken from reference 211)

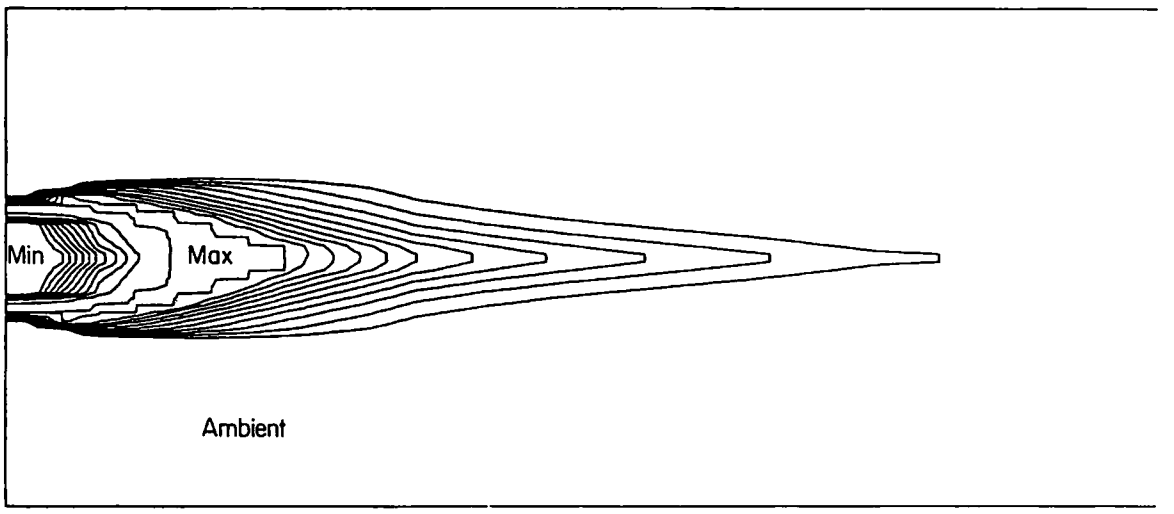


Figure 25 Second fluid temperature contours 600, (100), 1600 k. (Taken from reference 211)

mean flow are not sensitive to the inclusion of the inter-fluid heat-transfer model.

Figure 24 is taken from reference 211 and presents product mass-fraction, for a case of plane diffusion flame, simulated by a two-fluid model. Here fluid 1 refers to oxidant and fluid 2 to 'balloons' of gas containing fuel, products and excess oxidant. Finally, *Figure 25* presents 'fluid 2' temperature contours, for the same case.

The above results for the two-fluid turbulence model serve to demonstrate its potential for simulating accurately intermittency in free turbulent shear flows. Applications to free jets, and more extensive comparisons with data are reported elsewhere.[28,29,210]

Conclusions

The mathematical modelling and computer simulation of turbulent flows are now within the capabilities of modern mathematical and numerical methods. Several difficulties still surround turbulence modelling, and several research groups try to overcome them. The reviewer has tried to compile the progress and problems of turbulence simulations for engineering purposes. Although five parallel approaches for turbulence modelling have been pointed out in this review, only the turbulence transport field models have been explored in detail. This is because to date these models seem to offer a better compromise than the others between relative simplicity and economy and often reasonable accuracy.

Summarizing, the advantages and disadvantages of the various levels of turbulence models are as follows:

Integral equations and entrainment-law theories

- Useful simple prediction procedures have been based on them.
- The entrainment law is not the lowest member of a hierarchy because of its inextensibility. To make good the defects of these theories, one must abandon them in favour of more extensible approaches.

Mixing-length, l

For unbounded flows (mixing layers, jets)

- With more precise l-distribution prescriptions, it would be possible to produce better agreement between predictions and experiments. This would, however, reduce the utility of the model as a predictive tool; it is enough to be in doubt of l's value.
- The mixing-length model has proved very useful as a basis of computational procedures, and is still as good as some more elaborate methods. It is good enough for flows in which turbulence can be treated as isotropic and at a state of local equilibrium.
- Its main shortcoming is its lack of universality.

For bounded flows

- The mixing-length model is still among the best for predicting boundary-layer flows.
- The universality of the constants is fairly high.
- Boundary-layer separation under adverse pressure gradients remains difficult to predict; but this is true of even more advanced methods.
- The mixing-length model is not usable when no mixing-layer width can be identified, e.g. in sudden-enlargement flows.

Effective-viscosity models (e.g. direct prescription of ν_t as function of position or other flow properties)

- Suitable formulae for flows with recirculation are totally absent. It is not worth looking for better formulae. It is better to use a pde model to compute ν_t.
- Despite these limitations, it is useful to remember that ν_t/(flow width x velocity difference) = of order 0.01.
- This can be used as a check on more elaborate models. It can be used also to give starting values for calculation procedures of the iterative type, for which it is desirable to insert 'reasonable' values at the beginning, so as to accelerate convergence.

One-equation models

- The Prandtl k model is more realistic than the Prandtl l model, in that it allows for diffusion and convection of energy. It has not been greatly exercised, attention having been diverted to the Bradshaw k model and to two-equation models. Possibly it deserves more use than it has received.
- The main deficiency of these models is the need for a prescribed length-scale distribution; this is easily supplied only for flows of boundary-layer type.

- Somewhat more computation must be carried out as compared with mixing-length models; but this is no longer a very serious bar.
- Most workers have abandoned one-equation models in favour of two- or even more-equation models. It is possible, however, that one can do better with these models in most flows, because it may be easier after all to specify the length-scale distribution than to compute it with a pde. Further study of one-equation models does not appear to be a bad idea.

Two- and more-equation models

- The Kolmogorov energy ~ frequency model represents a major advance in turbulence modelling; for it permits the length scale to be predicted rather than presumed. This is true also for all other models that followed Kolmogorov's.
- The $k \sim w$ model has been successfully employed for the prediction of numerous turbulent flows. The chief disadvantage was that, in order to fit the logarithmic wall law, one constant must vary with distance from the wall. Because of this, preference was given to other second-equation variables. This disadvantage has now been removed and the model deserves more attention than it has received so far.
- The $k \sim kl$ model has been extensively and successfully employed for the prediction of free turbulent flows and those near walls. Its main defect is the necessity to introduce additional quantities to handle the region close to the wall. For this reason it is not used for elliptic flows, where walls often play an important part.
- The $k \sim \epsilon$ model is the currently most popular two-equation model, mainly for two reasons: (a), the ϵ equation may be derived from the *NS* equations (but this is also true of the w equation); (b), the 'Prandtl' number for ϵ has a reasonable value which fits the experimental data for the spread of the various entities at locations far from walls. The model still requires modifications for various effects (such as buoyancy, rapid compression, etc) and still requires further validation in elliptic 2D and 3D flows.* There is no reason to suppose that, if an equal amount of attention were given to it, another two-equation model would not perform as well, or even better.
- MTE (PK and ND) models relax the assumption of local equilibrium of turbulence by allowing transport of single velocity scale ($\sim \sqrt{k}$). Non-isotropic flows can be handled by MRS and ARS models which employ different velocity scales for different components of the Reynolds stresses; the latter allowing transport of only one velocity scale ($\sim \sqrt{k}$) characteristic of normal Reynolds stresses. For better performance these models are normally supplemented by a transport equation for a length scale characterized by an energy transfer rate. For recirculating flows the MRS models, however, perform only marginally better than PK ($k \sim \epsilon$) models, reflecting the unsatisfactory performance of the ϵ-equation which is common to both. This failure is attributed to the fact that the same energy transfer rate cannot be assumed across the entire turbulence energy spectrum. Multi-scale $k \sim \epsilon$ models, postulating different transfer rates for production and dissipation ranges, remove this deficiency and provide generalization to the standard $k \sim \epsilon$ model at the same level of mathematical complexity.
- In general, predictions of current transport models ($k \sim \epsilon$, $k \sim w$) agree fairly well with experimental data for:
  - 2D boundary layers and jets along plane walls
  - 2D jets, wakes, mixing layers, plumes
  - 2D flow in tubes, channels, diffusers and annuli
  - many 3D flows without strong swirl, density variations or chemical reaction
  - some flows influenced by buoyancy and low-Reynolds-number effects

  Ad hoc corrections must be made to the models or to the 'constants' in order to procure agreement with experiments on:
  - boundary layers on convex and concave walls
  - strongly swirling and recirculating flows
  - axi-symmetrical jets in stagnant surroundings
  - 3D wall jets
  - gravity-stratified flows
  - flows involving chemical reactions
  - two-phase flows
- Current transport models neglect intermittency (i.e. the ragged edges of jets and boundary layers), periodicity (i.e. the eddy-shedding propensity of wakes) and postulate, in general, gradient-induced diffusion, whereas other diffusion mechanisms exist. Furthermore, the absence of a direct means of establishing the constants delays progress.

The above disadvantages lead researchers towards more 'physical' models like large-eddy simulation and 'two-fluid' models.

Large-eddy simulations

- It involves the integration of the Navier–Stokes equations in time using an appropriate finite difference or spectral representation, and is therefore free from the closure difficulty. The technique of full simulation is only available for transition flows. Once turbulence is fully developed, the range of eddy sizes is too great to be represented on any computer now available or foreseen. The remedy is to represent the large eddies only and to account for their interaction with the small or subgrid eddies by means of a subgrid model. This reintroduces 'closure', but is more realistic and universal than the closures of the transport models.
- The technique is promising and is capable of supplementing experimental data (like providing information on the fluctuating pressure).
- The main disadvantages are:
  - it still requires too much computer time and storage
  - realistic subgrid models are not available near walls
  - the boundary conditions; the use of natural boundary conditions strains available computers to their limit, but present synthetic boundary conditions are questionable
- Work continuous at Queen Mary College, London (Prof. D. C. Leslie) and elsewhere. The researchers at Queen Mary College are currently applying the tech-

* It is pointed out that care must be taken to sort out discrepancies between prediction and experiment that arise separately from turbulence modelling and from numerical schemes. Therefore, for these flows turbulence-modelling research is likely to be coupled with research directed towards developing more accurate numerical prediction schemes.

nique to (a), fully developed flows in channels (including buoyancy-driven flows in vertical channels with heated/cooled walls) and; (b), the turbulent near-wake behind a flat plate at zero incidence with potential applications to wing design.

'Two-fluid' models

- Use is made of a newly-developed branch of computational fluid dynamics, that of multi-phase flow.[216,217] The basic idea arises from the fact that turbulent flows are 'spotty'. Fluid parcels ('blobs', 'spots', 'pockets', 'fragments') can be distinguished by reference to differences of velocity, vorticity, density, etc. 'Two-fluid' models idealize this spottiness; they allow for the co-existence of only two kinds of fluid (turbulent fluid and irrotational fluid drawn in from the surroundings), which can occupy (in turn) any location. The 'presence' probability of the turbulent fluid is interpreted as the intermittency of turbulence. 'Multi-fluid' models may also be devised.
- These models are promising. They can predict intermittency and therefore address directly important 'physical' aspects of turbulence. They are also economical; they involve twice as many equations as conventional models, but their solution is no problem on present day computers. The problem is the multiple interactions which occur between the 'two fluids' as they interchange mass, momentum and energy. The mass-transfer and size-change phenomena are of special importance in a two-fluid turbulence model; for turbulent fluid certainly engulfs nonturbulent fluid from its surroundings increasing in quantity as a consequence; and eddies grow not only for this reason but because of the distortion which the mean flow subjects them to. The question is: what laws of 'fluid interaction' should be built into two-fluid theory in order to cause it to represent the major features of turbulence? Some preliminary answers have already been given based on intuition; much more research is required.
- These models are currently undergoing development at the Computational Fluid Dynamics Unit, Imperial College, London (Prof. D. B. Spalding). Other 'intermittency' models are under development elsewhere (e.g. Kollmann, Von Karman Institute and University of California). It is still very early to pass judgement on these models, but they may become the standard practice of tomorrow. Finally, the best advice to be given to the would-be user is 'caution'. For, although there is now a considerable amount of evidence in support of the one- and two-equation models of turbulence, they are still unable to predict several important flows. The user will still have to make guesses in such cases; and he should exercise his intuition and physical insight in interpreting the models' predictions. Some further interesting reading may be found in references 218–230.

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Nomenclature

a_1	empirical parameter
b_{ij}	second-order anisotropy tensor
C	empirical constant in turbulence model
D/Dt	$= \partial/\partial t + u_j(\partial/\partial x_j)$ = total differential
$\mathscr{D}$	diffusion of dependent variable Φ
f	Kolmogorov's turbulence frequency; also empirical function
F_i	external body force (X_i-component); also inter-fluid friction forces
g	gravitational acceleration
k	$= \frac{1}{2}\overline{u_i' u_i'}$, kinematic turbulence kinetic energy
l	Prandtl's mixing length
L, L_k	length scale (characteristic of large eddies) of turbulence
p	instantaneous pressure
Q	set (ξ_i, t_i) of co-ordinates
Re	Reynolds number
s_{ij}	fluctuating symmetric strain-rate
S_{ij}	mean symmetric strain-rate
S_ϕ	source term for variable ϕ
t	time
T	temperature
u	instantaneous fluid velocity
u_i	instantaneous fluid velocity in i-direction
u_1, u_2 or u, v	velocities in directions 1 and 2
u_τ	shear velocity
U	mean velocity
U_1, U_2	mean velocities in directions 1 and 2
$\hat{V}$	velocity scale (characteristic of large eddies) of turbulence
w	mean-square vorticity fluctuations; also for weighting function for statistical averaging of fluctuating quantities
x_i	co-ordinates in tensor notation
x_1, x_2	co-ordinates in directions 1 and 2
y	lateral or radial coordinate
z	physical parameter corresponding to a length scale of turbulence

Greek symbols

α, α_1	empirical parameters
γ	turbulent diffusivity ($\equiv \Gamma/\rho$); also intermittency
Γ	turbulence diffusion coefficient
δ	shear layer thickness
δ_{ij}	Kronecker delta, = 1 for $i = j$ and 0 for $i \neq j$
$\delta(\xi)$	Dirac delta, = 1 for $\xi = 0$, = 0 for $\xi \neq 0$
Δt	increase in t
ϵ	energy transfer of turbulence into dissipation range, or dissipation rate of k
ϵ_p, ϵ_I	energy transfer rates out of production and inertial ranges, respectively
ϵ_{ijm}	alternating third-order tensor, = 1 if i, j, m are cyclic, -1 if i, j, m are anticyclic, and 0 if $i = j$ or m
η	similarity co-ordinate $\equiv y/\delta$
κ	von Kármán constant
μ, μ_t	absolute molecular and turbulent (or eddy) viscosities, respectively
μ^b	bulk-viscosity coefficient
ν, ν_t	kinematic molecular and turbulence (or eddy) viscosities, respectively
ξ_i	Cartesian co-ordinates

π_{ij}	pressure-strain terms
ρ	instantaneous density of the fluid
σ_t	turbulent Prandtl-Schmidt numbers
σ_k, σ_ϵ	empirical numbers similar to σ_t
σ_{ij}	stress-tensor components due to deformation and bulk dilatation
ϕ	instantaneous scalar quantity
Φ	dependent variable
ϕ_1	'presence probability' or intermittency
ω_m	fluctuating vorticity in direction x_m
Ω_{ij}	rotation tensor

Subscripts, superscripts and abbreviations

e	edge
i, j, l, m	directions of vectors
p	production range
t	turbulent
v	viscous
w	wall
$(')$	prime, denoting fluctuating component of a turbulence quantity
$(\bar{})$	overbar, denoting conventional (or temporal) averaging of turbulence quantities or their correlations
$(\ ,i)$	differential with respect to i
n	irrotational fluid
1	turbulent fluid in 'two-fluid' models
2	non-turbulent fluid in 'two-fluid' models
0	reference conditions
pde	partial-differential equation
ARS	algebraic-Reynolds-stress models
MVF	mean-velocity-field models
MTE	mean-turbulence-energy models
MRS	mean-Reynolds-stress models

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